II. CW-complex

정의 1 A ***CW-complex*** X is a Hausdorff space along with a family $\{e_\alpha\}$ of "open cells" such that the following conditions are satisfied.
Let $X^p = \bigcup\{e_\alpha |\ dim\ e_\alpha \le p\}$. ($p$-skeleton)
(1) $X = \coprod e_\alpha$(disjoint union).
(2) $\forall n$-cell e_α, $\exists$ a characteristic map $\varphi_\alpha : (D^n, \partial D^n) \to (X, X^{n-1})$ such that $\varphi_\alpha|_{\overset{\circ}{D^n}}$ is a homeomorphism onto e_α.
(3) (Closure finiteness) Each $\bar{e}_\alpha$ is contained in the union of finitely many open cells.
(4) (Weak topology) $A \subset X$ is closed if and only if $A \cap \bar{e}_\alpha$ is closed in $\bar{e}_\alpha$ for all α.

Note.
(1) Let $\dot{e}_\alpha = \bar{e}_\alpha - e_\alpha$. Then $\varphi_\alpha : (D^n, \partial D^n) \to (\bar{e}_\alpha, \dot{e}_\alpha)$ is onto.
증명 $\varphi_\alpha(D^n)$는 compact이고 따라서 closed이다. 정의의 조건 (2)로부터 $e_\alpha \subset \varphi_\alpha(D^n)$이므로,

$$\bar{e}_\alpha \subset \varphi_\alpha(D^n)$$

이다. 또한 φ_α가 continuous이므로

$$\varphi_\alpha(D^n) = \varphi_\alpha(\overline{\overset{\circ}{D^n}}) \subset \overline{\varphi_\alpha(\overset{\circ}{D^n})} = \bar{e}_\alpha$$

이다. 따라서 $\varphi_\alpha(D^n) = \bar{e}_\alpha$이고 $\varphi_\alpha(\partial D^n) = \dot{e}_\alpha$이다. □

(2) For a finite CW-complex, condition (3) and (4) are automatic.

정의 2 Let X be a CW-complex. A ***subcomplex*** of X is a subset Y along with a subfamily $\{e_\beta\}$ of the cells in X such that $Y = \bigcup e_\beta$ with $\bar{e}_\beta \subset Y$ for all β.

Note. A subcomplex Y is closed and a CW-complex in its own right.
증명 Y가 조건 (1)-(3)을 만족하는 것은 자명하다.
Claim If $B \subset Y$ with $B \cap \bar{e}_\beta$ is closed in $\bar{e}_\beta$ for all β, then B is closed in X.
pf) 조건 (4)에 의하여 $B \cup \bar{e}_\alpha$가 closed in X임을 보이면 된다. 조건 (3)과 Y의 성질에 의하여 $Y \cap \bar{e}_\alpha \subset e_1 \cap \cdots \cap e_k$인 $e_1, \cdots, e_k \subset Y$가 존재한다. 그런데 $B \cap \bar{e}_\alpha \subset Y \cap \bar{e}_\alpha$이므로 $B \cap \bar{e}_\alpha = ((B \cap \bar{e}_1) \cup \cdots \cup (B \cap \bar{e}_k)) \cap \bar{e}_\alpha$이고 따라서 B는 X에서 closed이다.

위의 Claim에서 조건 (4)가 만족됨을 알 수 있고 또한 $B = Y$로 하면 Y가 closed임을 얻는다. □

Example. $X^p = \bigcup\{e_\alpha |\ dim\ e_\alpha \leq p\}$= p-skeleton of X is a subcomplex (and hence closed).

정의 3 $dim\ X= \sup\{dim\ e_\alpha \mid e_\alpha \subset X\}$

정리 1 *Let X be a CW-complex with $\coprod e_\alpha = X$.*
(1) $f : X \to Y$ is continuous if and only if $f|_{\bar{e}_\alpha}$ is continuous for all α.
(2) $F : X \times I \to Y$ is continuous if and only if $F|_{\bar{e}_\alpha \times I}$ is continuous for all α.

증명 다음의 (참고)와 앞절의 정리*로부터 자명하다. □

(참고)
A space $X = \bigcup X_\alpha$ has a coherent topology with respect to X_α (or X is a coherent union of X_α) if
$A \subset X$ is closed $\Leftrightarrow A \cap X_\alpha$ is closed in X_α for all α
or equivalently, a natural projection $p : \coprod X_\alpha \to X$ is a quotient map
or equivalently, $f : X \to Y$ is continuous $\Leftrightarrow f|_{X_\alpha} : X_\alpha \to Y$ is continuous for all α.

증명 숙제 9. □

*정리(Theorem 20, of Munkres, p.113)
$p : X \to Y$, a quotient map. C : locally compact, Hausdorff
$\Rightarrow p \times id : X \times C \to Y \times C$ is a quotient map.