Homology of CW-complexes

1. **(Cellular chain complex)**
Let X be a CW-complex and X^p, p-skeleton of X.
Consider (X^p, X^{p-1}, X^{p-2}).

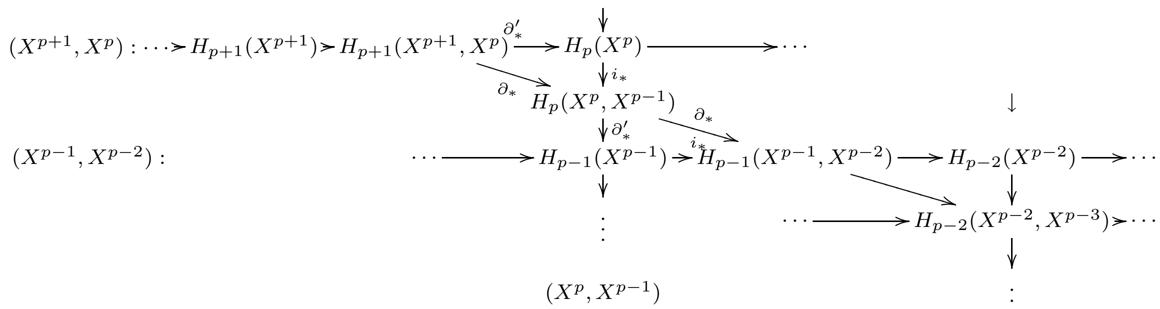
l.e.s : $\cdots \to H_p(X^{p-1}, X^{p-2}) \to H_p(X^p, X^{p-2}) \to H_p(X^p, X^{p-1}) \xrightarrow{\partial_*} H_{p-1}(X^{p-1}, X^{p-2}) \to \cdots$

Let $C_p := H_p(X^p, X^{p-1})$ and $\partial : C_p(X) \to C_{p-1}(X)$ be the connecting homomorphism ∂_* in the above l.e.s..
$\Rightarrow \{C_p(X), \partial\}$ is a chain complex and is called a cellular chain complex.

증명 Note

$$\begin{array}{ccccccccc}
\cdots \to & H_p(X^p, X^{p-2}) & \to & H_p(X^p, X^{p-1}) & \xrightarrow{\partial_*} & H_{p-1}(X^{p-1}, X^{p-2}) & \to \cdots \\
& \uparrow & & = \uparrow & & i_* \uparrow & \\
\cdots \longrightarrow & H_p(X^p) & \longrightarrow & H_p(X^p, X^{p-1}) & \xrightarrow{\partial'_*} & H_{p-1}(X^{p-1}) & \longrightarrow \cdots
\end{array}$$

(All homomorphisms are induced by inclusion maps)
$\Rightarrow \partial = \partial_* = i_* \circ \partial'_*$



위의 diagram에서 $H_p(X^{p+1}, X^p) \xrightarrow{\partial_*} H_p(X^p, X^{p-1}) \xrightarrow{\partial_*} H_{p-1}(X^{p-1}, X^{p-2})$ 즉, $C_{p+1}(X) \xrightarrow{\partial} C_p(X) \xrightarrow{\partial} C_{p-1}(X)$를 보면, diagram commutativity에 의해서 $\partial^2 = \partial_*^2 = 0$이 되는 것을 알 수 있다. 따라서 $\{C_p(X), \partial\}$는 chain complex가 된다.

□

2. $X^p = \coprod D^p_\alpha \cup_f X^{p-1}, f = \coprod \varphi_\alpha|_{\partial D_\alpha}, \bar{f} = \coprod \varphi_\alpha$. Then by the theorem of homology of adjuction space, $H_i(\coprod D_\alpha, \coprod \partial D_\alpha) \xrightarrow{\bar{f}_*, \cong} H_i(X^p, X^{p-1})$.

Consequently,

$$\begin{cases} H_i(X^p, X^{p-1}) = 0 & i \neq p \\ H_p(X^p, X^{p-1}) \cong \bigoplus_\alpha H_p(D^p_\alpha, \partial D^p_\alpha) \end{cases}$$

즉, $H_p(X^p, X^{p-1})$는 free abelian group generated by $\{\phi_{\alpha_*}(\gamma_\alpha)\}$,

where $H_p(D_\alpha, \partial D_\alpha) =< \gamma_\alpha >\cong \mathbb{Z}$이다.

이것은 다음 diagram으로 부터 알 수 있다. 아래의 diagram에서 $\partial = i_* \circ \partial_*$이다.

$$\begin{array}{ccccccc} \bigoplus H_p(D^p_\alpha, \partial D_\alpha) & \xrightarrow{\partial_*} & \bigoplus H_{p-1}(S^{p-1}_\alpha) = \bigoplus < \sigma_\alpha > & & & & \\ \bar{f}_* \downarrow \cong & & \downarrow f_* & & & & \\ H_p(X^p, X^{p-1}) & \xrightarrow[\partial_*]{} & H_{p-1}(X^{p-1}) & \xrightarrow[i_*]{} & H_{p-1}(X^{p-1}, X^{p-2}) & \longrightarrow & \cdots \end{array}$$

여기에서 $\partial(\bar{f}_*(\gamma_\alpha)) = i_* f_* \partial_*(\gamma_\alpha) = i_* f_*(\sigma_\alpha)$

3.

$$\begin{array}{ccccc} H_p(X) & \xleftarrow[(1),\cong]{incl_*} & H_p(X^{p+1}) & \xrightarrow[incl_*]{(2),\cong} & H_p(X^{p+1}, X^{p-2}) \\ & & \downarrow = & & \\ & & H_p(X^{p+1}, X^{-1}(=\emptyset)) & & \end{array}$$

증명 (1)

$$(X^{n+1}, X^n) \Rightarrow \cdots \to H_{p+1}(X^{n+1}, X^n) \to H_p(X^n) \to H_p(X^{n+1}) \to H_p(X^{n+1}, X^n) \to \cdots$$

If $p, p+1 < n+1$, i.e., $p < n$, then $H_p(X^n) \cong H_p(X^{n+1})$.

$$\therefore H_p(X^{p+1}) \xrightarrow{\cong} H_p(X^{p+2}) \xrightarrow{\cong} H_p(X^{p+3}) \xrightarrow{\cong} \cdots$$

Now consider $i_* : H_p(X^{p+1}) \to H_p(X)$.

i_* **is onto.**

$\forall a \in Z_p(X), |a|$:support of a, is compact, hence $|a| \subset X^N$ for some N. And the chain follows from the following commutative diagram.

$$\begin{array}{ccccc} H_p(X^{p+1}) & & \longrightarrow & & H_p(X)(\ni \alpha = \{a\}) \\ & \searrow^{\cong} & & \nearrow & \\ & & H_p(X^N)(\ni \alpha' = \{a\}) & & \end{array}$$

i_* is one-to-one.
$i_*(\beta)=0$. Then $i_*(\beta) =: \beta' = \{b'\} \Rightarrow b' = \partial c, c \in S_{p+1}(X)$.
Furthermore, $|c|$ is compact, hence $|c| \subset X^N$ for some N. And

$$\begin{array}{ccccc} H_p(X^{p+1}) & & \longrightarrow & & H_p(X) \\ & \searrow^{\cong} & & \nearrow & \\ & & H_p(X^N) & & \end{array} \qquad \begin{array}{ccccc} \beta & & \overset{i_*}{\longmapsto} & & \beta' = 0 \\ & \searrow & & \nearrow & \\ & & \beta'' = \{\partial c\} = 0 & & \end{array}$$

$\Rightarrow \beta = 0$

(2) (X^{p+1}, X^i, X^{i-1})로 부터 아래와 같은 l.e.s.를 얻는다.

$$\cdots \to H_p(X^i, X^{i-1}) \to H_p(X^{p+1}, X^{i-1}) \to H_p(X^{p+1}, X^i) \to H_{p-1}(X^i, X^{i-1}) \to \cdots$$

여기에서 만약 $i \neq p, p-1$인 경우 즉, $i < p-1$의 경우 $H_p(X^i, X^{i-1}) = H_{p-1}(X^i, X^{i-1}) = 0$이 되므로 $H_p(X^{p+1}, X^{i-1}) \cong H_p(X^{p+1}, X^i)$가 된다. 따라서 $H_p(X^{p+1}) = H_p(X^{p+1}, X^{-1}) \cong H_p(X^{p+1}, X^0) \cong \cdots \cong H_p(X^{p+1}, X^{p-2})$가 성립한다. □

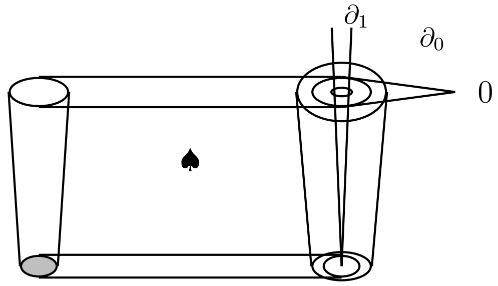
4.

정리 1 $H_p(C(X)) \overset{\lambda,\cong}{\to} H_p(X), \forall p$.
Furthermore, λ is natural, where $H_p(C(X))$ is homology of chain complex $\{C_p, \partial\} = \{H_p(X^p, X^{p-1}), \partial_*\}$.

증명 Consider $(X^{p+1}, X^p, X^{p-1}, X^{p-2})$.

$$\begin{array}{rcccccccc}
 & & & \downarrow & & \downarrow & & & \\
 & 0 & \to & H_{p+1}(X^{p+1}, X^p) & \xrightarrow{=} & H_{p+1}(X^{p+1}, X^p) & & & \\
 & & & \downarrow & \searrow & \downarrow \partial_1 & & & \\
(X^p, X^{p-1}, X^{p-2}): & 0 & \to & H_p(X^p, X^{p-2}) & \xrightarrow{\partial} & H_p(X^p, X^{p-1}) & \xrightarrow{\partial_0} & H_{p-1}(X^{p-1}, X^{p-2}) & \to \cdots \\
 & & & \downarrow & \spadesuit & \downarrow & \searrow^{\partial} & \downarrow = & \\
(X^{p+1}, X^{p-1}, X^{p-2}): & 0 & \to & H_p(X^{p+1}, X^{p-2}) & \to & H_p(X^{p+1}, X^{p-1}) & \to & H_{p-1}(X^{p-1}, X^{p-2}) & \to \cdots \\
 & & & \downarrow & & \downarrow & & & \\
 & & & 0 & & 0 & & &
\end{array}$$

$$(X^{p+1}, X^p, X^{p-2}) \hookrightarrow \quad (X^{p+1}, X^p, X^{p-1})$$



우선 위의 diagram으로부터 $H_p(C(X)) = ker\partial_0/im\partial_1$임을 안다. 이제 위의 그림으로부터 $ker\partial_0/im\partial_1$는 음영이 표시된 부분, 즉 $H_p(X^{p+1}, X^{p-2})$인 것을 알고 따라서 3의 내용에 의해서 $H_p(X)$인 것이 증명된다.

Remark
Let $f : X^{\mathrm{CW}} \to Y^{\mathrm{CW}}$ be a cellular map, i.e., $f(X^p) \subset Y^p, \forall p$. Then, $f : (X^p, X^{p-i}) \to (Y^p, Y^{p-i})$ and induces f_* on various homology of pairs and hence induces a chain map $f_* : C(X) \to C(Y)$.
Since everything is functorial and natural

$$\begin{array}{ccc} H_p(C(X)) & \xrightarrow[\cong]{\lambda_X} & H_p(X) \\ \downarrow f_* & & \downarrow f_* \\ H_p(C(Y)) & \xrightarrow[\lambda_Y]{\cong} & H_p(Y) \end{array}$$

commutes, i.e., λ is natural. □

따름정리 2 *Let X be an n-dimensional CW-complex. Then $H_q(X) = 0, q > n$.*