III. Orientation of Manifolds

III.1 Orientation

1. M : n-manifold, $x \in M, U$: open neighborhood of x
Let V be a coordinate chart s.t. $(V, U, x) \cong (\mathbb{R}^n, D^n, 0)$.
$H_n(M, M-x) \underset{i_*:excision}{\overset{\cong}{\leftrightarrow}} H_n(V, V-x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H_{n-1}(\mathbb{R}^n - 0) = \mathbb{Z}$

A choice of a generator in $H_n(M, M-x) \cong \mathbb{Z}$ is called an orientation at x.

$$\begin{array}{ccccccccc}
H_n(M, M-x) & \xleftarrow[i_*:\cong]{} & H_n(V, V-x) & \xrightarrow{\cong} & H_n(\mathbb{R}^n, \mathbb{R}^n - 0) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n - 0) & \xrightarrow{\cong} & \mathbb{Z} \\
\uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle i_*} & & \uparrow & & \uparrow & & \\
H_n(M, M-D) & \xleftarrow[i_*:\cong]{} & H_n(V, V-U) & \xrightarrow{\cong} & H_n(\mathbb{R}^n, \mathbb{R}^n - D) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n - D) & \xrightarrow{\cong} & \mathbb{Z}
\end{array}$$

$\rho^U_x := j_*$: "restriction to x" and denote by $\rho^U_x(\alpha) = \alpha|_x$

Note. $j_* :\cong$
$\Rightarrow$ (1)(uniqueness)$\forall \alpha, \beta \in H_n(M, M-U)$, $\alpha|_x = \beta|_x \Rightarrow \alpha = \beta$
(2)($\exists$ of continuation) $\forall \beta_x \in H_n(M, M-x), \exists \beta \in H_n(M, M-U)$
s.t. $\beta|_x = \beta_x$

In general, for $A \subset B \subset C \subset M$,
we have $(M, M-C) \hookrightarrow (M, M-B) \hookrightarrow (M, M-A)$.
$\Rightarrow \rho^B_A \cdot \rho^C_B = \rho^C_A$ or $\alpha_C|_B|_A = \alpha_C|_A$ and restriction homomorphism is natural with respect to homeomorphism.

2. An orientation on M is a "continuous" choice $\{\alpha_x\}$ of a generator α_x of $H_n(M, M-x)$ at each $x \in M$, i.e., $\forall x \in M, \exists U$, a ball neighborhood of x and a generator $\alpha \in H_n(M, M-U)$ s.t. $\rho^U_y(\alpha) = \alpha_y,\ \ \forall y \in U$.

M is (R-) *orientable* if $\exists$ an orientation on M.

(1) $M' \subset M$, an open submanifold. M orientable $\Rightarrow M'$: orientable.
$(H_n(M', M'-x) \underset{i_*}{\overset{\cong}{\to}} H_n(M, M-x))$
(2)$\forall M$ is $\mathbb{Z}/2$-orientable. (A choice of generator is unique.)

We can make continuity clear by viewing an orientation as a section.

Sheaf topology on $M_{\mathcal{O}} = \{\beta_x \in H_n(M, M-x) | x \in M\}$

Basis for the topology : Given $\beta_U \in H_n(M, M-U)$, $U^{open} \subset X$,
let $< \beta_U >= \{\beta_x \in M_{\mathcal{O}} | \ \beta_U|_x = \rho_x^U(\beta) = \beta_x\}$

Check.
(1) $\forall \beta_x \in M_{\mathcal{O}}$, $\exists$ coordinate ball neighborhood U and $\beta_U \in H_n(M, M-U)$ s.t. $\beta_U|_x = \beta_x$.
(2) $\beta_x \in < \beta_U > \cap < \beta_V >$
$\Rightarrow \exists W \subset U \cap V$ coordinate ball of x and β_W s.t. $\beta_W|_x = \beta_x$.
Show $< \beta_W > \subset < \beta_U > \cap < \beta_V >$:
$\beta_y \in < \beta_W > \Rightarrow \beta_W|_y = \beta_y \Rightarrow \beta_U|_W|_x = \beta_U|_x = \beta_x = \beta_W|_x \Rightarrow \beta_U|_W = \beta_W$
$\Rightarrow \beta_y = \beta_W|_y = \beta_U|_W|_y = \beta_U|_y \in < \beta_U >$ □

$M_{\mathcal{O}}$ with this topology is called the orientation sheaf of M.

3. Properties of $M_{\mathcal{O}}$

(1) $p : M_{\mathcal{O}} \to M$ is a covering.($M_{\mathcal{O}}$ is not connected in general.)
$\beta_x \mapsto x$

증명 p is continuous : $\forall \beta_x \in M_{\mathcal{O}}$ and V, a neighborhood of x, $\exists U$, a coordinate ball $\subset V$ s.t. $p(< \beta_U >) = U \subset V$.

p is open : p sends basic open sets $< \beta_U >$ to open sets U.

$\forall x \in M$, choose a coordinate ball neighborhood U, then $\{< \beta_U > | \beta_U \in H_n(M, M-U)\}$ evenly covers U :
disjoint: uniqueness로부터 $\alpha_U|_x = \beta_U|_x \Rightarrow \alpha_U = \beta_U \Rightarrow < \alpha_U >=< \beta_U >$
open :clear □

(2) $| \ | = \nu : M_{\mathcal{O}} \to \mathbb{Z}_{\geq 0}$ defined by $\beta_x = \nu(\beta_x) \cdot$a generator in $H_n(M, M-x) \cong \mathbb{Z}$ is continuous.

증명 $\forall \beta_x, \exists \beta_U (U$: coordinate ball) s.t. $\beta_U|_x = \beta_x$.
Suppose $\beta_U = n \cdot \alpha_U$, $\alpha_U =$ a generator of $H_n(M, M-U), n \geq 0$.
Thm. $\Rightarrow \beta_y \in < \beta_U > \Rightarrow \beta_y = \beta_U|_y = n \cdot \alpha_U|_y$
$\therefore \nu(\beta_y) = n, \forall y \in U$. □

(3) A section s of $p : M_{\mathcal{O}} \to M$ on $A \subset M$ is continuous iff s is locally constant, i.e., $\forall x \in A$, $\exists U$ and β_U s.t. $s(x) = \beta_U|_x$, , $\forall x \in A \cap U$.

증명 숙제 10. □

From now on, sections are always continuous.

(4) s, s': sections on a connected $A \subset M$
$s(a) = s'(a)$ for some $a \in A \Rightarrow s \equiv s'$
증명
$$\begin{array}{ccc} & & M_{\mathcal{O}} \\ & s \nearrow & \downarrow \text{(Uniqueness of Lifting)} \\ A_{cnt} & \underset{i}{\hookrightarrow} & M \end{array}$$
□

Note. $\beta_U|_{A\cap U}$ can be viewed as a section on $A \cap U$ and denote it by $\beta_{A\cap U}$.

4. We can rephrase the orientability of M as follows:
M is orientable if $\exists$ a global section $s : M \to M_{\mathcal{O}}$ with $\nu(s(x)) = 1$ and s is called an orientation.

More generally, M is orientable along $A \subset M$ if $\exists$ a section $s : A \to M_{\mathcal{O}}$ with $\nu(s(x)) = 1$.

(1) M is orientable iff $\exists$ a nowhere vanishing global section s:

Note. $s, s' \in \Gamma M$ = sections over M
$\Rightarrow s + s' \in \Gamma M$
ns (and $rs \in \Gamma M,\ r \in R$)

증명 May assume M is connected.
Suppose $\nu(s(x)) = |s(x)| = n \neq 0$. Then $s(x) = n\alpha_x$ for some generator α_x.
$M_{\mathcal{O}} \overset{\nu}{\to} \mathbb{Z}_{\geq 0}$
$s \uparrow \ \nearrow \nu \cdot s$ is continuous and M is connected. $\Rightarrow \nu(s(y)) = n, \forall y \in M$.
M
$\Rightarrow$ "$\frac{1}{n}s$" is a well-defined section and locally constant. (Use $\beta_U = s|_U$) □

(2) $M_{\mathcal{O}} - \nu^{-1}(0) (\cong M)$ is orientable :

증명 $x \in U = \frac{1}{2} - ball \subset V$ = coordinate unit ball.

$$\begin{array}{ccccccc}
H_n(M_{\mathcal{O}}, M_{\mathcal{O}} - <\beta_U>) & \xleftarrow{\cong} & H_n(<\beta_V>, <\beta_V> - <\beta_U>) & \xrightarrow{\cong} & H_n(<\beta_V>, <\beta_V> - \beta_x) & \xrightarrow{\cong} & H_n(M_{\mathcal{O}}, M_{\mathcal{O}} - \beta_x) \\
 & & \downarrow p_*:\cong & & \downarrow p_*:\cong & & \\
H_n(M, M-U) & \xleftarrow{\cong} & H_n(V, V-U) & \xrightarrow{\cong} & H_n(V, V-x) & \xrightarrow{\cong} & H_n(M, M-x)
\end{array}$$

$$\begin{array}{ccc}
"\beta_U" & \longrightarrow & "\beta_x" \neq 0 \\
 & \downarrow \quad \downarrow & \\
\beta_U & \longrightarrow & \beta_x
\end{array}$$

$\Rightarrow$ locally constant $\Rightarrow$ (1)로부터 clear. □

(3) Let M be connected and let $\bar{M}$ be a componenet of $M_{\mathcal{O}} - \nu^{-1}(0)$.
$\Rightarrow p : \bar{M} \to M$ is a covering. (at most two-fold)
p is a 1-fold covering (i.e. homeomorphism) iff M is orientable.
(This follows from a general fact from Covering Space Theory.)

증명 ($\Rightarrow$) Since p is a homeomorphism and $\bar{M}$ is orientable, M is orientable. In fact, p^{-1} is a non-vanishing section on M.
($\Leftarrow$) M : orientable $\Rightarrow \exists$ section s with $\nu(s(x)) = 1$.
Then for $\beta_x \in \bar{M}, \beta_x = n_0 s(x)$.
$\Rightarrow s' = n_0 s$ is a section and hence p is a homeomorphism. Note that since $s'(M)$ is a connected set intersecting a component , $s'(M) \subset \bar{M}$. □

Remark. The same argument shows that M : orientable $\Rightarrow \bar{M} = n_0 s(M)$ and hence $M_{\mathcal{O}} = \coprod_{n \in \mathbb{Z}} ns(M)$, i.e., $M_{\mathcal{O}} \cong M \times \mathbb{Z}$.

따름정리 1 *p is a 2-fold covering iff M is non-orientable.*
$\bar{M}$ is an orientable double covering of non-orientable M.

(4) $\pi_1 M$ does not have a subgroup of index 2. $\Rightarrow M$ is orientable. In particular, $\pi_1 M = 0 \Rightarrow M$ is orientable.

5 . M is *orientable along* $A \subset M$ if $\exists$ a section $s : A \to M_{\mathcal{O}}$ with $\nu(s(x)) = 1$.
Let $\Gamma A = \{$ sections on $A\}$: a group (or $R-$ module)

(1) M : orientable along $A \Rightarrow$

$$\begin{array}{ccc} p^{-1}(A) & \xrightarrow{\phi:\cong} & A \times \mathbb{Z} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & A & \end{array}$$

증명 $p^{-1}(A) \underset{s}{\overset{p}{\rightleftarrows}} A$ is a covering.
$\beta_x \in p^{-1}(A) \Rightarrow \beta_x = ns(x)$ and define $\phi(\beta_x) = (x, n)$.

ϕ is 1-1 and onto. : clear
$\forall x \in A, 3(3) \Rightarrow \exists\ U$, a coordinate ball neighborhood and $\alpha_U \in H_n(M, M-U)$,
s.t. $\alpha_U = s$ on $A \cap U$.
$\forall \beta_U$, if $\beta_U|_x = ns(x) = n\alpha_U|_x$ for some n, then $\beta_U = n\alpha_U = ns$ and

$$\begin{array}{ccc} <\beta_U> \cap p^{-1}(A) = <\beta_U|_{A\cap U}> = <\beta_{A\cap U}> & \xrightarrow{\phi} & (A \cap U, n) \\ {\scriptstyle p:\cong}\searrow & & \swarrow{\scriptstyle p_1:\cong} \\ & A \cap U & \end{array} \quad \text{commute.}$$

$\Rightarrow \phi$ is a local homeomorphism.
$\therefore \phi$ is a homeomorphism. □

따라서 다음 사실들이 성립한다.

(2) M : orientable along A and A : connected $\Rightarrow \Gamma A \cong \mathbb{Z}$(or R).
In general, $\Gamma A \cong \mathbb{Z}^k$, $k =$ the number of components of A.

(3) M : orientable $\Rightarrow M$ is orientable along $\forall A \subset M$.
In this case, $A^{connected} \Rightarrow \Gamma A \cong \mathbb{Z}$.

(4) $\bar{A}$: a component of $p^{-1}(A) - \nu^{-1}(0) \Rightarrow p : \bar{A} \to A$ is 1 or 2-fold covering and orientable iff p is homeomorphism. (same proof as 4(3))
M : non-orientable along $A \Rightarrow \Gamma A = 0$.