III.2 $H_q(M^n, M-A),\quad q \geqslant n$

idea : Compare $H_n(M, M-A)$ with ΓA.

6. Suppose $U^{open} \subset M$. Then $\beta_U \in H_n(M, M-U)$ can be viewed as a section as before and we have a homomorphism $j_U : H_n(M, M-U) \to \Gamma U$.

$$\beta_U \mapsto j_U(\beta_U) : x \mapsto \beta_U|_x$$

In general, $\forall A \subset M$, does $j_A : H_n(M, M-A) \to \Gamma A$(defined by "$\beta$" $= j_A(\beta) : x \mapsto \beta|_x$) define a homomorphism, i.e., is $j_A(\beta)$ continuous section on A?

증명 Want "β" is locally constant, i.e., $\forall a \in A$, $\exists V$ and β_V s.t. $\beta|_x = \beta_V|_x \ , \forall x \in A \cap V$.
Recall : Can represent $\beta = \{b\}$ with $\partial b \subset M - A$.
$|\partial b|$: compact $\Rightarrow$ $U = M - |\partial b|$ is open and choose V, a coordinate ball neighborhood of a with $V \subset U$.

$$\begin{array}{ccccccc}
(M, M-U) & \to & (M, M-V) & \quad\Rightarrow\quad & \beta' = \{b\} & \dashrightarrow & \exists \beta_V \\
\downarrow & & \downarrow \cong on H_n & & \downarrow & & \updownarrow \cong \\
(M, M-A) & \to & (M, M-a) & & \beta'|_A = \beta & \to & \beta|_a
\end{array}$$

And $\forall x \in A \cap V,\ \ \beta_V|_x = \beta'|_V|_x = \beta'|_x = \beta'|_A|_x = \beta|_x$. □

7. When is $j_A : H_n(M, M-A) \to \Gamma A$ an isomorphism?
Know : true if $A = U$, a coordinate ball.

$$\begin{array}{ccccc}
H_n(V, V-U) & \xrightarrow{excision:\cong} & H_n(M, M-U) & \xrightarrow{j_U} & \Gamma U \\
\downarrow \psi_*:\cong & & & & \downarrow \psi_*:\cong \\
\mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \backslash D) & & \xrightarrow{j_D:\cong} & & \Gamma D = \mathbb{Z}
\end{array}$$

Also note $H_q(M, M-U) \cong H_q(\mathbb{R}^n, \mathbb{R}^n \backslash D) \cong \widetilde{H_{q-1}}(S^{n-1}) = 0$ if $q > n$.

Let $M = \mathbb{R}^n$.
If A is a "nice " compact set, then j_A is $\cong$.
e.g. $A = D, [0,1] \times [0,1], [0,1]$, point, $\cdots$ etc.
But note that if $A = M = \mathbb{R}^1$, $\ H_1(\mathbb{R}^1, \mathbb{R}^1 \backslash \mathbb{R}^1) = H_1(\mathbb{R}^1) = 0$ but $\Gamma \mathbb{R}^1 = \mathbb{Z}$.

Note. A: closed. Then

$$\begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ & {\searrow}^{j_A} \quad {\nearrow}_{\subset} & \\ & \Gamma_c A & \end{array}$$

i.e., $j_A(\beta) \in \Gamma_c A$, $\forall \beta$ where $\Gamma_c A$ consists of sections with compact support.

증명Let $\beta = \{b\}$: relative cycle $\Rightarrow |b|$: compact.
Then $\forall x \in (M - |b|) \cap A, \beta|_x = 0$ since $|b| \subset M - x$. Think of this in chain level.

$$\begin{array}{ccccccccc} 0 & \to & S_n(M-A) & \to & S_n(M) & \to & S_n(M)/S_n(M-A) & \to & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \to & S_n(M-x) & \to & S_n(M) & \to & S_n(M)/S_n(M-x) & \to & 0 \end{array}$$

$\therefore \beta$ has a support $\subset |b| \cap A$: compact $\square$

So the right statement is $j_A : H_n(M, M-A) \to \Gamma_c A$.
Furthermore, j_A is natural:

$$B \subset A \subset M \Rightarrow \quad \begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ \downarrow \rho^A_B & & \downarrow \textit{restriction} \\ H_n(M, M-B) & \xrightarrow{j_B} & \Gamma B \end{array} \quad \text{commute.}$$

Exercise. **숙제 11.**

$$f : (M, A) \xrightarrow{\cong} (N, B) \Rightarrow \quad \begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ \downarrow f_* : \cong & & \downarrow \text{"}f_*\text{"} \\ H_n(N, N-B) & \xrightarrow{j_B} & \Gamma B \end{array} \quad \text{commute.}$$

8. (Theorem) Let M be an n-dimensional manifold and $A^{\text{closed}} \subset M$. Then
(1) $H_q(M, M-A) = 0$ for $q > n$
(2) $H_n(M, M-A) \cong \Gamma_c A$

증명

보조정리 1 *(MV) A, B closed $\subset M$. If the theorem is true for A, B and $A \cap B$, then so is for $A \cup B$.*

증명 A, B : closed $\Rightarrow M - A, M - B$ open with $(M-A) \cap (M-B) = M - A \cup B$
$(M-A) \cup (M-B) = M - A \cap B$

relative MV:

$$\begin{array}{lccccccccc} \text{hypothesis} \to & 0 & \to & H_n(M, M - A \cup B) & \to & H_n(M, M-A) \oplus H_n(M, M-B) & \to & H_n(M, M - A \cap B) & \to & 0 \\ & \downarrow \cong & & \downarrow j & & \downarrow \cong & & \downarrow \cong & & \\ \text{exact}: & 0 & \longrightarrow & \Gamma(A \cup B) & \longrightarrow & \Gamma(A) \oplus \Gamma(B) & \longrightarrow & \Gamma(A \cap B) & \longrightarrow & 0 \end{array}$$

$$s \longmapsto (s|_A, s|_B)$$

$$(a, b) \longmapsto a|_{A \cap B} - b|_{A \cap B}$$

(i) follows from relative MV-sequence.
(ii) follows from 5-lemma. □

보조정리 2 *If $M = \mathbb{R}^n$ and A is a compact subset of $\mathbb{R}^n$, then the theorem is true.*

증명 Know : The theorem is true for a "nice" compact set $A \subset \mathbb{R}^n$, for instance A =rectangle.
By lemma 1, the theorem is true if A is a finite union of rectangles by induction on number of rectangles. □

이제 compact set A에 대하여 정리가 성립함을 보이자.
먼저 $j_A : H_n(M, M - A) \to \Gamma A = \Gamma_c A$가 onto임을 보인다.
(i) For $s \in \Gamma A$, there exists an open set U containing A such that s can be extended to $\bar{s}$ on U.
pf) $s(A)$ is compact $\Rightarrow s(A)$ lies in finitely many sheets of $\mathbb{R}^n_{\mathcal{O}} \cong \mathbb{R}^n \times \mathbb{Z}$.
Let A_i=$s^{-1}(\mathbb{R}^n \times i)$ for $i \in \mathbb{Z}$. A_i is compact.
So there exists an open set U_i containing A_i such that U_i's are pairwise disjoint and s can be extended to $\bar{s}$ over $U = \bigcup U_i$.

(ii) Cover A by finitely many rectangles in U and let A' be the union of rectangles.

Then the following diagram commutes.

$$\begin{array}{ccc} H_n(M, M-A') & \xrightarrow[\cong]{j_{A'}} & \Gamma A' \ni \bar{s}|_{A'} \\ \downarrow & & \downarrow \text{restriction} \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \ni s \end{array}$$

임의의 $s \in \Gamma A$에 대하여 (i)에 의하여 $\bar{s}$가 존재하므로 $\bar{s}|_{A'}$은 s에 대응되는 $\Gamma A'$의 원소이다. 또한 $j_{A'}$은 isomorphism이므로 j_A는 onto이다.

이제 j_A이 1-1이고 $H_q(M, M-A) = 0$ for $q > n$임을 보이자.
Let $\alpha \in H_q(M, M-A)$ and assume $j_A(\alpha) = 0$ if $q = n$.
Suppose $\alpha = \{a\}$ with $\partial a \subset M - A$. Since $|\partial a|$ is compact, $V = M - |\partial a|$ is open. Let $\alpha' = \{a\} \in H_q(M, M-V)$.

$(q = n)$: Let $U \subset V$ be an open set containing A each of whose components intersects A. Let A' be a finite union of rectangles which covers A and is contained in U.
Then $j_U(\alpha'|_U)$ is a section on U which has a zero on each of its component.
($\because$ Each component intersects A and $j_A(\alpha) = 0$.)
By uniqueness of sections on connected sets, $j_U(\alpha'|_U) = 0$.

$$\begin{array}{ccc} H_n(M, M-U) & \xrightarrow{j_U} & \Gamma U \\ \downarrow & & \downarrow \\ H_n(M, M-A') & \xrightarrow[\cong]{j_{A'}} & \Gamma A' \\ \downarrow & & \downarrow \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \end{array} \qquad \begin{array}{ccc} \alpha'|_U & \longmapsto & 0 \\ \downarrow & & \downarrow \\ \alpha'|_{A'} & \longmapsto & 0 \\ \downarrow & & \downarrow \\ \alpha & \longmapsto & 0 \end{array}$$

Therefore by the above diagram, $\alpha'|_{A'} = 0$ and hence $\alpha = 0$.
$(q > n)$: The theorem is true for A'. So $\alpha'|_{A'} = 0$ and hence $\alpha = 0$. □

정리의 증명

Step 1. $A^{\text{compact}} \subset M$:
A is a finite union of compact sets each of which is contained in a coordinate ball neighborhood($\approx \mathbb{R}^n$) and apply lemma 1 and lemma 2.

Step 2. $A \subset U^{\text{open}} \subset \overline{U}^{\text{compact}} \Rightarrow$ The theorem is true for $A \subset U(= M)$[1]
where A is a closed subset of U:

[1] manifold의 open subset은 manifold이므로 $U = M$으로 하여도 무방하다.

$\partial U = \overline{U} - U$ is compact and $U - \overline{A} = U - A$.
Consider $(M, M - \partial U, M - (\partial U \cup \overline{A}))$.
$(q > n)$:

$$H_{q+1}(M, M-\partial U) \to H_q(M-\partial U, M-(\partial U \cup \overline{A})) \to H_q(M, M-(\partial U \cup \overline{A})) \to \cdots$$

By excision theorem, $H_q(M - \partial U, M - (\partial U \cup \overline{A})) = H_q(U, U - A)$.
By step 1, $H_{q+1}(M, M - \partial U) = 0$ and $H_q(M, M - (\partial U \cup \overline{A})) = 0$.
Therefore $H_q(U, U - A) = 0$.
$(q = n)$:

$$\begin{array}{ccccccc} 0 \longrightarrow & H_n(U, U-A) & \longrightarrow & H_n(M, M-(\partial U \cup \overline{A})) & \longrightarrow & H_n(M - \partial U) \\ & \downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle \cong (\text{step 1})} & & \downarrow{\scriptstyle \cong (\text{step 1})} \\ 0 \longrightarrow & \Gamma_c A & \xrightarrow[\text{extension by 0 outside support}]{} & \Gamma(\partial U \cup \overline{A}) & \xrightarrow[\text{restriction}]{} & \Gamma(\partial U) \end{array}$$

So, j_A is an isomorphism by 5 lemma.

Step 3.(general case)
Show $j_A : H_n(M, M - A) \to \Gamma_c A$ is onto:
$\forall s \in \Gamma_c A$, let supp $s = K$.
Then K is compact and $\exists U$ such that $K \subset U \subset \overline{U}^{\text{compact}}$
Let $A' = A \cap U$.

$$\begin{array}{ccc} H_n(U, U-A') & \xrightarrow{\cong (\text{step 2})} & \Gamma_c A' \ni s|_{A'} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \text{extension by 0 outside } K} \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma_c A \ni s \end{array}$$

So, j_A is onto.

Show j_A is 1-1 and $H_q(M, M - A) = 0$ if $q > n$:
$\alpha \in H_q(M, M - A)$ and assume $j_A(\alpha) = 0$ if $q = n$.
If $\alpha = \{a\}$ with $\partial a \subset M - A$, $|a|$: compact $\Rightarrow |a| \subset U \subset \overline{U}^{\text{compact}}$.
Let $A' = A \cap U$. Apply the above diagram.
$q = n :$ Since $|a| \subset U$, $\alpha' = \{a\} \in H_n(U, U - A')$. So in the above diagram, $i_*(\alpha') = \alpha$. Since $j_A(\alpha) = 0$ and $0|_{A'} = 0$, $j_{A'}(\alpha') = 0$. So $\alpha' = 0$ and hence $\alpha = 0$.
$q > n :$ By step 2, $\alpha' = \{a\} \in H_q(U, U - A') = 0$. So $\alpha = 0$. $\square$

9. Consequences of the theorem

(1) Let A be connected and closed but not compact. Then $H_n(M, M-A) = 0$. In particular if M is connected but not compact then $H_n(M) = 0$.

증명 If $s \in \Gamma_c A$, $\nu \circ s : A \to \mathbb{Z}^{\geq 0}$ is continuous and $= 0$ at some $a \notin \text{supp } s$. So $\Gamma_c A = 0$ □

(2) If M is orientable along A, A is compact and has k components, then $H_n(M, M-A) \cong \mathbb{Z}^k$.

증명 $\Gamma_c A = \Gamma A \cong \mathbb{Z}^k$ by 5. (2). □

(3) If $A \subset \mathbb{R}^n$ is compact and has k components, then $\widetilde{H}_{n-1}(\mathbb{R}^n - A) \cong \mathbb{Z}^k$.

증명 From homology sequence of pair $(\mathbb{R}^n, \mathbb{R}^n - A)$, we get

$$0 \to H_n(\mathbb{R}^n, \mathbb{R}^n - A) \to \widetilde{H}_{n-1}(\mathbb{R}^n - A) \to 0$$

So $\widetilde{H}_{n-1}(\mathbb{R}^n - A) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - A) \cong \mathbb{Z}^k$ by (2). □

(4) If M is connected and closed[2], then $H_n(M) = \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable} \\ 0, & \text{if } M \text{ is non-orientable} \end{cases}$

증명 Clear from 5. (3) and (4). □

Remark. For a PID R, $H_n(M) = \begin{cases} R, & \text{if } M \text{ is } R-\text{orientable} \\ 0, & \text{if } M \text{ is not } R-\text{orientable} \end{cases}$

10. Fundamental class of M and degree

A choice of generating section is an orientation and the corresponding homology class ζ[3] is called the ***fundamental (orientation) class*** of M, i.e., $\zeta|_x \in H_n(M, M-x)$ is the preferred orientation at x for all $x \in M$.

Let M^n, N^n be oriented closed connected manifolds. For $f : M \to N$, if $f_*(\zeta_M) = k \cdot \zeta_N$, k is called the ***degree*** of f.

숙제 14 $\forall y \in N$, "regular value" i.e., $\forall x \in f^{-1}(y)$, f is a homeomorphism on a neighborhood U_x ($f|_{U_x} : U_x \xrightarrow{\cong} V_y$). Then

$$deg f = \sum_{x \in f^{-1}(y)} deg_x f$$

[2] A manifold M is closed if it is compact without boundary.
[3] $[M]$으로 쓰기도 한다.

where $deg_x f$ is defined by $f|_{U*} : H_n(U, U - x) \to H_n(V, V - y)$.
In particular, if $p : M \to N$ is a k-fold covering, $degp = k$ (with respect to the induced orientation on M).

숙제 15 (22.43) (22.49) (22.50)*