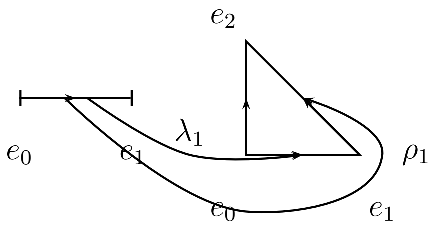
IV.3 Cup and Cap Product

Cup Product
coefficient: R, a commutative ring with 1.

정의 1 $a \in S^p(X), b \in S^q(X)$, define a cup product of a and b, $a \cup b \in S^{p+q}(X)$:
Let $\lambda_p : \triangle_p \to \triangle_{p+q}$ be a linear map determined by
$(e_0, \cdots, e_p) \mapsto (e_0, \cdots, e_p) < \triangle_{p+q}$ (λ_p is called "front p-face")
$\rho_q : \triangle_q \to \triangle_{p+q}$
$(e_0, \cdots, e_q) \mapsto (e_p, \cdots, e_{p+q}) < \triangle_{p+q}$ (ρ_q is called "back q-face")

예



Notation: $a \in S^p, x \in S_p \Rightarrow a(x) =: (x, a)$: pairing.
This induces a pairing on homology - cohomology level. [1]

Define $(\sigma, a \cup b) := (\sigma\lambda_p, a)(\sigma\rho_q, b)$ and extend linearly on S_{p+q}, so that $a \cup b \in S^{p+q}$, where σ is a singular $(p+q)$-simplex.
(1) $\cup$ is bilinear : clear

(2) $\cup$ is associative : $(a \cup b) \cup c = a \cup (b \cup c), \quad a \in S^p, b \in S^q, c \in S^r$:
$(\sigma, (a \cup b) \cup c) = (\sigma\lambda_{p+q}, a \cup b)(\sigma\rho_r, c) = (\sigma\lambda_{p+q}\lambda_p, a)(\sigma\lambda_{p+q}\rho_q, b)(\sigma\rho_r, c)$
$(\sigma, a \cup (b \cup c)) = (\sigma\lambda_p, a)(\sigma\rho_{q+r}, b \cup c) = (\sigma\lambda_p, a))(\sigma\rho_{p+q}\lambda_q, b)(\sigma\rho_{q+r}\rho_r, c)$
and it is easy to check $\sigma\lambda_{p+q}\rho_q = \sigma\rho_{q+r}\lambda_q$ and $\sigma\rho_r = \sigma\rho_{q+r}\rho_r$.

(3) $1 \cup a = a \cup 1 = a, \quad 1 \in S^0(X)$ with $(x, 1) = 1, \forall x \in S_0(X)$

$\therefore$ Let $S^* = \bigoplus_p S^p$. Then $\cup$ is defined on S^* by extending linearly, i.e., $a = \sum_p a^p \in$

[1] $(\xi, \alpha) = ([x], [a]) = (x, a)$
$(i)(x + \partial y, a) = (x, a) + (\partial y, a) = (x, a)(\because (\partial y, a) = (y, \delta a) = 0)$
$(ii)(x, a + \delta b) = (x, a) + (x, \delta b) = (x, a)(\because (x, \delta b) = (\partial x, b) = 0)$

$S^*, b = \Sigma_q b^q \in S^* \Rightarrow a \cup b = \Sigma_{p,q}(a^p \cup b^q)$ and $\cup$ is an associative product with 1 on S^* giving a ring structure.(or R-algebra structure)

2. Derivation property : $\delta(a \cup b) = \delta a \cup b + (-1)^p a \cup \delta b$.

증명 $(\sigma_{p+q+1}, \delta(a \cup b)) = (\partial\sigma, a \cup b) = \sum_0^{p+q+1} (-1)^i (\sigma f^i_{p+q+1}, a \cup b)$ [2]

$= \sum_0^p (-1)^i (\sigma f^i_{p+q+1}\lambda_p, a)(\sigma f^i_{p+q+1}\rho_q, b) + \sum_{p+1}^{p+q+1} (-1)^i (\sigma f^i_{p+q+1}\lambda_p, a)(\sigma f^i_{p+q+1}\rho_q, b)$

Note. $0 \le i \le p+1 \Rightarrow \sigma f^i_{p+q+1}\lambda_p = \sigma\lambda_{p+1} f^i_{p+1}$

$0 \le i \le p \Rightarrow \sigma f^i_{p+q+1}\rho_q = \sigma\rho_q$

$$\Rightarrow \sum_0^p = (\underbrace{\sum_0^{p+1} (-1)^i \sigma\lambda_{p+1} f^i_{p+1}}_{=\partial(\sigma\lambda_{p+1})}, a)(\sigma\rho_q, b) - (-1)^{p+1}(\sigma f^{p+1}_{p+q+1}\lambda_p, a)(\sigma\rho_q, b)$$

$$= (\sigma, \delta a \cup b) + (-1)^p (\sigma\lambda_p, a)(\sigma\rho_q, b)$$

Similarly, $p \le i \le p+q+1 \Rightarrow f^i_{p+q+1}\rho_q = \rho_{q+1} f^{i-p}_{q+1}$

$p+1 \le i \le p+q+1 \Rightarrow f^i_{p+q+i}\lambda_p = \lambda_p$

$$\therefore \sum_{p+1}^{p+q+1} = (\sigma\lambda_p, a)(\underbrace{\sum_p^{p+q+1} (-1)^i \sigma\rho_{q+1} f^{i-p}_{q+1}}_{= \sum_0^{q+1} (-1)^{j+p}\sigma\rho_{q+1} f^j_{q+1} = (-1)^p \partial(\sigma\rho_{q+1})}, b) - (-1)^p (\sigma\lambda_p, a)(\sigma\rho_q, b)$$

$$= (-1)^p (\sigma, a \cup \delta b) - (-1)^p (\sigma\lambda_p, a)(\sigma\rho_q, b)$$

□

S^* : a ring with respect to $\cup$

$Z^* = \oplus Z^p$: subring by derivation property

$B^* = \oplus B^p$: ideal in Z^*

$\Rightarrow \cup$ is induced on $Z^*/B^* = H^* = \oplus H^p$: a "graded ring"(algebra) with 1.

(Note. 1 is a cocycle since $\delta 1(e) = 1(\partial e) = 1(e_1 - e_0) = 1 - 1 = 0$)

3. Let $f : X \to Y$

$\Rightarrow f^\sharp(a \cup b) = f^\sharp(a) \cup f^\sharp(b)$ and hence $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$

$(f^\sharp = \tilde{f}_\sharp : S^*(X) \leftarrow S^*(Y)$ and $f^* : H^*(X) \leftarrow H^*(Y))$

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[2] $f^i_p : \triangle_{p-1} \longrightarrow \triangle_p$

$(e_0, \cdots, e_{p-1}) \mapsto (e_0, \cdots, \hat{e_i}, \cdots, e_p)$

증명$(\sigma, f^\sharp(a\cup b)) = (f_\sharp(\sigma), a\cup b) = (f_\sharp(\sigma)\lambda_p, a)(f_\sharp(\sigma)\rho_q, b)$
$= (f\cdot\sigma\cdot\lambda_p, a)(f\cdot\sigma\cdot\rho_q, b) = (f_\sharp(\sigma\cdot\lambda_p), a)(f_\sharp(\sigma\cdot\rho_q), b)$
$= (\sigma\cdot\lambda_p, f^\sharp a)(\sigma\rho_q, f^\sharp b) = (\sigma, f^\sharp a\cup f^\sharp b)$ □

$\therefore S^*, H^*$: contravariant functor $\mathcal{T}opology \rightarrow \mathcal{R}ing(\mathcal{R} - algebra)$

4. (anti-) commutativity

$\alpha\in H^p, \beta\in H^q \Rightarrow \alpha\cup\beta = (-1)^{pq}\beta\cup\alpha$
This formula is not true on cochain level.(but almost true)
But true in simplicial case :
Let $\theta_p : \triangle_p \rightarrow \triangle_p$
$(e_0, \cdots, e_p) \mapsto (e_p, \cdots, e_0)$
To get the same oriented simplex, we have to multiply $\epsilon_p = (-1)^{\frac{p(p+1)}{2}}$ so that $\epsilon_p\theta_p =$ id on the simplicial chain complex.

Let $\theta(\sigma) = \epsilon_p\sigma\cdot\theta_p$ for both simplicial and singular case.
Then θ is a natural chain map :
(1) Show $\theta\partial = \partial\theta$:
$\partial\theta(\sigma) = \epsilon_p\partial(\sigma\cdot\theta_p) = \epsilon_p\sum_0^p(-1)^i\sigma\theta_p\cdot f_p^i = \epsilon_p\Sigma(-1)^i\sigma f_p^{p-i}\theta_{p-1} = \epsilon_p\sum_0^p(-1)^{p-j}\sigma f_p^j\theta_{p-1} = (-1)^p\epsilon_p\sum_0^p(-1)^j\sigma f_p^j\theta_{p-1} = (-1)^p\epsilon_p\epsilon_{p-1}\theta(\partial\sigma) = \theta(\partial\sigma).$

(2) Naturality :

$$\begin{array}{ccc} S(X) & \xrightarrow{f_\sharp} & S(Y) \\ \downarrow{\scriptstyle\theta_X} & & \downarrow{\scriptstyle\theta_Y} \\ S(X) & \xrightarrow{f_\sharp} & S(Y) \end{array}$$

$\theta(f_\sharp\sigma) = \epsilon_p f_\sharp\sigma\cdot\theta_p = \epsilon_p f\cdot\sigma\cdot\theta_p$
$= \epsilon_p f_\sharp(\sigma\cdot\theta_p) = f_\sharp(\epsilon_p\sigma\theta_p) = f_\sharp(\theta(\sigma)).$

Now, $(\sigma, \tilde\theta(a\cup b)) := (\theta(\sigma), a\cup b) = \epsilon_{p+q}(\sigma\theta_{p+q}, a\cup b)$
$= \epsilon_{p+q}(\sigma\cdot\theta_{p+q}\lambda_p, a)(\sigma\cdot\theta_{p+q}\rho_q, b)$
$= \epsilon_{p+q}(\sigma\cdot\rho_p\cdot\theta_p, a)(\sigma\cdot\lambda_q\cdot\theta_q, b)$
$= \epsilon_{p+q}\epsilon_p\epsilon_q(\theta(\sigma\rho_p), a)(\theta(\sigma\lambda_q), b)$
$= (-1)^{pq}(\sigma\rho_q, \tilde\theta(a))(\sigma\lambda_p, \tilde\theta(b))$
$= (-1)^{pq}(\sigma, \tilde\theta(b)\cup\tilde\theta(a))$

$\therefore \tilde\theta(a\cup b) = (-1)^{pq}\tilde\theta(b)\cup\tilde\theta(a)$

In simplicial case, $\theta = \text{id} \Rightarrow \tilde{\theta} = \text{id} \Rightarrow a \cup b = (-1)^{p+q} b \cup a$
In singular case, θ is not id but $\theta^* = \text{id}$ on H^*, since $\theta : S(X) \to S(X)$ is a natural chain map and hence $\theta \simeq \text{id}$ by Acyclic model theorem.

$\therefore \alpha \cup \beta = (-1)^{pq} \beta \cup \alpha.$

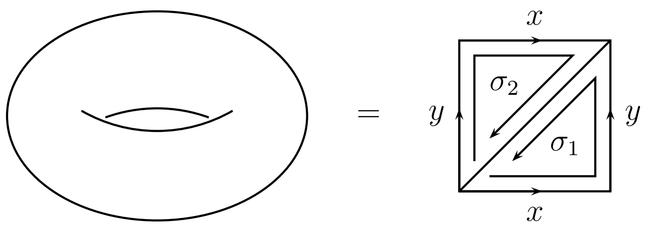
Recall Acyclic model theorem.

Acyclic Model Theorem

Let $\mathcal{T}$ be a category and $\mathcal{C}$ be a category of chain complexes and chain maps.
Let $\mathcal{S}$ and $\mathcal{S}' : \mathcal{T} \to \mathcal{C}$ be functors and $\mathcal{M} \subset Ob(\mathcal{T})$.
(1)$\mathcal{S}'$ is acyclic relative to $\mathcal{M}$, i.e., $\mathcal{S}'(M)$ is acyclic for $\forall M \in \mathcal{M}$.
(2)$\mathcal{S}$ is free relative to $\mathcal{M}$, i.e., $\forall p, \exists$ an indexed family $\{M_\alpha\}_{\alpha \in J_p}$ and $\{i_\alpha\}_{\alpha \in J_p},\ \ M_\alpha \in \mathcal{M}, i_\alpha \in S_p(M_\alpha)$ such that the indexed family $\{S(\sigma) i_\alpha\}_{\alpha \in J_p},\ \ \sigma \in hom(M_\alpha, X)$ is a basis for $S_p(X)$.

Then (i) $\exists$ a natural transformation $\tau : \mathcal{S} \to \mathcal{S}'$ which induces a given natural transformation $\tau_0 : H_0(\mathcal{S}) \to H_0(\mathcal{S}')$.
(ii) Given two each natural transformation τ and τ', $\ \tau \simeq \tau'$.

5. Examples
(1) $T = S^1 \times S^1 \Rightarrow H^2 = \mathbb{Z}, H^1 = \mathbb{Z} \oplus \mathbb{Z}, H^o = \mathbb{Z}$



σ_1과 σ_2를 위 그림과 같은 2-simplex라 하자.(standard 3-simplex의 vertex 0,1,2를 화살표 순서로 보내는 simplex이다.)
$\partial(\sigma_1 - \sigma_2) = 0$이므로 $\zeta = \{\sigma_1 - \sigma_2\} \in H_2$로 두면 이는 H_2의 generator이므로 T^2의 fundamental orientation class가 된다.
$H^1 \cong \text{Hom}(H_1, \mathbb{Z})$이므로 $H_1 = \mathbb{Z} \oplus \mathbb{Z} = \langle \{x\}, \{y\} \rangle$라 하면 $(\{x\}, \{y\})$의 dual basis $\alpha = \{a\}, \beta = \{b\}$가 H^1의 basis가 된다.

따라서,

$$\begin{aligned}(\zeta, \alpha\cup\beta) &= (\sigma_1 - \sigma_2, a\cup b)\\ &= (\sigma_1, a\cup b) - (\sigma_2, a\cup b)\\ &= (\sigma_1\lambda_1, a)(\sigma_1\rho_1, b) - (\sigma_2\lambda_1, a)(\sigma_2\rho_1, b)\\ &= (x,a)(y,b) - (y,a)(x,b)\\ &= (\{x\},\alpha)(\{y\},\beta) - (\{y\},\alpha)(\{x\},\beta)\\ &= 1\end{aligned}$$

이므로, $\alpha\cup\beta$는 H^2의 generator이다.

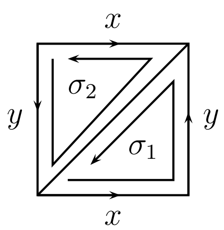
Ring structure of $H^*(T^2)$:
This is generated by α and β with the relation $\alpha\cup\alpha = 0$, $\beta\cup\beta = 0$, $\alpha\cup\beta = -\beta\cup\alpha$.
따라서 다음을 알 수 있다.

$$H^*(T,\mathbb{R}) \cong \Lambda(\mathbb{R}^2)$$

(이 때, cup product은 wedge product에 대응된다.)

숙제 20. Compute the cohomology ring of $\Sigma_2 = T\sharp T$.

(2) K=Klein bottle. $\Rightarrow H^2 = 0, H^1 = \mathbb{Z}\oplus\mathbb{Z}/2, H^o = \mathbb{Z}$
Universal coefficient theorem으로부터 $H^*(K;\mathbb{Z}/2) : H^2 = \mathbb{Z}/2$, $H^1 = \mathbb{Z}/2\oplus\mathbb{Z}/2$, $H^0 = \mathbb{Z}/2$임을 알 수 있다.



왼쪽 그림과 같이 σ_1과 σ_2를 정의하면

$$\partial(\sigma_1+\sigma_2) = 2y \equiv 0 \text{ (mod 2)}$$

임을 알 수 있다. 따라서 $\zeta = \{\sigma_1+\sigma_2\}$로 하면 H_2의 generator가 된다.

앞서와 마찬가지로 α, β를 $\{x\}$, $\{y\}$의 dual이라 하고 계산해 보면 다음 사실을 알 수 있다.

Ring structure of $H^*(K;\mathbb{Z}/2)$:
$H^*(K;\mathbb{Z}/2)$ is a ring generated by α, β with the relations : $\alpha^2 = 0, \alpha\beta = \beta\alpha = \beta^2$.

숙제 21. Compute the ring structure of $H^*(P^2;\mathbb{Z}/2)$.

6. Relative cup product
(1) $H^p(X,A)\times H^q(X)\xrightarrow{\cup} H^{p+q}(X,A)$ is well defined:
Note. An element of $H^p(X,A)$ can be represented by a cocyle in $S^p(X)$ which vanishes on $S_p(A)$:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & S^{p+1}(A) & \longleftarrow & S^{p+1}(X) & \longleftarrow & S^{p+1}(X,A) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\delta} & & \\
0 & \longleftarrow & S^p(A) & \xleftarrow{\ \widetilde{i}\ } & S^p(X) & \xleftarrow{\ \widetilde{p}\ } & S^p(X,A) & \longleftarrow & 0
\end{array}$$

위 diagram에서 $\widetilde{i}$는 restriction map이다. 따라서 $S^p(X,A)$의 cocycle은 $S^p(X)$의 cocycle 중 $S_p(A)$에 restrict하면 0이 되는 것들이다.

Suppose $a\in S^p(X,A)\subset S^p(X)$ (vanishing on A). Then $a\cup b$ also vanishes on A since front p-face of $\sigma\in S_{p+q}(A)$ also lies in $S_p(A)$.

(2) $H^p(X,A)\times H^q(X,A)\xrightarrow{\cup} H^{p+q}(X,A)$ is a restriction of (1).

Cap product

7. The cap product is a bilinear pairing,

$$\cap : S_{p+q}(X)\times S^p(X)\longrightarrow S_q(X)$$

given by $(\sigma\cap a, b)=(\sigma, a\cup b)$, i.e., $\cap a$ is the adjoint of $a\cup$, or $\sigma\cap a=(\sigma\lambda_p, a)\sigma\rho_q$.
($\Rightarrow \cap : S_*(X)\times S^*(X)\to S_*(X)$: extend linearly)

8. (0) $\sigma\cap 1=\sigma$
(1) $\sigma\cap(a\cup b)=(\sigma\cap a)\cap b$
(2) $\partial(\sigma\cap a)=(-1)^p(\partial\sigma\cap a-\sigma\cap\delta a)$, $\sigma\in S_{p+q}$, $a\in S^p$
증명 (0) clear from $1\cup b=b$.
(1) $((\sigma\cap a)\cap b, c)=(\sigma\cap a, b\cup c)=(\sigma, a\cup(b\cup c))=(\sigma,(a\cup b)\cup c)=(\sigma\cap(a\cup b), c)$
(2) $(\partial(\sigma\cap a), b)=(\sigma\cap a,\delta b)=(\sigma, a\cup\delta b)$
$=(-1)^p((\sigma,\delta(a\cup b))-(\sigma,\delta a\cup b))=(-1)^p((\partial\sigma\cap a, b)-(\sigma\cap\delta a, b))$ □

위의 (2)에 의하여 다음을 알 수 있다.
Cap product induces a bilinear pairing,

$$\cap : H_{p+q}(X)\times H^p(X)\longrightarrow H_q(X)$$

($\because \{\sigma\} \in H_{p+q}(X)$, $\{a\} \in H^p(X)$라 하면 (2)에 의하여

$$\partial(\sigma \cap a) = 0$$

이므로 $\sigma \cap a$은 cycle이 된다. 또한 (2)에서

$$(\partial\tau) \cap a = \pm\partial(\tau \cap a) \pm \tau \cap \delta a = \pm\partial(\tau \cap a)$$

이고

$$\sigma \cap \delta b = \pm\partial(\sigma \cap b) \pm \partial\sigma \cap b = \pm\partial(\sigma \cap b)$$

이므로 homology에서의 map이 잘 정의된다.)

9. $f : X \to Y \Rightarrow f_\sharp(\sigma \cap f^\sharp a) = f_\sharp\sigma \cap a$. (Same for f_* and f^*)

증명

$$\begin{aligned}(f_\sharp(\sigma \cap f^\sharp a), b) &= (\sigma \cap f^\sharp a, f^\sharp b)\\ &= (\sigma, f^\sharp a \cup f^\sharp b)\\ &= (\sigma, f^\sharp(a \cup b))\\ &= (f_\sharp\sigma, a \cup b)\\ &= (f_\sharp\sigma \cap a, b)\end{aligned}$$

□

10. Relative cap product

$$\cap : H_{p+q}(X, A) \times H^p(X, A) \longrightarrow H_q(X)(\longrightarrow H_q(X, A))$$

and $\quad \cap : H_{p+q}(X, A) \times H^p(X) \longrightarrow H_q(X, A)$

are well-defined.

증명 (1) $z \in Z_{p+q}(X, A)$, $c \in Z^p(X, A)$라 두면

$$\partial z \in S(A)$$

이고

$$c \text{ is a cocycle in } Z^p(X) \text{ which vanishes on } S_p(A).$$

임을 안다. 따라서

$$\partial(z \cap c) = \pm(\partial z) \cap c \pm z \cap (\delta c) = 0^3$$

이므로 $z \cap c$는 X의 cocycle이 되어 증명이 끝난다.

(2) $z \in Z_{p+q}(X, A)$, $c \in Z^p(X)$라 두면

$$\partial(z \cap c) = \pm(\partial z) \cap c \pm z \cap (\delta c) = \pm(\partial z) \cap c$$

에서 $\partial(z \cap c)$가 A의 chain이 되어 $z \cap c$는 relative cycle이 된다. □

[3] $\because \delta c = 0$ and $\partial z \in A$