V. Poincare Duality

V.I Direct Limit

1.(Directed set)
A **directed set** is a partially ordered set $\{J, \leq\}$ such that $\forall\alpha, \beta \in J, \exists\gamma \in J$ such that $\alpha \leq \gamma, \beta \leq \gamma$.

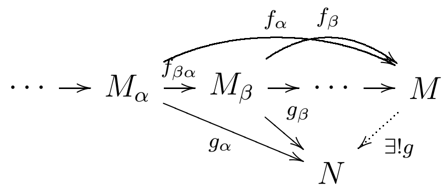
Examples
(1) a totally ordered set
(2) a power set $\mathcal{P}(X)$ with $\subset$(or $\supset$)
(3) Let X be a space. $J = \{\mathcal{A} | \mathcal{A}$ is an open covering of$X\}$ and $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{B}$ is a refinement of $\mathcal{A}$
($\{A \cap B | A \in \mathcal{A}, B \in \mathcal{B}\}$ is a common refinement of $\mathcal{A}$ and $\mathcal{B}$)
For example, $\{$a partition $\mathcal{P}$ on$[a, b]\}$ is a directed set.

(4) Let (X, x_0) be a space with a base point. $J = \{(\widetilde{X}, \widetilde{x}_0; p)$: a covering space of $X\}$ with $X_1 \leq X_2$ if there exists a covering from X_2 to X_1. For $X_1, X_2 \in J$, we put $X_3 = \{(x_1, x_2) \in X_1 \times X_2 | p_1 x_1 = p_2 x_2\}$, then $X_3 \in J$ such that $X_1 \leq X_3$ and $X_2 \leq X_3$.

2. Let J be a directed set. **A direct system of R-modules and homs** is an indexed family of R-modules $\{M_\alpha\}_{\alpha \in J}$ and homs $\{f_{\beta\alpha} : M_\alpha \to M_\beta, \forall\alpha \leq \beta\}$ such that
(1)$f_{\alpha\alpha} = id., \forall\alpha \in J$
(2)$f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}, \forall\alpha \leq \beta \leq \gamma$

Remark
1. In general, a **direct system** in a category $\mathcal{C}$ is defined in the same way : M_α is an object in $\mathcal{C}$ and $f_{\beta\alpha}$ is a morphism : $M_\alpha \to M_\beta$
2. An **inverse system** is defined dually; $g_{\alpha\beta} : M_\beta \to M_\alpha, \forall\alpha \leq \beta$ with (1) and (2).

3. A **direct limit** of a direct system $\{M_\alpha, f_{\beta\alpha}\}$ is M and $\{f_\alpha : M_\alpha \to M\}_{\alpha \in J}$ with the property that $f_\beta f_{\beta\alpha} = f_\alpha$ which is universal, i.e.,
$\forall N$ together with $\{g_\alpha : M_\alpha \to N\}$ satisfying $g_\beta f_{\beta\alpha} = g_\alpha$, there exists unique $g : M \to N$ such that $g_\alpha = g \circ f_\alpha, \forall\alpha$.



Notation $M = \varinjlim M_\alpha$

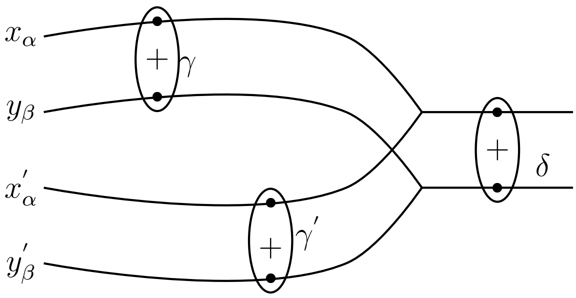
Remark Inverse limit is defined similarly by reversing the arrows.

4. Existence and uniqueness of direct limit

♣ Existence ♣

Let $M = \coprod_\alpha M_\alpha / \sim$, where $x_\alpha \sim x_\beta$ for $x_\alpha \in M_\alpha, x_\beta \in M_\beta$ if and only if $f_{\gamma\alpha}(x_\alpha) = f_{\gamma\beta}(x_\beta)$for some $\gamma \geq \alpha, \beta$, and $f_\alpha : M_\alpha \to M$ given by $f_\alpha(x_\alpha) = \{x_\alpha\}$(: equivalence class of x_α)

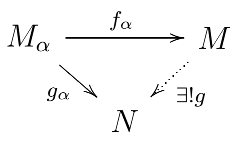
Module structure on M : $\{x_\alpha\}+\{y_\beta\} := \{f_{\gamma\alpha}(x_\alpha)+f_{\gamma\beta}(y_\beta)\}$ for some $\gamma \geq \alpha, \beta$. Is it well-defined? It is clear by the following figure.



Similarly $r\{x_\alpha\} := \{rx_\alpha\}$.

$f_\alpha : M_\alpha \to M$ is a homomorphism.(clear)

Universal property : Given N and $g_\alpha : M_\alpha \to N$, need to show that there exists unique homomorphism g such that the following diagram commutes.



Define $g\{x_\alpha\} = gf_\alpha(x_\alpha) = g_\alpha(x_\alpha)$.

Well-definedness : $x_\alpha \sim x_\beta \Rightarrow f_{\gamma\alpha}(x_\alpha) = f_{\gamma\beta}(x_\beta) \Rightarrow g_\gamma f_{\gamma\alpha}(x_\alpha) = g_\gamma f_{\gamma\beta}(x_\beta) \Rightarrow g_\alpha(x_\alpha) = g_\beta(x_\beta)$
Clearly g is a homomorphism.
g is unique : $g' f_\alpha(x_\alpha) = g_\alpha(x_\alpha) = g f_\alpha(x_\alpha)$ and $\forall x \in M, x = f_\alpha(x_\alpha)$for some α and x_α

숙제 22 Do the same for $\mathcal{T}op$.

Another description for R-module case
Let $M = \bigoplus_\alpha M_\alpha / N$, where $N =< i_\beta f_{\beta\alpha}(x_\alpha) - i_\alpha(x_\alpha) | x_\alpha \in M_\alpha, \forall \alpha \leq \beta >$ (submodule) and $i_\alpha : M_\alpha \hookrightarrow \bigoplus_\alpha M_\alpha$

♠ Uniqueness ♠
Let M, N be direct limits of $\{M_\alpha, f_{\beta\alpha}\}$. Then

$$\begin{array}{ccc} M_\alpha & \xrightarrow{f_\alpha} & M \\ & \searrow^{g_\alpha} & \exists! g \downarrow \uparrow \exists! f \\ & & N \end{array}$$

$f \circ g : M \to M$ with $(f \circ g) f_\alpha = f g_\alpha = f_\alpha \Rightarrow f \circ g = id.$ by the uniqueness of such homomorphism.

Similarly $g \circ f = id$.

숙제23
Let $\cdots \hookrightarrow X_i \hookrightarrow X_{i+1} \hookrightarrow \cdots$.
Describe $\varinjlim X_i$ in $\mathcal{S}$(the category of sets), $\mathcal{T}$(topological spaces) and $R-\mathcal{M}$od.

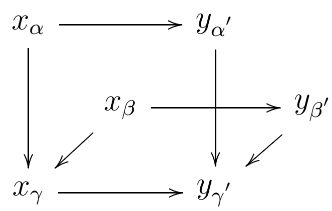
Note
(1) If $f_{\beta\alpha}$ is an isomorphism $\forall \alpha, \beta$, then $\varinjlim M_\alpha \cong M_\alpha$.($\because f_\alpha : M_\alpha \to M$ is an isomorphism.)
(2) If $f_{\beta\alpha}$ is 0, $\forall \alpha, \beta$, then $\varinjlim M_\alpha = 0$.

5.Let $\phi : \{M_\alpha, f_{\beta\alpha}\}_J \to \{N_\gamma, g_{\delta\gamma}\}_K$ be a morphism, i.e., $\exists$"ϕ" : $J \to K$, an order preserving map and $\forall \alpha \in J, \exists \phi_\alpha : M_\alpha \to N_{\phi(\alpha)}$ such that

$$\begin{array}{ccc} M_\alpha & \xrightarrow{f_{\beta\alpha}} & M_\beta \\ \phi_\alpha \downarrow & & \downarrow \phi_\beta \\ N_{\phi(\alpha)} & \xrightarrow[g_{\phi(\beta)\phi(\alpha)}]{} & N_{\phi(\beta)} \end{array} \qquad \text{commutes}$$

Then ϕ induces hom $\phi = \varinjlim \phi_\alpha : \varinjlim M_\alpha \to \varinjlim N_\gamma$ given by $\phi\{x_\alpha\} = \{\phi_\alpha(x_\alpha)\}$

Well-definedness : $x_\alpha \sim x_\beta \overset{?}{\Rightarrow} \phi_\alpha(x_\alpha) \sim \phi_\beta(x_\beta)$
Put $\phi_\alpha(x_\alpha) = y_{\alpha'}, \phi_\beta(x_\beta) = y_{\beta'}$. Since $x_\alpha \sim x_\beta$, there exists γ such that $f_{\gamma\beta}(x_\beta) = f_{\gamma\alpha}(x_\alpha)(:= x_\gamma)$. By the commutativity of the above diagram, $\phi_\gamma(x_\gamma) = g_{\gamma'\alpha'}(y_{\alpha'}) = g_{\gamma'\beta'}(y_{\beta'})(:= y_{\gamma'})$. Hence $y_{\alpha'} \sim y_{\beta'}$. See the following diagram.



ϕ is a homomorphism :
$\phi(\{x_\alpha\}+\{y_\beta\}) = \phi\{f_{\gamma\alpha}(x_\alpha)+f_{\gamma\beta}(y_\beta)\} = \{\phi_\gamma(f_{\gamma\alpha}(x_\alpha)+f_{\gamma\beta}(y_\beta))\} = \{g_{\gamma'\alpha'}\phi_\alpha(x_\alpha)+ g_{\gamma'\beta'}\phi_\beta(y_\beta)\} = \{\phi_\alpha(x_\alpha)\} + \{\phi_\beta(y_\beta)\} = \phi\{x_\alpha\} + \phi\{y_\beta\}$
$\phi(r\{x_\alpha\}) = \phi\{rx_\alpha\} = \{\phi_\alpha(rx_\alpha)\} = \{r\phi_\alpha(x_\alpha)\} = r\{\phi_\alpha(x_\alpha)\} = r\phi\{x_\alpha\}$

Note Let $K = J$ and "ϕ" $= id$. Then the followings hold clearly.
(1) If each ϕ_α is 1-1, then ϕ is 1-1.
(2) If each ϕ_α is onto, then ϕ is onto.
(3) If each ϕ_α is 0, then ϕ is 0.

Note $\varinjlim$ is a covariant functor from the category of direct systems of R-modules to R-modules;

$$\begin{array}{ccccc} \{M_\alpha, f_{\beta\alpha}\}_J & \overset{\phi_\alpha}{\to} & \{N_\gamma, g_{\delta\gamma}\}_K & \overset{\psi_\gamma}{\to} & \{P_\lambda, h_{\mu\lambda}\}_L \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\quad\phi\quad} & N & \xrightarrow{\quad\psi\quad} & P \end{array}$$

Where $\phi\{x_\alpha\} = \{\phi_\alpha(x_\alpha)\}$ and $\psi_\gamma\{y_\gamma\} = \{\psi_\gamma(y_\gamma)\}$. Hence by defining $(\psi \circ \phi)_\alpha = \psi_{\alpha'} \circ \phi_\alpha$, we easily check the second condition for functor.

6. (Cofinal Subset)

J: a directed set, $J_0 \subset J$ is *cofinal* in J if $\forall\alpha \in J,\ \exists\alpha_0 \in J_0$ s.t. $\alpha \le \alpha_0$.
Let $\{M_\alpha, f_{\beta\alpha}\}_J$ be a direct system.
Then $i : J_0 \hookrightarrow J$ induces an isomorphism : $\varinjlim_{J_0} M_\alpha \xrightarrow{\cong} \varinjlim_{J} M_\alpha$

증명 5.$\Rightarrow \exists$ a homomorphism $\phi(= i_*) : \underset{J_0}{\underset{\longrightarrow}{\lim}} M_\alpha = M_0 \to \underset{J}{\underset{\longrightarrow}{\lim}} M_\alpha = M$.

ϕ is onto : $\forall \{x_\alpha\} \in M,\ \exists \alpha_0 \geq \alpha \Rightarrow x_\alpha \sim f_{\alpha_0 \alpha}(x_\alpha) = x_{\alpha_0} \in M_{\alpha_0}$
$\Rightarrow \phi\{x_{\alpha_0}\}_0 = \{\phi_{\alpha_0}(x_{\alpha_0})\} = \{x_{\alpha_0}\} = \{x_\alpha\}$

ϕ is 1-1 : $\phi\{x_{\alpha_0}\}_0 = \{\phi_{\alpha_0}(x_{\alpha_0})\} = \{x_{\alpha_0}\} = 0$
$\Rightarrow \exists \beta (\geq \alpha_0)$ s.t. $f_{\beta\alpha}(x_{\alpha_0}) = x_\beta = 0$
$\Rightarrow \exists \beta_0 \in J_0$ s.t. $\beta_0 \geq \beta$ and $x_{\beta_0} := f_{\beta_0 \beta}(x_\beta) = f_{\beta_0 \alpha_0}(x_{\alpha_0}) = 0$
$\Rightarrow \{x_{\alpha_0}\}_0 = \{x_{\beta_0}\}_0 = 0$ in M_0. □

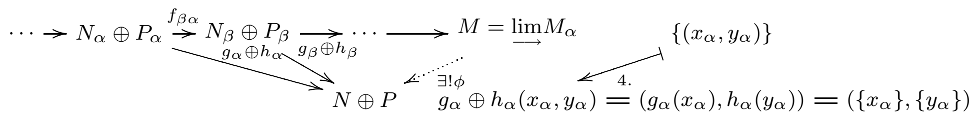
따름정리 1 *Suppose* $\exists \alpha_0 \in J$ *s.t.* $M_\alpha \cong M_{\alpha_0},\ \forall \alpha \geq \alpha_0 \Rightarrow \underset{\longrightarrow}{\lim} M_\alpha \cong M_{\alpha_0}$

증명 Let $J_0 = \{\alpha \in J \mid \alpha \geq \alpha_0\}$. Then J_0 is cofinal and use 4. Note(i). □

따름정리 2 *If* J *has a maximal element* α_0*, then* $\{\alpha_0\}$ *is cofinal and* $\underset{\longrightarrow}{\lim} M_\alpha \cong M_{\alpha_0}$.

7. ($\underset{\longrightarrow}{\lim}$ preserves direct sum.)
Given $\{N_\alpha, g_{\beta\alpha}\}_J$ and $\{P_\alpha, h_{\beta\alpha}\}_J$ with $\underset{\longrightarrow}{\lim} N_\alpha = N$ and $\underset{\longrightarrow}{\lim} P_\alpha = P$,
$\{N_\alpha \oplus P_\alpha, g_{\beta\alpha} \oplus h_{\beta\alpha}\}_J$ is also a direct system with $\underset{\longrightarrow}{\lim}(N_\alpha \oplus P_\alpha) = \underset{\longrightarrow}{\lim} N_\alpha \oplus \underset{\longrightarrow}{\lim} P_\alpha$.
Let $N_\alpha \oplus P_\alpha = M_\alpha$ and $g_{\beta\alpha} \oplus h_{\beta\alpha} = f_{\beta\alpha}$.

In fact,



And ϕ is 1-1 and onto : clear.

8. ($\underset{\longrightarrow}{\lim}$ preserves s.e.s.)

Given
$$\begin{array}{ccccccccc} 0 & \to & N_\alpha & \xrightarrow{\phi_\alpha} & M_\alpha & \xrightarrow{\psi_\alpha} & P_\alpha & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N & \xrightarrow{\phi} & M & \xrightarrow{\psi} & P & \to & 0 \end{array}$$
s.e.s. of direct system (over the same index set J) is exact.

증명 ϕ is 1-1, ψ is onto, $\psi \cdot \phi = 0$: clear from 5.

Suppose $\psi\{x_\alpha\} = \{\psi_\alpha(x_\alpha)\} = 0$. Then $h_{\beta\alpha}\psi_\alpha(x_\alpha) = \psi_\beta(x_\beta) = 0$, for some $\beta \geq \alpha$,where $x_\beta = f_{\beta\alpha}(x_\alpha)$

$\Rightarrow \exists y_\beta \in N_\beta$ s.t. $\phi_\beta(y_\beta) = x_\beta = f_{\beta\alpha}(x_\alpha)$
$\Rightarrow \phi\{y_\beta\} = \{\phi_\beta(y_\beta)\} = \{f_{\beta\alpha}(x_\alpha)\} = \{x_\alpha\}$ □

따름정리 3 *(1) Homology functor (of chain complexes) commutes with* $\varinjlim$.
(i.e.,$\varinjlim_\alpha H(C^\alpha) = H(\varinjlim C^\alpha)$, $\{C^\alpha, f_{\beta\alpha}\}$ *a direct system of chain complexes.)*
(2) $J = \{K \subset X \mid K \text{ is compact}\}$ *(with* $\leq = \subset$*)* $\Rightarrow \varinjlim_J H_*(K) = H_*(X)$
(3) $\bigcup_n U_n$*: increasing union of open sets* $U_n \Rightarrow H_*(\bigcup U_n) = \varinjlim_{U_n} H_*(U_n)$

증명(1)

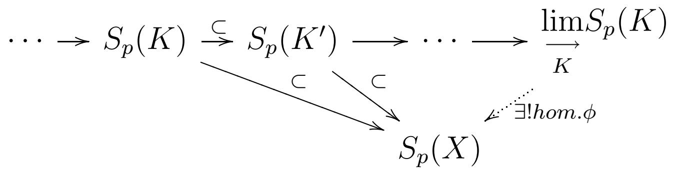
$$
\begin{array}{ccccccccc}
\vdots & & & & & & & & \vdots \\
\downarrow & & & & & & & & \downarrow \\
\mathcal{C}^\alpha : & \cdots \to & C_p^\alpha & \xrightarrow{\partial_\alpha} & C_{p-1}^\alpha & \to \cdots & \Rightarrow & & H_p(\mathcal{C}^\alpha) \\
\downarrow f_{\beta\alpha} & & \downarrow f_{\beta\alpha} & & \downarrow f_{\beta\alpha} & & & & \downarrow f_{\beta\alpha *} \\
\mathcal{C}^\beta : & \cdots \to & C_p^\beta & \xrightarrow{\partial_\beta} & C_{p-1}^\beta & \to \cdots & \Rightarrow & & H_p(\mathcal{C}^\beta) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow ? \\
\mathcal{C} : & \cdots \to & C_p & \xrightarrow{\partial} & C_{p-1} & \to \cdots & \Rightarrow & & H_p(\mathcal{C})
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & & & C_{p+1}^\alpha & & \\
 & & & & & \overset{\subset}{\nearrow} & & & \\
0 \to & Z_p^\alpha & \to & C_p^\alpha & \xrightarrow{\partial_\alpha} & B_{p+1}^\alpha & \longrightarrow & 0 & \\
 & \vdots & & \vdots & & \vdots & & & \\
 & \downarrow & & \downarrow & & \downarrow & \varinjlim C_{p+1}^\alpha & & \\
 & & & & & \overset{\subset}{\nearrow} & & & \\
0 \to & \varinjlim Z_p^\alpha & \to & \varinjlim C_p^\alpha & \xrightarrow{\partial} & \varinjlim B_{p+1}^\alpha & \longrightarrow & 0 & \text{: s.e.s.}
\end{array}
$$

$\Rightarrow Z_p = ker\partial = \varinjlim Z_p^\alpha$ and $B_{p+1} = im\partial = \varinjlim B_{p+1}^\alpha$ (Note that $\varinjlim$ inclusion = inclusion from 5.)

$$
\begin{array}{lccccccccc}
\text{Now} & 0 \to & B_p^\alpha & \to & Z_p^\alpha & \to & H_p^\alpha & \to 0 & & \\
 & & \vdots\downarrow & & \vdots\downarrow & & \vdots\downarrow & & & \\
 & 0 \to & B_p & \to & Z_p & \to & \varinjlim H_p & \to 0 & & \Rightarrow \varinjlim H_p^\alpha = H_p
\end{array}
$$

(2)



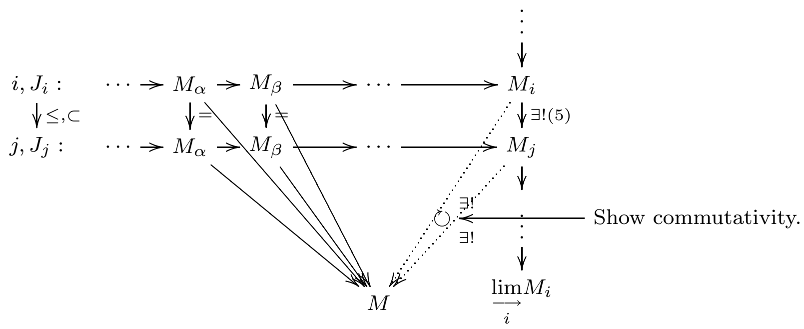
ϕ is clearly 1-1 since each inclusion is 1-1 and onto since $supp\sigma$ is compact. Now apply (1).

(3) 마찬가지. □

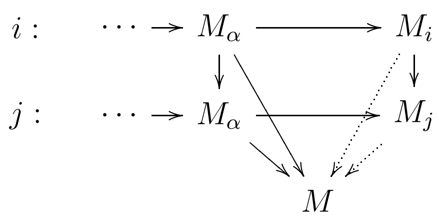
9. (Iterated Limit)
Let $\{M_\alpha, f_{\beta\alpha}\}_J$ be a direct system with $\varinjlim_J M_\alpha = M$. Let I be a directed set s.t. $\forall i \in I,\ \exists$ a directed set $J_i \subset J$ with the property (*) $\left\{ \begin{array}{c} i \le j \Rightarrow J_i \subset J_j \\ \bigcup_{i \in I} J_i = J \end{array} \right\}$

Then $\varinjlim_i \varinjlim_{J_i} M_\alpha = M$.

증명Let $M_i := \varinjlim_{J_i} M_\alpha$ and show $\varinjlim_i M_i = M$.



commutativity :



이 5면체에서 맨 오른쪽 면만 빼고 다 commute한다.
→가 onto이므로 맨 오른쪽 면도 commute한다.

Then ∃! a map : $\varinjlim_i M_i \to M$ which is clearly onto and 1-1.(Using the 2nd condition of (*).) □

Note. The condition (*) can be relaxed to the condition :

$$\left\{ \begin{array}{ll} i \le j \Rightarrow \exists \phi_{ji} : J_i \to J_j, & \text{directed set morphism} \\ \bigcup_{i \in I} J_i = J_0, & \text{a cofinal subset of J} \end{array} \right\}$$