V.2 Poincaré Duality

Let M be a n-dimensional manifold (without boundary), R-orientable. (R : PID, if M is non-orientable over $\mathbb{Z}$, take $R = \mathbb{Z}/2$.)
If M is compact, we will show the following isomorphism,

$$H^q(M) \xrightarrow[\cong]{\text{P.D.}} H_{n-q}(M)$$

If M is not compact, the above duality as stated is not true. For instance,

$$M = \mathbb{R} \Rightarrow H^1(\mathbb{R}) = 0, H_0(\mathbb{R}) = \mathbb{Z}$$

In this case, the correct duality is

$$H_c^q(M) \cong H_{n-q}(M)$$

where $H_c^q(M)$ is cohomology with compact support.

예를 들어 $M = \mathbb{R}^1$인 경우에 다음과 같이 주어진 1-cochain Dx, $x \in \mathbb{R}$를 생각해 보자.

$$D(x)(e) := H(e(1)) - H(e(0))$$

여기서 $H \in S^0(\mathbb{R})$는 함수 $H(y) = H_x(y) = \begin{cases} 1 & y \geq x \\ 0 & y < x \end{cases}$ 에 의해 결정된 0-cochain이다. 그러면 $Dx = \delta H$이다. 따라서 Dx는 cocycle이다.
또한 $Dx = \delta H$이므로 $H^1(\mathbb{R})$에서는 $\{Dx\} = 0$이다. 그러나, H가 compact support를 갖지 않으므로 $H_c^1(\mathbb{R})$에서는 $\{Dx\} \neq 0$이다. 사실상 D는 $H_0(\mathbb{R})$과 $H_c^1(\mathbb{R})$ 사이의 duality isomorphism을 induce한다.

1. $H_c^q(X)$
Let X be a space and $J = \{K^{\text{compact}} \subset X\}$ with $K \leq K'$ (if $K \subset K'$) be a directed set.
Then $\{H^q(X, X-K), i_{K'K} = i^* : H^q(X, X-K) \to H^q(X, X-K')$ where $i : (X, X-K) \hookrightarrow (X, X-K')\}$ is a direct system.

Define

$$H_c^q(X) = \varinjlim_K H^q(X, X-K)$$

Note. Since homology commutes with direct limit[1], if we let

$$S_c^*(X) = \varinjlim_K S^*(X, X-K),$$

[1] 앞절의 8.Cor 1.

then $H_c^q(X)$ is q-th homology of $S_c^*(X)$.
Recall $S^q(X, X-K)$ is a collection of cochains in $S^q(X)$ which vanish on $S_q(X-K)$.

2. $f: X \to Y$가 주어졌을때,
Note that $f(X-K) \nsubseteq Y - f(K)$.
If f is *proper*, i.e. f^{-1}(compact) =compact. Then $\forall L^{\text{compact}} \subset Y$

$$f : X - f^{-1}(L) \to Y - L.$$

So f induces

$$f^* : H^q(Y-L) \to H^q(X - f^{-1}(L)) \to H_c^q(X)$$

And f^* induces

$$"f^{*}" : H_c^q(Y) = \varinjlim_L H^q(Y, Y-L) \to H_c^q(X)$$

3. Let K be a compact subset of an R-orientable manifold M.
Recall $H_n(M, M-K) \cong \Gamma K (\cong R^k, k =$number of components of K.)
Let $\zeta_K \in \Gamma K$ be a restriction of an orientation of M(as a section) on K.

$$\zeta_K\text{: "fundamental class of } H_n(M, M-K)\text{"}$$

(If M is compact $\zeta_K = i^*(\zeta_M), \zeta_M$: fundamental class of M)
Consider

$$\begin{array}{rccc} \zeta_K\cap: & H^q(M, M-K) & \to & H_{n-q}(M) \\ & a & \mapsto & \zeta_K \cap a \end{array}$$

If $K \subset K'$,

$$\begin{tikzcd} H^q(M, M-K) \arrow[d] \arrow[dr, "\zeta_K\cap"] & \\ H^q(M, M-K') \arrow[r, "\zeta_{K'}\cap"] \arrow[d, dotted] & H_{n-q}(M) \\ H_c^q(M) \arrow[ur, dotted, "\exists! D"'] & \end{tikzcd}$$

IV.3 cup and cap product의 9번에 의하여 위 삼각형이 commute하고 따라서 D가 induce된다. 이 때 D가 바로 duality homomorphism이다.

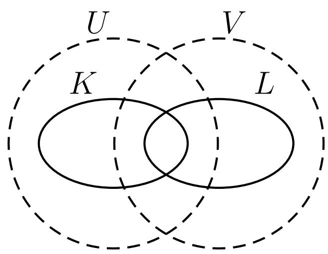
4. **정리** [**Poincaré duality**] Let M be an R-orientable n-manifold. Then

$$D : H^q_c(M) \to H_{n-q}(M)$$

is an isomorphism.

정리를 증명하기 위하여 먼저 다음의 Lemma를 증명한다.

Lemma 1. (MV) If theorem holds for open sets U, V and $U \cap V$, then theorem holds for $U \cup V$.
증명 Let $K^{\text{compact}} \subset U$, $L^{\text{compact}} \subset V$, $B = U \cap V$, $Y = U \cup V$.



Consider cohomology MV-sequence for triple $(Y, Y - K, Y - L)$:

$$\begin{tikzcd}
H(Y, Y - K \cup L) & H(Y, Y - K) \oplus H(Y, Y - L) \arrow[l] \arrow[d, phantom, "\wr\|\ \text{excision}"] & H(Y, Y - K \cap L) \arrow[l] \arrow[d, phantom, "\wr\|\ \text{excision}"] \\
 & H(U, U - K) \oplus H(V, V - L) & H(B, B - K \cap L)
\end{tikzcd}$$

and homology sequence for a pair (U, V)

$$H(B) \to H(U) \oplus H(V) \to H(Y)$$

Lemma는 다음의 key diagram으로부터 얻어진다.

$$\begin{tikzcd}
H^{q+1}(B, B - K \cap L) \arrow[d, "\zeta_{K\cap L}\cap"] \arrow[dr, phantom, "(3)"] & H^q(Y, Y - K \cup L) \arrow[l, "\delta^*"'] \arrow[d, "\zeta_{K\cup L}\cap"] \arrow[dr, phantom, "(2)"] & H^q(U, U - K) \oplus H^q(V, V - L) \arrow[l] \arrow[d, "\zeta_K\cap \oplus \zeta_L\cap"] \arrow[dr, phantom, "(1)"] & H^q(B, B - K \cap L) \arrow[l] \arrow[d, "\zeta_{K\cap L}\cap"] \\
H_{n-q-1}(B) & H_{n-q}(Y) \arrow[l, "\partial_*"'] & H_{n-q}(U) \oplus H_{n-q}(V) \arrow[l] & H_{n-q}(B) \arrow[l]
\end{tikzcd}$$

Claim. 위 diagram에서 (1),(2)은 commute하고 (3)은 up to sign으로 commute한다.

위 Claim을 증명하고 key diagram에 direct limit을 취하면 다음과 같다.[2]

$$\begin{tikzcd}
H_c^{q+1}(B) \arrow[d, "\cong"', "D"] & H_c^q(Y) \arrow[l] \arrow[d, "D"] \arrow[l, phantom, shift right=6, "\pm\circlearrowleft"] & H_c^q(U)\oplus H_c^q(V) \arrow[l] \arrow[d, "\cong"', "D\oplus D"] \arrow[l, phantom, shift right=6, "\circlearrowleft"] & H_c^q(B) \arrow[l] \arrow[d, "\cong"', "D"] \arrow[l, phantom, shift right=6, "\circlearrowleft"] \\
H_{n-q-1}(B) & H_{n-q}(Y) \arrow[l] & H_{n-q}(U)\oplus H_{n-q}(V) \arrow[l] & H_{n-q}(B) \arrow[l]
\end{tikzcd}$$

이 diagram에서 commute up to sign이라도 5-lemma는 여전히 성립하므로 5-lemma에 의하여 원하는 결과를 얻는다.

따라서 Claim만 증명하면 된다.
V-part of (1) Commutes (U-part도 비슷하게):

$$\begin{tikzcd}
 & H^q(V,V-L) \arrow[dd] & & H^q(Y,Y-K\cap L) \arrow[ll] \arrow[dl, "\cong"', "\text{excision}"] \\
H^q(V,V-K\cap L) \arrow[ur] \arrow[rr, "\cong", "\text{excision}"'] \arrow[dd] & & H^q(B,B-K\cap L) \arrow[ul] \arrow[dd] \arrow[dl, phantom, "\circlearrowleft(\text{Want})"] & \\
 & H_{n-q}(V) & & \\
H_{n-q}(V) \arrow[ur, equal] & & H_{n-q}(B) \arrow[ll] \arrow[ul] &
\end{tikzcd}$$

위 diagram에서 나머지 면들이 모두 commute함은 쉽게 확인할 수 있다. (윗면과 아랫면은 정의에 의하여, 옆면들은 inclusion과 cap product가 commute한다는 사실에 의하여 commute한다.) 따라서 (1)은 commute한다.

U-part of (2) commutes (V-part도 마찬가지):

$$\begin{tikzcd}
H^q(Y,Y-K\cup L) \arrow[dd] & & H^q(Y,Y-K) \arrow[ll] \arrow[dl, "\cong"', "\text{excision}"] \arrow[dd] \\
 & H^q(U,U-K) \arrow[ul] \arrow[dd] \arrow[dl, phantom, "\circlearrowleft(\text{Want})"] & \\
H_{n-q}(Y) \arrow[rr, equal] & & H_{n-q}(Y) \\
 & H_{n-q}(U) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

마찬가지로 나머지 면들이 모두 commute함을 알 수 있고 따라서 (2)가 commute한다.

---

[2] 여기서 Y의 임의의 compact set이 $K\cup L$의 꼴로 나타남을 증명할 수 있다. (Exercise)

(3) commutes up to sign :
다음 diagram을 2층으로 놓고 δ, ∂을 추적하면 원하는 결과를 얻는다. 부호는 $(-1)^q$만큼 차이남을 확인할 수 있다.(Exercise)

$$\begin{tikzcd}
0 & S^q(Y, Y-K\cup L) \arrow[l] \arrow[dd, "\zeta\cap"] & S^q(Y,Y-K)\oplus S^q(Y,Y-L) \arrow[l] \arrow[d, "i^\sharp\oplus i^\sharp"] & S^q(Y,Y-K\cap L) \arrow[l] \arrow[dd, "\zeta\cap"] \arrow[dr, "\text{excision}", "\cong"'] & 0 \arrow[l] \\
 & & S^q(U,U-K)\oplus S^q(V,V-L) \arrow[d, "\zeta\cap\oplus\zeta\cap"] & & S^q(B,B-K\cap L) \arrow[d, "\zeta\cap"] \\
 & S_{n-q}(Y) & S_{n-q}(U)\oplus S_{n-q}(V) \arrow[l, "i_\sharp\oplus i_\sharp"'] \arrow[r, "i_\sharp\oplus i_\sharp"] & S_{n-q}(Y) & S_{n-q}(B) \arrow[l, "i_\sharp"']
\end{tikzcd}$$

□

Lemma 2.
Let $\{U_i\}$ be a system of open sets totally ordered by inclusion and let $U = \cup U_i$. If theorem is true for each U_i, then true for U.

증명 By the iterated limit argument proved earlier(앞절 9), $H_c^q(U) = \varinjlim H_c^q(U_i)$ and the corollary of 8(앞절) implies $H_{n-q}(U) = \varinjlim H_{n-q}(U_i)$. Then by using the fact 5(앞절), we can easily check the lemma. See the following diagram.

$$\begin{tikzcd}
H_c^q(U_i) \arrow[r, "D_i", "\cong"'] \arrow[d] & H_{n-q}(U_i) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
H_c^q(U) \arrow[r, dotted, "D", "\cong"'] & H_{n-q}(U)
\end{tikzcd}$$

□

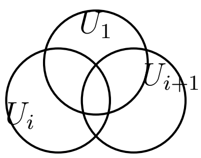
Lemma 3.
Theorem holds for $U^{\text{open}} \subset \mathbb{R}^n$.

증명
Step 1 U is convex (so that $U \cong \mathbb{R}^n$) :
Let $B^n = B(0,r)$, n-ball $\subset \mathbb{R}^n$. Then

$$
\begin{tikzcd}
H^q(\mathbb{R}^n,\mathbb{R}^n-B) \arrow[r, "\zeta_B\cap", "\cong"'] \arrow[d] & H_{n-q}(\mathbb{R}^n) \\
\vdots \arrow[d] & \\
H^q_c(\mathbb{R}^n) \arrow[uur, dotted, "\exists! D", "\cong"'] &
\end{tikzcd}
$$

Show $\zeta_B\cap : H^q(\mathbb{R}^n,\mathbb{R}^n-B) \cong H_{n-q}(\mathbb{R}^n)$: If $q \neq n$, then both are 0 and if $q = n$, then consider as follows.
Note that $H^n(\mathbb{R}^n,\mathbb{R}^n-B) \cong \mathrm{Hom}(H_n(\mathbb{R}^n,\mathbb{R}^n-B),R)$. Let $\bar{\zeta}$ be a dual of ζ_B. Then $H^n(\mathbb{R}^n,\mathbb{R}^n-B) =< \bar{\zeta} >$. Furthermore, $(\zeta_B \cap \bar{\zeta}, 1) = (\zeta_B, \bar{\zeta} \cup 1) = (\zeta_B, \bar{\zeta}) = 1 \Rightarrow \zeta_B\cap$ is an isomorphism.
Combining the fact $\{B(0,r) | r \in \mathbb{R}\}$ is cofinal in $J = \{K^{\text{cpt.}} \subset \mathbb{R}^n\}$, it is clear $D : H^q_c(\mathbb{R}^n) = \varinjlim_B H^q(\mathbb{R}^n,\mathbb{R}^n-B) \cong H_{n-q}(\mathbb{R}^n)$.

Step 2 U is a finite union of convex open sets.
Induction on the number of convex open sets and apply lemma 2.



$U_{i+1} \cap (U_1 \cup \cdots \cup U_i) = (U_{i+1} \cap U_1) \cup \cdots \cup (U_{i+1} \cap U_i)$ is a union of i convex sets and holds by the induction hypothesis.

Step 3 U : arbitrary open set in $\mathbb{R}^n$.
Let $\mathcal{W} = \{\mathring{B}(x,r) | x \in U$ with rational coordinate and $\mathring{B} \subset U, r \in \mathbb{Q}\} = \{W_1, W_2, \cdots\}$ and let $U_1 = W_1, \cdots, U_n = W_1 \cup \cdots \cup W_n, \cdots$ and clearly $U = \bigcup_{n=1}^{\infty} U_n$. Now apply lemma 2.

$\square$

Proof of Thoerem
$\mathcal{U} = \{U^{\text{open}} \subset M | \text{Theorem holds for} U\}$: partially ordered set with inclusion. By lemma 2 and lemma 3, $\mathcal{U}$ satisfies the hypothesis of Zorn's lemma. Then there exists a maximal element $U \in \mathcal{U}$.

For any coordinate neighborhood V, theorem holds for V and $V \cap U$ by lemma 3. Then theorem holds for $V \cup U$ by lemma 1 and $V \subset U$ by maximality. Hence $U = M$.

Consequences and applications

5. **따름정리** (1) Let M^n be connected and R-orientable. Then $H_c^n(M) \cong \mathbb{R}$.

(2) Let M^n be closed and orientable. $\Rightarrow \begin{cases} \beta_q = \beta_{n-q} & \beta_i = i-\text{th Betti Number} \\ T_q \cong T_{n-q-1} & T_i = \text{torsion part of } H_i(M,\mathbb{Z}) \end{cases}$

(3) Let M^n be closed and orientable. Then $H_{n-1}(M,\mathbb{Z})$ is free abelian.

(4) Let M^{2k+1} be closed (and orientable). Then $\chi(M) = 0$.

증명

(1) $H_c^n(M) \cong H_0(M) \cong \mathbb{R}$.

(2) Note that if M is compact, then $H_*(M)$ is finitely generated.

우선 poincaré duality에 의해서 $H^q(M) \cong H_{n-q}(M)$ 임을 안다. 그리고 the universal coefficient theorem에 의해서 $H^q(M) = \mathrm{Hom}(H_q(M),\mathbb{Z}) \bigoplus \mathrm{Ext}(H_{q-1};\mathbb{Z})$. 이제 $\mathrm{Hom}(H_q(M),\mathbb{Z})$를 생각해보자. $H_q(M)$에 Hom-functor를 취하면 $H_q(M)$의 free part만 남고 torsion part는 없어진다. 이것은 $\forall d(\neq 0)$에 대해서 $\mathbb{Z}/d \to \mathbb{Z}$는 0밖에 없기 때문이다. 반대로 $H_{q-1}(M)$에 Ext를 취하면 $H_{q-1}(M)$의 free part는 없어지고 torsion part만이 남는다. 따라서, $\beta_{n-q} = \beta_q$이고 $T_{n-q} = \mathrm{Ext}(H_{q-1}(M),\mathbb{Z}) = T_{q-1}$이 성립한다.

Remark If M is compact, $H_*(M)$ is finitely generated.(나중에 증명)

(3) Clear from (2)

(4) $\chi(M) = \sum(-1)^q \beta_q = 0$

□

6. Intersection pairing

Let M be a closed, connected and orientable manifold with R: P.I.D. and fundamental class ζ.

Consider

$$I : H^{n-q}(M) \bigotimes H^q(M) \xrightarrow{\cup} H^n(M) \underset{\cong}{\xrightarrow{<\zeta,\ >}} \mathbb{Z}(\text{or}R)$$
$$a \otimes b \mapsto a \cup b$$

If a is a torsion element, i.e. $ra = 0$ for some $r \neq 0$,

$0 = ra \cup b = r(a \cup b) \Rightarrow <\zeta, a \cup b> = 0$.

Similarly for b.

Hence I induces "I" : $H^{n-q}(M)/T^{n-q} \bigotimes H^q(M)/T^q \xrightarrow{"\cup"} \mathbb{Z}(\text{or}R)$.

This pairing is non-degenerate,i.e., $I(\forall\alpha,\beta) = 0 \Rightarrow \beta = 0$ and $I(\alpha,\forall\beta) = 0 \Rightarrow \alpha = 0$.

증명

Consider

$$
\begin{array}{ccccc}
H^{n-q}(M)\bigotimes H^q(M) & \xrightarrow{\cup} & H^n(M) & \underset{\cong}{\xrightarrow{<\zeta,>}} & \mathbb{Z}(\text{or}R) \\
\zeta\cap\otimes id. \downarrow\cong & & \zeta\cap\downarrow\cong & & \downarrow= \\
H_q(M)\bigotimes H^q(M) & \xrightarrow{\cap} & H_0(M) & \underset{<,1>}{\xrightarrow{\cong}} & \mathbb{Z}(\text{or}R)
\end{array}
$$

$$<,>$$

$$
\begin{array}{ccccc}
a\otimes b & \longmapsto & a\cup b & \longmapsto & <\zeta, a\cup b> \\
\downarrow & & \downarrow & & \downarrow= \\
(\zeta\cap a)\otimes b & \mapsto & (\zeta_{\cap}a)\cap b & \mapsto & <(\zeta\cap a)\cap b, 1> (=<\zeta\cap a, b>)
\end{array}
$$

$$<,>$$

This induces

$$
\begin{array}{ccc}
H^{n-q}(M)/T\bigotimes H^q(M)/T & \rightarrow & \mathbb{Z}(\text{or}R) \\
\downarrow & & \downarrow= \\
H_q(M)/T\bigotimes H^q(M)/T & \longrightarrow & \mathbb{Z}(\text{or}R)
\end{array}
$$

Let $\alpha = \{a\}, \beta = \{b\}, a \in H^{n-q}(M), b \in H^q(M)$ and b torsion free. And suppose

$$0 = I(\forall\alpha, \beta) =<\zeta, \forall a\cup b>=<\zeta\cap\forall a, b>$$

Since, $H^q(M) \cong \text{Hom}(H_q(M), \mathbb{Z})\bigoplus \text{Ext}(H_{q-1}(M), \mathbb{Z})=$ Torsion free part $\bigoplus$ Torsion part, $b = b^{'}\oplus b^{''} \Rightarrow b = b^{'}$.
Hence $0 =<\zeta\cap\forall a, b>=<\zeta\cap\forall a, b^{'}>\Rightarrow b^{'} = 0 \Rightarrow b = 0$, since $\zeta\cap\forall a$ represents every elements in $H_q(M)$ by Poincaré duality and $b^{'} \in \text{Hom}(H_q(M), \mathbb{Z})$. Hence $b = b^{'} = 0$ and $\beta = 0$.

□

Remark
If we consider with a field coefficient R, Tor =0 and have a non-degenerate pairing for a R-orientable manifold M,

$$
\begin{aligned}
I : H^{n-q}(M;R)\bigotimes H^q(M;R) &\rightarrow R \\
a\otimes b &\mapsto <\zeta, a\cup b>
\end{aligned}
$$

Remark
If $R = \mathbb{R}$, then $I(a, b) = \int_M a\wedge b$.

따름정리 Let M^{4k+2} be closed and orientable. Then
(1) dim $H^{2k+1}(M;\mathbb{R}) =$ even.
(2) $\chi(M;\mathbb{R}) =$ even.

증명
$I : H^{2k+1}(M;\mathbb{R}) \bigotimes H^{2k+1}(M;\mathbb{R}) \to \mathbb{R}$ is non-degenerate and note that $I(a,b) = -I(b,a)$. Now (1) follows from the following fact :
Fact If I is a skew-symmetric bilinear form (2-form), then there exists a basis $e = \{e_1, \cdots, e_n\}$ such that I can be represented as

$$\left(\begin{array}{ccc|c} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 & \\ & \ddots & & 0 \\ 0 & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ \hline & 0 & & 0 \end{array}\right)$$

where $I(e_i, e_j) = a_{ij}$.
Proof of fact
Choose $x_1 \in \mathbb{R}^n$ such that $I(x_1, y) \neq 0$ for some y. Let y_1 be such that $I(x_1, y_1) = 1$. Consider $\theta : \mathbb{R}^n \to \mathbb{R}^2$ given by $\theta(x) = (I(x, x_1), I(x, y_1)$. Then $\theta(x_1) = (0,1)$ and $\theta(y_1) = (-1,0)$. Clearly $\ker\theta = \mathbb{R}^{n-2}$ and repeat this process until I becomes trivial.
(2) follows from (1) immediately ◻

Remark We will show later that $\chi(M) = \chi(M;\mathbb{R})$.

7. (Cohomology ring of projective spaces)
$M = \mathbb{C}P^n$
Recall : $H_q(Z, Y) \cong H_q(X, A), \quad Z = X \underset{f}{\cup} Y$
$$\Rightarrow H^q(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \cong H^q(D^{2n}, S^{2n-1}) = \left\{ \begin{array}{ll} \mathbb{Z} & , if \;\; q = 2n \\ 0 & , otherwise \end{array} \right\}$$
Long exact sequence of pair $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$
$\Rightarrow H^q(\mathbb{C}P^n) \xrightarrow{i^*} H^q(\mathbb{C}P^{n-1})$ is $\cong$ for $q \leq 2n-2$.

Consider $\mathbb{C}P^2$ first :

$$\begin{array}{ccc} H^2(\mathbb{C}P^2) & \xrightarrow{i^*:\cong} & H^2(\mathbb{C}P^1) \\ \Big\| & & \Big\|{\scriptstyle let} \\ <\text{"}a\text{"}> & \longleftrightarrow & <a> \end{array}$$

Poincaré Duality $:< a >= H^2(\mathbb{C}P^2) \xrightarrow{\zeta\cap:\cong} H_2(\mathbb{C}P^2)$

$\Rightarrow \zeta \cap a$ genererates $H_2(\mathbb{C}P^2)$
$\Rightarrow \pm 1 =< \zeta \cap a, a >=< \zeta, a \cup a >=< \zeta, a^2 >$
$\Rightarrow a^2$ generates $H^4(\mathbb{C}P^2)$

Similarly let $M = \mathbb{C}P^3$,
then $H^2(\mathbb{C}P^3) =< a >$ and $H^4(\mathbb{C}P^3) \cong H^4(\mathbb{C}P^2) =< a^2 >= \mathbb{Z}$

Poincaré Duality : $H^4(\mathbb{C}P^3) \xrightarrow{\zeta\cap:\cong} H_2(\mathbb{C}P^3)$
$\Rightarrow \zeta \cap a^2$ generates $H_2(\mathbb{C}P^3)$
$\Rightarrow \pm 1 =< \zeta \cap a^2, a >=< \zeta, a^3 >$
$\Rightarrow a^3$ is a generator of H^6.
$\vdots$
Inductively we can conclude as follows.

정리. $H^*(\mathbb{C}P^n)$ is a ring generated by $a \in H^2(\mathbb{C}P^n)$ with $a^{n+1} = 0$, i.e., truncated polynomial ring(or algebra) generated by a of degree 2 with height $n+1$.

숙제 24. Show the same thing for $H^*(\mathbb{H}P^n, \mathbb{Z})$ and $H^*(\mathbb{R}P^n, \mathbb{Z}/2)$.($n = \infty$ 일 때도 마찬가지)

8. 따름정리 Suppose $f : P^n \to P^m$, $n > m$. Then $f_q^* = 0$ for $q > 0$.
증명 $f^* : H^*(P^m) \to H^*(P^n)$ (Use $\mathbb{Z}/2$-coefficient for $\mathbb{R}P^n$) is a ring homomorphism, where $H^*(P^m) =< a | a^{m+1} = 0 >$ and $H^*(P^n) =< b | b^{n+1} = 0 >$.
$\Rightarrow 0 = f^*(a^{m+1}) = (f^*(a))^{m+1}$
$\Rightarrow f^*(a) = rb = 0$
$\Rightarrow f^* \equiv 0$ except at 0-dimension. □

따름정리 P^m is not a retract of P^n if $n > m$.
증명 If $P^m \underset{r}{\overset{i}{\rightleftarrows}} P^n$ with $r \cdot i = id$, then $i^* \cdot r^* = id$ and r^* is 1-1. Then this is a contradiction.($\because r^* = 0$) □

9. (Borsuk-Ulam)
(1) $n > m \geq 0, \nexists$ anti-pode preserving map g s.t. $g(-x) = -g(x), \ \forall x \in S^n$.
(2)$n \geq k, \ \ f : S^n \to \mathbb{R}^k \Rightarrow \exists x \in S^n$ s.t. $f(x) = f(-x)$

증명 (1) : Suppose g is anti-pode preserving. $\Rightarrow$

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p"] & S^m \arrow[d, "p"] \\ P^n \arrow[r, dotted, "f"] \arrow[ur, dotted, "\tilde{f}"] & P^m \end{tikzcd}$$

Claim. $f_* : \pi_1(P^n) \to \pi_1(P^m)$ is 0 : $m = 1 : f_* : \mathbb{Z}_2 \to \mathbb{Z}$: clear
$m > 1 : f^* = 0 (* \neq 0)$

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, equal] & \pi_1(P^n) \arrow[r, "f_*"] \arrow[d, "\chi:\cong"] & \pi_1(P^m) \arrow[r, equal] \arrow[d, "\chi:\cong"] & \mathbb{Z}/2 \\ \mathbb{Z}/2 \arrow[r, equal] & H_1(P^n) \arrow[r, "f_*=H_1(f)"] & H_1(P^m) \arrow[r, equal] & \mathbb{Z}/2 \end{tikzcd}$$

Since $f^* = 0$ on $H^1 \Rightarrow f_* = 0$ on H_1 and hence on π_1.

$\therefore \exists \widetilde{f}$: a lifting of $f \Rightarrow \widetilde{f} \cdot p$ and g: two liftings of $f \cdot p$
$\Rightarrow \widetilde{f} \cdot p = g$. But $\widetilde{f} \cdot p \neq g$ since g is 1 to 1 while $\widetilde{f} \cdot p$ is 2 to 1 on each fiber $p^{-1}(x)$.

(2) Suppose not. Then let $g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$ and apply (1). $\square$

따름정리 [**Ham sandwich Theorem**]
Let $A_1, \cdots, A_n$ be bounded measurable subsets of $\mathbb{R}^n$. Then $\exists$ a hyperplane that bisects each of A_i.

증명 Let N be the north pole of $S^n \subset \mathbb{R}^{n+1}$. Then each $x \in S^n$ determines a unique hyperplane in $\mathbb{R}^{n+1}$ passing through N and perpendicular to x. This hyperplane divides S^n into two parts and consider the part containing x. The image of this part under stereographic projection is a half space of $\mathbb{R}^n$ and denote it by H_x.
Let $f_i(x)$ be the measure of $A_i \cap H_x$. Then note that $f_i(-x)$ is the measure of $A_i \cap H_x^c$. Now let $f : S^n \to \mathbb{R}^n$ be given by $x \mapsto (f_1(x), \cdots, f_n(x))$.
Apply Bolsuk-Ulam theorem to get $x \in S^n$ s.t. $f(x) = f(-x)$. $\square$

10. ANR (Absolute Neighborhood Retract) and AR (Absolute Retract)

X^{normal} is AR if

$$\begin{tikzcd} \forall Y^{normal} \arrow[r, dotted, "\exists \bar{f}:extension\ of\ f"] & X \\ B^{closed} \arrow[u, "\cup"] \arrow[ur, "f"'] & \end{tikzcd}$$

e.g. Tietz extension theorem$\Rightarrow$ I^n: AR.

X^{normal} is ANR if

$$\begin{tikzcd}
\forall Y^{normal} & \exists U^{open} \arrow[l, "\supset"'] \arrow[r, dotted, "\exists \bar{f}: extension\ of\ f"] & X \\
B^{closed} \arrow[u, "\cup"] \arrow[ur, "\subset"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

e.g. S^n : ANR. (Consider $S^n \subset D^{n+1}$: AR and let $U = \bar{f}^{-1}(D - 0)$)

Theorem A. Every paracompact manifold is an ANR.
Reference: 책 26장 17.6 for compact case.
2.7 of Munkres, Elementary Differential Topology.

Theorem B. Every paracompact n-manifold can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset.
Reference: p.315 of Munkres, a first course Topology or also the above book of Munkres.

따름정리 M: a compact manifold $\Rightarrow H_*(M)$ is finitely generated.

증명 Thm B. $\Rightarrow M \subset \mathbb{R}^N$
Thm A. $\Rightarrow \exists U$ a neighborhood of M in $\mathbb{R}^N$ and $\exists$ a retraction $r : U \to M$.
Choose a finite simplicial complex K s.t. $M \subset |K| \subset U$ so that $r| : |K| \to M$ gives a retraction.
Know $H_*(|K|)$ is finitely generated (using simplicial homology or CW-homology) and $H_*(M) \xrightarrow{i_*} H_*(|K|) \xrightarrow{r_*} H_*(M)$ and $r_* \cdot i_* = id$
$\Rightarrow r_*$ is onto. $\Rightarrow H_*(M)$ is finitely generated. $\square$