VI. Other duality

VI.1 Alexander duality

1.

정리 1 *Let (X, A) be a compact pair. Then there exists a long exact sequence*

$$\cdots \longrightarrow H_c^q(X-A) \xrightarrow{i} H^q(X) \xrightarrow{j} \check{H}^q(X) \xrightarrow{\delta} H_c^{q+1}(X-A) \longrightarrow \cdots .$$

먼저 map들이 어떻게 정의되었는지 살펴보자.

(1) $i : U^{\text{open}} \subset X \Rightarrow \exists$a canonical homomorphism $i : H_c^q(U) \to H^q(X)$:
임의의 compact set $K \subset U$에 대하여 아래 diagram이 commute하므로 i가 induce된다.

$$\begin{array}{ccc} H^q(U, U-K) & \xleftarrow[\cong]{\text{excision}} & H^q(X, X-K) \\ \downarrow & \searrow \quad \circlearrowleft & \downarrow i^* \\ H_c^q(U) & \xrightarrow[\exists!\text{"}i\text{"}]{} & H^q(X) \end{array}$$

(2) j:

$$\begin{array}{ccc} H^q(X) & \xrightarrow{j^*, j=\text{inclusion}} & H^q(V) \\ & \exists!\text{"}j\text{"} \searrow \quad \circlearrowleft & \downarrow \\ & & \varinjlim_{V^{\text{open}} \supset A} H^q(V) := \check{H}^q(A) \end{array}$$

(3) δ:
$K = X - V$, $U = X - A$라 두면 다음 diagram을 얻는다.

$$\begin{array}{ccccc} H^q(V) & \xrightarrow{\delta} & H^{q+1}(X, V) & \xrightarrow[\cong]{\text{excision}} & H^{q+1}(U, U-K) \\ \downarrow & & & & \downarrow \\ \check{H}^q(A) & & \xrightarrow[\exists!\text{"}\delta\text{"}]{} & & H_c^{q+1}(U) \end{array}$$

이 때, δ가 natural하므로 "δ"가 induce된다.

증명

$$\begin{array}{ccccccccc}
 & & H^q(X,V) & & & & H^{q+1}(X,V) & & \\
 & & \downarrow\cong & & & & \downarrow\cong & & \\
\cdots \to & & H^q(U,U-K) & \to & H^q(X) \to H^q(V) & \to & H^{q+1}(U,U-K) & \to & \cdots \\
 & & \vdots\downarrow & & \vdots\downarrow \quad \vdots\downarrow & & \vdots\downarrow & & \\
\cdots \longrightarrow & & H^q_c(X-A) & \xrightarrow{i} & H^q(X) \xrightarrow{j} \check{H}^q(X) & \xrightarrow{\delta} & H^{q+1}_c(X-A) & \longrightarrow & \cdots
\end{array}$$

$\varinjlim$은 exact functor이므로 위 diagram과 같이 (X,V)의 cohomology long exact sequence로부터 원하는 exact sequence를 얻는다. □

2. (Alexander duality) $X = M^n$: R-orientable compact manifold, $A^{\text{cpt}} \subset M$.
Consider

$$\begin{array}{ccc}
H^q(V) & \xrightarrow{\zeta_A \cap} & H_{n-q}(V, V-A) \\
\vdots\downarrow & \searrow & \cong \downarrow \text{excision} \\
\check{H}^q(A) & \xrightarrow[\exists! D_A]{} & H_{n-q}(M, M-A)
\end{array}$$

where $\zeta_A \in H_n(V, V-A)$, restriction of orientation class $\zeta_A \in H_n(M, M-A)$.
Then

$$D_A : \check{H}^q(A) \xrightarrow{\cong} H_{n-q}(M, M-A) \qquad \forall q$$

증명 다음 diagram에 5-lemma를 적용하면 증명이 끝난다.

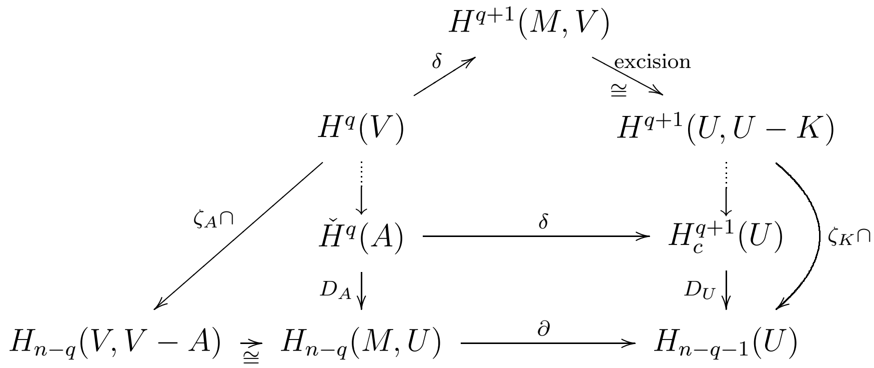
$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^q_c(U) & \longrightarrow & H^q(M) & \longrightarrow & \check{H}^q(A) & \xrightarrow{\delta} & H^{q+1}_c(U) & \longrightarrow \cdots \\
 & D_U \downarrow \cong & (1) & D_M \downarrow \cong & (2) & D_A \downarrow ?? & (3) & D_U \downarrow \cong & \\
\cdots \longrightarrow & H_{n-q}(U) & \longrightarrow & H_{n-q}(M) & \longrightarrow & H_{n-q}(M,U) & \xrightarrow{\partial} & H_{n-q-1}(U) & \longrightarrow \cdots
\end{array}$$

U는 앞서와 마찬가지로 $M-A$이고 D_U와 D_M은 Poincaré duality map들이다.
따라서 (1), (2), (3)이 commute한다는 것만 보이면 된다. (1)은 cap product의 naturality에 의하여 commute한다.
(2)는 다음 diagram에서

$$\begin{array}{ccccc}
H^q(M) & \longrightarrow & H^q(V) & \cdots\cdots\to & \check{H}^q(A) \\
D_M \downarrow \cong & \circlearrowleft & \downarrow \zeta_A \cap & & D_A \downarrow \\
H_{n-q}(M) & \longrightarrow & H_{n-q}(V, V-A) & \xrightarrow[\text{excision}]{\cong} & H_{n-q}(M,U)
\end{array}$$

cap product의 naturality에 의하여 왼쪽 사각형이 commute하고 이것의 direct limit이 (2)이므로 commute한다.

(3)은 up to sign으로 commute하는데 다음 diagram에서 확인할 수 있다.

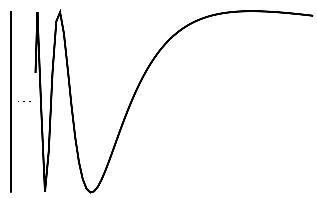


여기서 $V = M - K$이다. chain level에서 $\partial(\zeta\cap a) = (-1)^q(\partial\zeta\cap a + \zeta\cap\delta a) = (-1)^q(\zeta\cap\delta a)$이므로 up to sign으로 commute함을 확인할 수 있다. □

3.(Taut embedding) When is $\check{H}^q(A) = H^q(A)$?

e.g. non-taut embedding

$$A = \{(x, \sin\frac{1}{x}\} \cup \{(1,y) \mid |y| \le 1\}$$

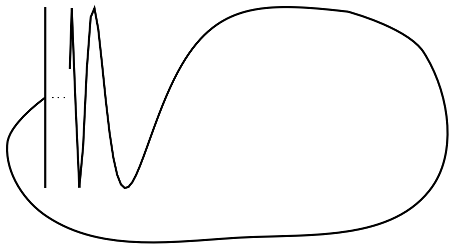


A는 $\mathbb{R}^2$의 closed subset이고 다음이 성립한다.

(1) number of path components of $A = 2$
(2) number of connected components of $A = 1$

(1)에 의하여 $H^0(A) = \mathbb{Z}^2$이다. 그러나 (2)에서 임의의 open set $U \supset A$에 대하여 A는 U의 한 component에 속해있으므로 $\check{H}^0(A) = \varinjlim H^0(U) = \mathbb{Z}$이다. 따라서 A는 taut embedding이 아니다.

또, $A =$



이라면 $\check{H}^1(A) = \mathbb{Z}$, $H^1(A) = 0$임을 확인할 수 있다.

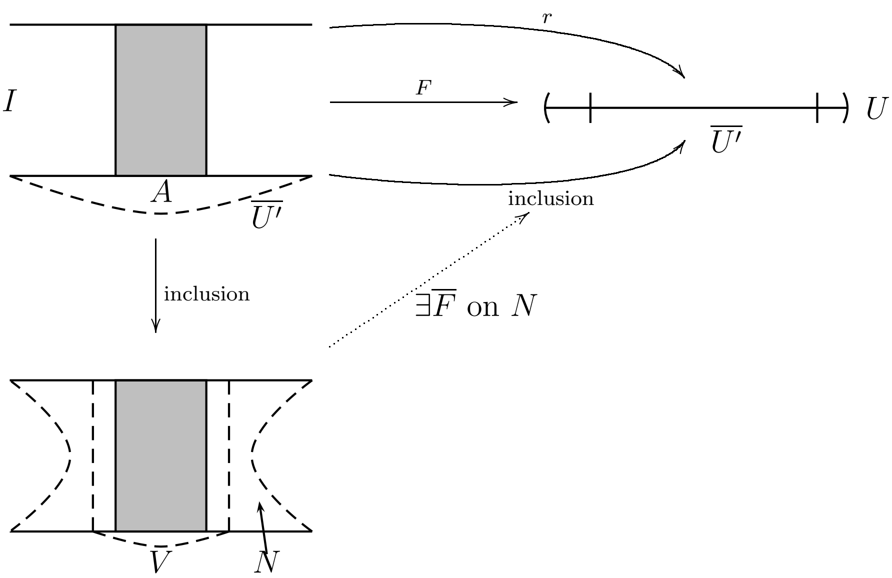
Let $A^{\text{closed}} \subset X^{\text{normal}}$ and suppose A is ANR. Then there exists U^{open} such that $A \underset{i}{\overset{r}{\rightleftarrows}} U$ with $ri = id$.

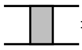
$$\Longrightarrow \quad H^q(A) \underset{r^*}{\overset{i^*}{\leftrightarrows}} H^q(U), \quad H^q(U) \dashrightarrow \check{H}^q(A), \quad \exists! \kappa : \check{H}^q(A) \dashrightarrow H^q(A)$$

and κ is onto.

Furthermore, let X be a binormal ANR. (i.e. $X \times I$ is also normal.)

$A \subset U^{\text{open}} \subset X^{\text{normal}} \Rightarrow \exists U'$ open such that $A \subset U' \subset \overline{U'} \subset U$.
Consider



,where N is an open neighborhood of  $=: C$

F를 위의 그림과 같이 C에서 U로 가는 map으로서 0-level에서는 inclusion으로 1-level에서는 r로, $A \times I$ 에서는 A로의 projection으로 정의한다. $\overline{U'} \times I$는 normal space $X \times I$의 closed subset이므로 normal이고 C는 $\overline{U'} \times I$의 closed subset이다. X가 ANR이므로 U도 ANR이다. 따라서 N에서 정의된 F의 extension $\overline{F}$가 존재한다.

$V^{\text{open}} \supset A$를 $V \times I \subset N$이 되도록 잡으면 $\overline{F}|_{V \times I}$는 $j : V \hookrightarrow U$와 $ir' : V \xrightarrow{r'} A \overset{i}{\hookrightarrow} U, r' = r|_V$ 사이에 homotopy를 준다. 따라서 $r'^* i^* = j^*$이다.

이제 κ가 1-1임을 보이자. $\kappa(x) = 0$이라 하면, 어떤 U에 대하여 $x = \{x_u\}, x_u \in H^q(U)$이고, $i^*(x_u) = 0$이다. 따라서 $j^*(x_u) = r'^* i^*(x_u) = 0$이고 결국 $x = \{j^*(x_u)\} = 0$임을 알 수 있다.

Note. The product of Paracompact space and compact Hausdorff is paracompact.
(Munkres, p.259)

In particular κ becomes an isomorphism for $A^{\text{closed, ANR}}$ in paracompact manifold X. (eg. A is a closed submanifold of X.)

Remark. The proof of 3 shows in particular that if $\exists V \supset A$ such that A is a deformation retract of V then A is taut.

4. If X is compact and $\check{H}^q(A) \cong H^q(A)$, then we have

$$\begin{array}{ccccccccc}
 & & H^q(U, U-K) & \overset{\cong}{\to} & H^q(X, X-K) & & & & \text{where } X-K=V \\
 & & \downarrow & \swarrow i^* & & & & & \text{and } X-A=U \\
\cdots \to & \check{H}^{q-1}(A) & \xrightarrow{\delta} & H^q_c(X-A) & \longrightarrow & H^q(X) & \longrightarrow & \check{H}^q(A) & \to \cdots \\
 & \downarrow \kappa:\cong & & \downarrow \exists! & & \downarrow id:\cong & & \downarrow \kappa:\cong & \\
\cdots \to & H^{q-1}(A) & \xrightarrow{\delta} & H^q(X, A) & \longrightarrow & H^q(X) & \longrightarrow & H^q(A) & \to \cdots
\end{array}$$

Diagram commutes since all the maps involved are induced by inclusion and δ is natural. Then by 5 lemma, $H^q_c(X - A) \cong H^q(X, A)$.

숙제 25. Show that $H^q_c(\mathbb{R}^n) \cong H^q(S^n, \infty) = \widetilde{H^q}(S^n)$.

5. Let M be a compact n-manifold, $A^{closed\,ANR} \subset M$. Then

$$\begin{array}{ccc} H_c^q(M-A) & \xrightarrow{\cong} & H^q(M,A) \\ \downarrow{\scriptstyle P.D:\cong} & \swarrow{\scriptstyle \text{"Lefschetz duality"}:\cong} & \\ H_{n-q}(M-A) & & \end{array}$$

e.g. $H^q(M,\partial M) \cong H_{n-q}(M-\partial M) \cong H_{n-q}(M)$ (∂M has a collar)

Remark. (K,L): compact pair in M, a compact manifold.
relative A.D. : $\check{H}^q(K,L) \xrightarrow{\cong} H_{n-q}(M-L, M-K)$
In particular, if $M=K$, then $\check{H}^q(M,L) \xrightarrow{\cong} H_{n-q}(M-L)$.**(숙제 26.)**

Application of Alexander Duality

6. $A^{compact} \subset \mathbb{R}^n \subset S^n$
$\Rightarrow$ (1)$\check{H}^q(A) \cong \widetilde{H}_{n-q-1}(\mathbb{R}^n - A)$
(2)$\check{\widetilde{H}}^q(A) \cong \widetilde{H}_{n-q-1}(S^n - A)$, where $\check{\widetilde{H}}^q(A) = \underset{\overrightarrow{V}}{lim}\tilde{H}^q(V)$

증명(1) and (2) :

$$\begin{array}{ccccccc} \check{H}^q(A) & \xrightarrow{A.D:\cong} & H_{n-q}(S^n, S^n - A) & \xrightarrow{excision:\cong} & H_{n-q}(\mathbb{R}^n, \mathbb{R}^n - A) & \xrightarrow{\cong} & \widetilde{H}_{n-q-1}(\mathbb{R}^n - A) \\ & & \downarrow{\scriptstyle (*):\cong} & & & & \\ & & \widetilde{H}_{n-q-1}(S^n - A) & & & & \end{array}$$

(*) : reduced long exact sequence of pair where $q \neq 0$

$q=0$ case :
$\rightarrow \underset{(\overset{open}{=}0)}{\widetilde{H}_n(S^n - A)} \rightarrow \underset{(=R)}{\widetilde{H}_n(S^n)} \rightarrow H_n(S^n, S^n - A) \xrightarrow{\partial} \widetilde{H}_{n-1}(S^n - A) \rightarrow \underset{(=0)}{\widetilde{H}_{n-1}(S^n)}$
$\Rightarrow \widetilde{H}_{n-1}(S^n - A) \cong H_n(S^n, S^n - A)/R.$

Now note that $\check{H}^0(A)/R \cong \check{\widetilde{H}}^0(A)$.
Then by A.D.,$H_n(S^n, S^n - A) \cong \check{H}^0(A)$ and $\widetilde{H}_{n-1}(S^n - A) \cong \check{\widetilde{H}}^0(A)$. □

7. $A \subset \mathbb{R}^n \subset S^n$, A : compact $(n-1)$-manifold.
$\Rightarrow$ the number of components of $\mathbb{R}^n - A$= (the number of components of A)+1

증명 $\widetilde{H}_0(\mathbb{R}^n - A;\mathbb{Z}/2) \cong H^{n-1}(A;\mathbb{Z}/2) \overset{P.D.}{=} H_0(A,\mathbb{Z}/2) = (\mathbb{Z}/2)^k$ where $k=$ the number of components of A $\square$

8. A non-orientable closed M^n can not be embedded in $\mathbb{R}^{n+1}$.
증명 May assume M is connected.
M is non-orientable.
$\Rightarrow H_n(M) = 0 \Rightarrow rk(H^n(M;\mathbb{Z})) = 0$ and hence $H^n(M;\mathbb{Z}) \cong \widetilde{H}_0(\mathbb{R}^{n+1} - M;\mathbb{Z}) = 0$
$\therefore \mathbb{R}^{n+1} - M$ has 1 component. This is a contradiction to 7. $\square$

9. $A =$ a link with k components in $\mathbb{R}^3$
$\Rightarrow H_*(\mathbb{R}^3 - A) = H_*(\mathbb{R}^3$- trivial link with k component$)$
증명 $\widetilde{H}_{n-q-1}(\mathbb{R}^3 - A) = H^q(A) = H^q($trivial link$)=\widetilde{H}_{n-q-1}(\mathbb{R}^3-$trivial link with k component$)$. $\square$