VI.2 Lefschetz Duality

Induced orientation on the boundary

1. Let M be an R-orientable manifold with boundary $\partial M \neq \phi$
$x \in \partial M, W$ a coordinate neighborhood of $x \cong \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$
$\supset V$ a coordinate neighborhood of $x \cong D^n_+ = \{x \in D^n \mid x_n \geq 0\}$.

Want to extent the orientation on $\mathring{V}$ to ∂V locally first in a compatible way to obtain a global extension :

orientation $\in \Gamma\mathring{V} \xleftarrow[j_{\mathring{V}}]{\cong} H_n(M, M - \mathring{V})$

$\xrightarrow{Want\ "\partial"}$ orientation $\in \Gamma(\partial V) \xleftarrow[j_{\partial \mathring{V}}]{\cong} H_{n-1}(\partial M, \partial M - \partial V)$:

Note that $j_{\mathring{V}}$ is $\cong$ since

$$\begin{tikzcd}
\text{orientation} \in \Gamma\mathring{V} & H_n(M, M - \mathring{V}) \arrow[l, "j_{\mathring{V}}"'] \\
H_n(\mathbb{R}^n, \mathbb{R}^n - \mathring{V}) \arrow[u, "j:\cong"] & H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \mathring{V}) \arrow[l, "excision:\cong"'] \arrow[u, "excision:\cong"']
\end{tikzcd}$$

and similarly $\Gamma(\partial V) \xleftarrow[j_{\partial V}]{\cong} H_{n-1}(\partial M, \partial M - \partial V)$

Consider the long exact sequence of the triple $(M, M - \mathring{V}, M - V)$ and note that $M - V$ is a strong deformation retract of M.

$$\begin{tikzcd}
 & & \Gamma\mathring{V} \arrow[dddl, dotted] & & \\
0 \arrow[r, equal] & H_n(M, M - V) \arrow[r] & H_n(M, M - \mathring{V}) \arrow[u, "\cong"] \arrow[r, "\partial:\cong"] & H_{n-1}(M - \mathring{V}, M - \mathring{V} - \partial V) \arrow[r] & 0 \\
 & & & H_{n-1}(U, U - \partial V) \arrow[u, "excision:\cong"] \arrow[d, "\partial W \text{is a s.d.r. of} U"] & \\
 & R \cong \Gamma(\partial V) & H_{n-1}(\partial M, \partial M - \partial V) \arrow[l, "\cong"'] & H_{n-1}(\partial W, \partial W - \partial V) \arrow[l, "excision:\cong"'] &
\end{tikzcd}$$

Here $U = (W - \mathring{V}) \cap$ "$\{x \in \mathbb{R}^n | 0 \leq x_n < 1\}$."
Note the construction of $\partial_V : \Gamma\mathring{V} \to \Gamma(\partial V)$ is compatible:

i.e., let $x \in U \subset V$, where $x \in \partial M$ and U is a half ball neighborhood of x contained in V , then

$$
\begin{tikzcd}
 & \Gamma\mathring{V} \arrow[dl, "\partial_V:\cong"'] \arrow[ddd, "r"] & & & H_n(M,M-\mathring{V}) \arrow[llll] \arrow[dl, "\partial:\cong"] \arrow[ddd] \\
 & & & H_{n-1}(M-\mathring{V},M-V) \arrow[ddd] & \\
\Gamma(\partial V) \arrow[ddd, "r:retraction"] & & H_{n-1}(\partial M,\partial M-\partial V) \arrow[ll, "\cong"'] \arrow[ur, "excision:\cong"] \arrow[ddd] & & \\
 & \Gamma\mathring{U} \arrow[dl, "\partial_U:\cong"'] & & & H_{n-1}(M,M-\mathring{U}) \arrow[llll] \arrow[dl, "\partial:\cong"] \\
 & & & H_n(M-\mathring{U},M-U) & \\
\Gamma(\partial U) & & H_{n-1}(\partial M,\partial M-\partial U) \arrow[ll] \arrow[ur, "excision:\cong"] & &
\end{tikzcd}
$$

want commutitivity

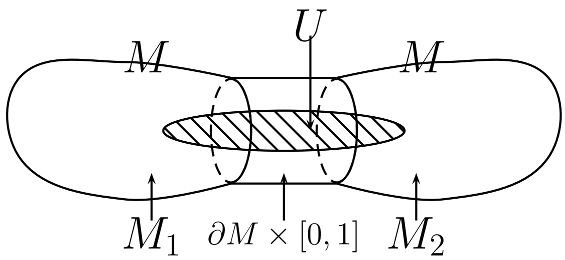
of left face

$\Rightarrow$ commutes(다른 면이 모두 commute하므로)

$\therefore$ we have a well-defined map $\partial : \Gamma M \to \Gamma(\partial M)$
$\therefore$ M : R-orientable $\Rightarrow$ ∂M : R-orientable.

2. Double of M

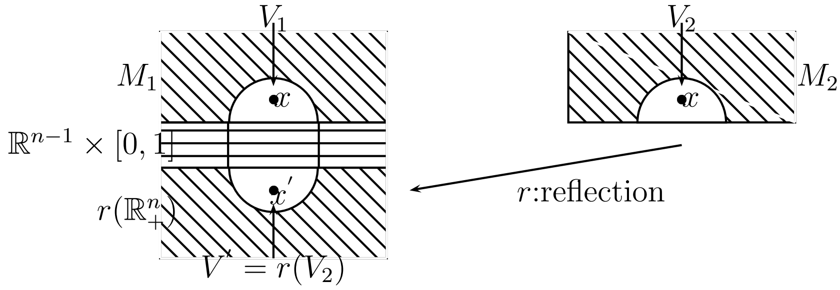
Consider the double of M, $DM = M_1 \coprod \partial M \times [0,1] \coprod M_2 / \sim$, where $M_1 = M_2 = M$ and $x \in \partial M_1 \Rightarrow x \sim (x,0) \in \partial M \times [0,1], x \in \partial M_2 \Rightarrow x \sim (x,1) \in \partial M \times [0,1]$.



Note If M is R-orientable, DM is R-orientable. (unique orientation compatible with that of M)

Proof

U: "coordinate neighborhood across the boundary" can be obtained as follow:
Given $x \in V \subset M$,



$$\Rightarrow \quad \begin{tikzcd} H_n(M_2, M_2 - x) = H_n(V_2, V_2 - x) \arrow[r, "r_*"', "\times(-1)"] \arrow[d, "\cong"'] & H_n(V', V' - x') \arrow[d, "\cong"'] \\ H_{n-1}(S^{n-1}) \arrow[r, "r_*", "\deg r = -1"'] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} H_n(M_1, M_1 - x) \arrow[r, "id."] & H_n(DM, DM - x) \arrow[dr, "id."] & \\ & & H_n(DM, DM - U) = R \\ H_n(M_2, M_2 - x) \arrow[r, "\times(-1)"'] & H_n(DM, DM - x') \arrow[ur, "id."'] & \end{tikzcd}$$

$\Rightarrow$ (M_1, s) and $(M_2, -s)$ can be spliced together to give an orientation of DM.

□

3. (Fundamental orientation class for ∂M)

Let $M(\partial M \neq \emptyset)$ be compact, R-orientable. (i.e.,$\mathring{M}$ is R-orientable.)

$$\begin{tikzcd} H_n(M, M - \mathring{M}) \arrow[r, "j"] \arrow[d, "\cong\ \text{excision}"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma\mathring{M} \\ H_n(DM, DM - \mathring{M}) \arrow[ur, "j_{\mathring{M}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \\ H_n(DM, DM - M) \arrow[u, "i_*", "\cong"'] \arrow[r, "j_M", "\cong"'] & \Gamma M \arrow[uu, "r\text{:restriction}"'] \end{tikzcd}$$

Here i_* is an isomorphism since $DM - \mathring{M}$ and $DM - M$ are homotopically equivalent to M_2, and j_M is isomorphism since M is compact.

Note that r is an isomorphism since an orientation on $\mathring{M}$ can be extended uniquely on DM.(**by 2.**)

Hence j is an isomorphism.

$\therefore$ An orientation of $\mathring{M}$, $s \in \Gamma\mathring{M}$ determines a unique class $\zeta \in H_n(M, \partial M)$ via $j(\cong)$, so that $\zeta|_{x \in \mathring{M}} = s(x) \in H_n(\mathring{M}, \mathring{M} - x)$.

Now consider

$$\begin{tikzcd}
 & H_n(M, M-\mathring{V}) \arrow[rr, "\cong"] \arrow[dd, "\cong"', "\mathrm{exc}^{-1}\circ\partial"] & & \Gamma\mathring{V} \arrow[dd, "\cong"', "\partial_V"] \\
\zeta \in H_n(M,\partial M) \arrow[ur] \arrow[rr, "j", "\cong"'] \arrow[dd, "\partial"] & & \Gamma M \arrow[ur, "r"'] \arrow[dd] & \\
 & H_{n-1}(\partial M, \partial M - \partial V) \arrow[rr, "\cong"] & & \Gamma(\partial V) \\
\partial\zeta = \zeta_{\partial M} \in H_{n-1}(\partial M) \arrow[ur] \arrow[rr, "j", "\cong"'] & & \Gamma(\partial M) \arrow[ur, "r"'] &
\end{tikzcd}$$

diagram에서 옆면과 위아래면은 commutativity는 자명하고, 뒷면의 commutativity는 1의 내용으로 부터 성립한다. 따라서 앞면도 commute하다.(r을 작용했을 때 일치하면 section의 uniqueness에 의해 일치한다.)

4. Lefschetz Duality

Let M be compact, R-orientable manifold with $\partial M \neq \emptyset$. Consider

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q-1}(M) \arrow[r] \arrow[d, "\zeta\cap"'] \arrow[dr, phantom, "(1)"] & H^{q-1}(\partial M) \arrow[r, "\delta"] \arrow[d, "\cong,\mathrm{P.D}"', "\partial\zeta\cap"] \arrow[dr, phantom, "(2)"] & H^q(M,\partial M) \arrow[r] \arrow[d, "\zeta\cap"] \arrow[dr, phantom, "(3)"] & H^q(M) \arrow[r] \arrow[d, "\zeta\cap"] & \cdots \\
\cdots \arrow[r] & H_{n-q+1}(M,\partial M) \arrow[r, "\partial"'] & H_{n-q}(\partial M) \arrow[r] & H_{n-q}(M) \arrow[r] & H_{n-q}(M,\partial M) \arrow[r] & \cdots
\end{tikzcd}$$

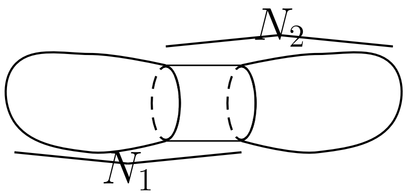
,where $\zeta \in H_n(M, \partial M)$ is the fundamental orientation class.
(1) is commutative up to sign $(-1)^{q-1}$: Note that we have the following on the chain level.
$\partial(\zeta \cap a) = (-1)^{q-1}(\partial\zeta \cap a - \zeta \cap \delta a)$. $\qquad (*)$
(2) $\partial\zeta \cap a = \zeta \cap \delta a$: $\quad (*)$ on the chain level $\Rightarrow !$.
(3) Clear.

Show $\zeta_\cap : H^q(M) \to H_{n-q}(M, \partial M)$ is an isomorphism :

$$H^q(M) = \check{H}^q(M) \overset{\cong,\text{A.D.}}{\rightarrow} H_{n-q}(DM, DM - M) \overset{\text{excision, cf.3}}{\cong} H_{n-q}(M, \partial M)$$

and note $\zeta \in H_n(M, \partial M) = \Gamma \overset{\circ}{M}$ is a restriction of $\zeta_M \in H_n(DM, DM - M)$ and A.D. map is essentially $\zeta_{M\cap}$.
Now 5-lemma $\Rightarrow \zeta_\cap : H^q(M, \partial M) \to H_{n-q}(M)$ is also an isomorphism.

Application
5. Let M be a compact manifold with ∂M.
(1) $\chi(DM) = 2\chi(M) - \chi(\partial M)$.
(2) $(M : R-$ orientable $\Rightarrow)\chi(\partial M) =$ even.
Proof



Use MV-sequence : "$N_1 \cap N_2 \to N_1 \bigoplus N_2 \to DM$".
$\Rightarrow \chi(N_1) + \chi(N_2) = \chi(N_1 \cap N_2) + \chi(DM) \Rightarrow (1)$
(2) n: even $\Rightarrow \dim(\partial M) =$ odd $\Rightarrow \chi(\partial M) = 0$.
n : odd $\Rightarrow \chi(DM) = 0 \Rightarrow \chi(\partial M) = 2\chi(M)$.

□

Example $\mathbb{R}P^{2n}(\chi = 1)$, $\mathbb{C}P^{2n}(\chi = 2n+1)$ and $\mathbb{H}P^{2n}(\chi = 2n+1)$ can not be a boundary of some manifolds.

6. (Signature of M^{4k})
Let M^{4k} be R-orientable, closed (and connected).
Consider $I : H^{2k}(M) \bigotimes H^{2k}(M) \to R$
$\Rightarrow I$ is a symmetric bilinear pairing.
Let $R = \mathbb{R} \Rightarrow I$ is a non-degenerated symmetric bilinear form.
Signature of $M = \sigma(M) :=$ signature of I.

Linear Algebra
Let V^n be a vector space over $\mathbb{R}$.
$b : V \bigotimes V \to \mathbb{R}$, a symmetric bilinear form.
Choose $e_1 \in V$ such that $b(e_1, e_1) \neq 0$. (If not, $b \equiv 0$). May assume $b(e_1, e_1) = \pm 1$.
Let $\epsilon^1 : V \to \mathbb{R}, \epsilon^1(x) = b(x, e_1)$. Then ϵ^1 is onto. $\Rightarrow V_1 = \ker \epsilon^1 = \langle e_1 \rangle^\perp$.
Apply the same argument to V_1 to get e_2 with $b(e_2, e_2) = \pm 1$.

$$\Rightarrow b = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & \text{\Large 0} & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & \text{\Large 0} & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix} \qquad b_{ij} = b(e_i, e_j)$$

sign $b :=$ $\sharp$ of positive eigenvalues of $b -$ $\sharp$ of negative eigenvalues of $b = r - s$ and rk $b = r + s$.

Witt index Let b be a non-degenerated symmetric bilinear form on V.
$\nu(b) :=$ Witt index of $b = \dim$(maximal totally isotropic subspace U of V i.e., $\forall x \in U \Rightarrow b(x,x) = 0 (b|_U \equiv 0)$).

Then $\nu(b) = \frac{1}{2}(n - |\text{sign} b|) = \sharp$ of (1,-1) pairs in $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & \text{\Large 0} & \\ & & 1 & & & \\ & & & -1 & & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & -1 \end{pmatrix}$.

Proof Let U be a maximal totally isotropic subspace and $\{x_1, \cdots, x_\nu\}$ be a basis for U.
Let $\alpha_1 : U \to \mathbb{R}$ be the dual of x_1, i.e., $\alpha_1(x_1) = 1, \alpha_1(x_i) = 0, i \geq 2$ and extend $\alpha_1 : V \to \mathbb{R}$ trivially ($V = U \bigoplus U'$ and $\alpha_1(U') = 0$)
b: non-degenerate $\Rightarrow \exists y_1 \in V$ such that $\alpha_1 = b(\ , y_1)$ so that $b(x_1, y_1) = 1,\ b(x_i, y_1) = 0, i \geq 2$ and may assume $b(y_1, y_1) = 0$.
$(b(y_1 + ax_1, y_1 + ax_1) = b(y_1, y_1) + 2ab(x_1, y_1) + a^2 b(x_1, x_1) = b(y_1, y_1) + 2a$ and let $a = -\frac{1}{2} b(y_1, y_1))$.

Let $H_1 = \text{span}\{x_1, y_1\}$ and consider $\theta_1 : V \to \mathbb{R}^2$ given by $\theta_1(v) = (b(v, x_1), b(v, y_1))$.
Then $\theta_1(x_1) = (0,1),\ \theta_1(y_1) = (1,0)$, so $\ker \theta_1 = H_1^{\perp}$ and $\{x_2, \cdots, x_\nu\} \subset \ker \theta_1$.

Apply the same argument to x_2 in $H_1^{\perp}$, etc., finally to get $V = H_1 \bigoplus \cdots \bigoplus H_\nu \bigoplus W$(orthogonal direct sum) for some W.

Now W does not have an isotropic vector by the maximality of U.
$\Rightarrow b|_W$ is definite and $b|_{H_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to $\{\frac{x_i+y_i}{\sqrt{2}}, \frac{x_i-y_i}{\sqrt{2}}\}$. $\square$

정리 1 *Let M^{4k+1} be compact and $\mathbb{R}$-orientable. Then*
*(1) dim $H_{2k}(\partial M)$ = 2 dim(im j_*) = 2dim(*ker j_**) = 2dim(im j^*), where $j_* : H_{2k}(\partial M) \to H_{2k}(M)$.*
(2) $\sigma(\partial M) = 0$.

Proof

$$\begin{tikzcd}
H_{2k+1}(M, \partial M) \arrow[d, "\partial"'] \arrow[r, phantom, "\circlearrowleft" below=12pt] & H^{2k}(M) \arrow[l, "\cong"', "\mathrm{L.D}"] \arrow[r, "\cong"] \arrow[d, "j^*"] \arrow[r, phantom, "\circlearrowleft" below=12pt] & \mathrm{Hom}(H_{2k}(M), \mathbb{R}) \arrow[d, "\widetilde{j_*}"] \\
H_{2k}(\partial M) \arrow[d, "j_*"] & H^{2k}(\partial M) \arrow[l, "\cong"', "\mathrm{P.D}"] \arrow[r, "\cong"] & \mathrm{Hom}(H_{2k}(\partial M), \mathbb{R}) \\
H_{2k}(M) & &
\end{tikzcd}$$

dim $H_{2k}(\partial M)$ = dim $H^{2k}(\partial M)$
dim(im j^*) = dim (im ∂) =dim (ker j_*) and
dim(im j^*) = dim(im $\widetilde{j_*}$) = dim(im j_*)
$\therefore$ dim $H_{2k}(\partial M)$ = dim(im j_*) + dim (ker j_*) = 2dim(im j_*) = 2dim(im j^*)
Now note $I \equiv 0$ on im $j^* \subset H^{2k}(\partial M)$ since
$I(j^*a, j^*b) = \langle \zeta_{\partial M}, j^*a \cup j^*b \rangle = \langle \zeta_{\partial M}, j^*(a \cup b) \rangle = \langle j_*\zeta_{\partial M}, a \cup b \rangle = \langle j_*\partial\zeta, a \cup b \rangle = 0$
Hence $\nu(I) = \frac{1}{2}\mathrm{dim}H^{2k} \Rightarrow \sigma(\partial M) = \sigma(I) = 0$.

$\square$

Cobordism theory Reference

Milnor, "Characteristic classes"
Stong, "Cobordism theory"