VII. Products

VII.1 The Künneth formulas and torsion product

Want a formula relating the homology of $X \times Y$ to the homology of X and Y.

정리 1 (The Künneth formula) *R: PID. There exists a natural s.e.s*

$$0 \to \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y) \to H_n(X \times Y) \to \bigoplus_{p=0}^{n-1} Tor(H_p(X), H_{n-p-1}(Y)) \to 0$$

which splits (but not naturally).

This result will be proved in 2 stages :
(1) The Eilenberg-Zilber theorem : *There exists a natural chain homotopy equivalence between $S(X \times Y)$ and $S(X) \otimes S(Y)$.*

(2) Algebraic Künneth formula: *Let $\mathcal{C}, \mathcal{C}'$ be chain complexes over PID R. Suppose at least one of $\mathcal{C}$ and $\mathcal{C}'$ is a free chain complex. Then there exists a natural s.e.s*

$$0 \to \bigoplus_{p=0}^{n} H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}') \to H_n(\mathcal{C} \otimes \mathcal{C}') \to \bigoplus_{p=0}^{n-1} Tor(H_p(\mathcal{C}), H_{n-p-1}(\mathcal{C}')) \to 0$$

which splits (but not naturally) if both $\mathcal{C}$ and $\mathcal{C}'$ are free.

Tensor product

Work in the category of R-modules(R: commutative ring with 1) and write $A \otimes B$ for $A \otimes_R B$.

1. (1) Let $f : A \to B$, $f' : A' \to B'$ be homomorphisms.
$\Rightarrow \exists!$ homomorphism $f \otimes f' : A \otimes A' \to B \otimes B'$
$a \otimes a' \mapsto f(a) \otimes f'(a')$

(2) $A \xrightarrow{f} B \xrightarrow{g} C$, $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ homomorphisms
$\Rightarrow (g \circ f) \otimes (g' \circ f') = (g \otimes g') \circ (f \otimes f') : A \otimes A' \to B \otimes B' \to C \otimes C'$

(3) $f : A \to B$, $f' : A' \to B'$: surjective

$\Rightarrow f \otimes f'$ is surjective and $ker(f \otimes f')$ is generated by the elements of the form $a \otimes a'$ with $a \in kerf$ or $a' \in kerf'$.

증명 $f \otimes f' : A \otimes A' \to B \otimes B'$. For all $b \otimes b' \in B \otimes B'$, there exists $a \in A$, $a' \in A'$ such that $f(a) = b$ and $f'(a') = b'$. Hence, $f \otimes f'$ is surjective.
Let $K = \langle a \otimes a' | a \in kerf \text{ or } a' \in kerf' \rangle$. Clearly $K \subset kerf \otimes f'$.
Consider

$$\begin{array}{ccc} A \otimes A' & \xrightarrow{\quad f \otimes f' \quad} & B \otimes B' \\ & {\scriptstyle p} \searrow \quad \circlearrowleft \quad \swarrow \exists\varphi & \\ & A \otimes A'/K & \end{array}$$

and define $\varphi(b \otimes b') = (a \otimes a')$ where $f(a) = b$ and $f'(a') = b'$. Then φ is well-defined since

$$a \otimes a' - a_0 \otimes a_0' = (a - a_0) \otimes a' + a_0 \otimes (a' - a_0') \in K$$

Therefore $ker(f \otimes f') \subset K$. $\square$

In particular, $A \xrightarrow{f} B \to 0 \Rightarrow A \otimes G \xrightarrow{f \otimes id} B \otimes G \to 0$.

2. Exactness of $\otimes G$

(1) $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$: exact

$\Rightarrow$ $A \otimes G \xrightarrow{f \otimes id} B \otimes G \xrightarrow{g \otimes id} C \otimes G \longrightarrow 0$: exact

(2) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$: split exact

$\Rightarrow$ $0 \longrightarrow A \otimes G \xrightarrow{f \otimes id} B \otimes G \xrightarrow{g \otimes id} C \otimes G \longrightarrow 0$: split exact

증명 (1) 1.(3) $\Rightarrow g \otimes id$ is onto and $ker(g \otimes id) = \langle b \otimes x | b \in kerg \rangle = im(f \otimes id)$.
(2) There exists p such that $p \circ f = id$. So

$$(p \otimes id) \circ (f \otimes id) = (p \circ f) \otimes id = id.$$

Therefore $f \otimes id$ is 1-1 and the sequence is split exact. $\square$

<u>Remark</u> $A \otimes R \cong A \cong R \otimes A$

$a \otimes r \to ra \leftarrow r \otimes a$

$a \otimes 1 \leftarrow a \to 1 \otimes a$

e.g. $\mathbb{Z} \otimes \mathbb{Z}/2 = \mathbb{Z}/2$

$$(\, 0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \,) \otimes \mathbb{Z}/2$$

$$\Rightarrow\ \mathbb{Z}/2 \xrightarrow[\text{0-map}]{\times 2} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

In general,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

$$\overset{\otimes G}{\Rightarrow}\ \mathbb{Z}\otimes G \longrightarrow \mathbb{Z}\otimes G \longrightarrow \mathbb{Z}/n\otimes G \longrightarrow 0$$

$$\begin{array}{ccc} \mathbb{Z}\otimes G & & \mathbb{Z}\otimes G \\ \| & & \| \\ G & \xrightarrow{\times n} & G \end{array}$$

Therefore $\mathbb{Z}/n \otimes G \cong G/nG$.

Exercise. $\mathbb{Z}/n \otimes \mathbb{Z}/m = \mathbb{Z}/d$, $d = (n, m)$

Recall : $A \otimes B \cong B \otimes A$, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$,
$(\oplus A_\alpha) \otimes B \cong \oplus(A_\alpha \otimes B)$, $A \otimes (\oplus B_\alpha) \cong \oplus(A \otimes B_\alpha)$
Use this to compute $\otimes$ of finitely generated abelian groups or R-modules. (R: PID)

3. If G is free, then $\otimes G$ preserves s.e.s.
(G : free $\Rightarrow G = \oplus R$ and

$$0 \to A \to B : \text{ exact} \Rightarrow\ 0 \to \underset{\underset{A}{\|}}{A \otimes R} \to \underset{\underset{B}{\|}}{B \otimes R} : \text{ exact}$$

$$\Rightarrow\ 0 \to A \otimes G \to B \otimes G : \text{ exact})$$

4. (Homology with coefficient G)
$\mathcal{C}$: a chain complex (over R) , G : R-module.
$\Rightarrow \mathcal{C} \otimes G : \cdots \to C_p \otimes G \to C_{p-1} \otimes G \to \cdots$ is a chain complex.
$\Rightarrow H(C; G) :=$ homology of $\mathcal{C} \otimes G :=$ homology of $\mathcal{C}$ with coefficient G.

$f : \mathcal{C} \to \mathcal{C}'$, a chain map $\Rightarrow f_* : H(\mathcal{C}; G) \to H(\mathcal{C}'; G)$.
$\phi : G \to G'$, an R-module homomorphism $\Rightarrow \phi_* : H(\mathcal{C}; G) \to H(\mathcal{C}; G')$.

If $0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0$ is a split s.e.s.(e.g. $\mathcal{E}$ is free), by snake lemma there exists a natural l.e.s.,

$$\cdots \to H_p(\mathcal{C}; G) \to H_p(\mathcal{D}; G) \to H_p(\mathcal{E}; G) \xrightarrow{\partial} H_{p-1}(\mathcal{C}; G) \to \cdots .$$

If $\mathcal{C}$ is free and $0 \to G' \to G \to G'' \to 0$ is a s.e.s., by snake lemma there exists a natural l.e.s.,

$$\cdots \to H_p(\mathcal{C};G') \to H_p(\mathcal{C};G) \to H_p(\mathcal{C};G'') \xrightarrow{\beta_*} H_{p-1}(\mathcal{C};G') \to \cdots .$$

In this case, β_* is called Bockstein homomorphism.

Torsion product

5. Let

$$\cdots \to F_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} A \to 0$$

be a free resolution of A.

Define $\mathrm{Tor}_p(A,G) := H_p(F;G) = H_p(F \otimes G)$
$= \text{homology of } \cdots \to F_p \otimes G \to \cdots \to F_1 \otimes G \to F_0 \otimes G \to 0.$

This is well-defined by comparison theorem. (Ext^p의 경우와 마찬가지)

Note. (1) $\mathrm{Tor}_0(A,G) \cong A \otimes G$:

$$F_1 \otimes G \xrightarrow{\partial \otimes 1} F_0 \otimes G \xrightarrow{\epsilon \otimes 1} A \otimes G \longrightarrow 0$$

is exact. So $\mathrm{Tor}_0(A,G) = F_0 \otimes G/im(\partial \otimes 1) = F_0 \otimes G/ker(\epsilon \otimes 1) = A \otimes G$.

(2) If $p \geq 1$ and F is free, $\mathrm{Tor}_p(F,G) = 0$ and $\mathrm{Tor}_p(G,F) = 0$.
A free resolution of F is

$$\cdots \longrightarrow 0 \longrightarrow F \xrightarrow{id} F \longrightarrow 0$$

So if $p \geq 1$, $C_p = 0$ and

$$\mathrm{Tor}_p(F,G) = 0.$$

And since $\otimes$free preserves resolution,

$$\mathrm{Tor}_p(G,F) = 0$$

(3) If R is a PID, $\mathrm{Tor}_p(A,G) = 0$ for $p \geq 2$.
A free resolution of A is

$$0 \longrightarrow \underset{\substack{\parallel \\ ker\epsilon}}{R} \longrightarrow \underset{\substack{\parallel \\ free}}{F} \xrightarrow{\epsilon} A \longrightarrow 0.$$

So if $p \geq 2$, $C_p = 0$ and
$$\mathrm{Tor}_p(A, G) = 0.$$
In this case we simply denote $\mathrm{Tor}(A, G)$ or $A * G$ for $\mathrm{Tor}_1(A, G)$.

(4) $\mathrm{Tor}_p(A, G)$ is a covariant functor in both variables.

6. If
$$0 \to A \to B \to C \to 0$$
is a short exact sequence, then there exists a natural long exact sequence,
$$\cdots \to \mathrm{Tor}_n(A, G) \to \mathrm{Tor}_n(B, G) \to \mathrm{Tor}_n(C, G) \to \mathrm{Tor}_{n-1}(A, G) \to$$
$$\cdots \to \mathrm{Tor}_1(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$$

증명 There exist free resolutions such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

And apply snake lemma. Naturality is same as before. □

In particular, if R is a PID, the l.e.s becomes
$$0 \to A * G \to B * G \to C * G \to A \otimes G \to B \otimes G \to C \otimes G \to 0.$$

7.If
$$0 \to A \to B \to C \to 0$$
is a short exact sequence, then there exists a natural long exact sequence,
$$\cdots \to \mathrm{Tor}_n(G, A) \to \mathrm{Tor}_n(G, B) \to \mathrm{Tor}_n(G, C) \to \mathrm{Tor}_{n-1}(G, A) \to$$
$$\cdots \to \mathrm{Tor}_1(G, C) \to G \otimes A \to G \otimes B \to G \otimes C \to 0$$

증명 $F \to G \to 0$: free resolution of G.
$\Rightarrow 0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$: s.e.s. of chain complex.
Apply the snake lemma. □

8. (1) $\mathrm{Tor}_n(A,B) \cong \mathrm{Tor}_n(B,A)$ for all n.
(2) $\mathrm{Tor}_n(\bigoplus A_\alpha, B) \cong \bigoplus \mathrm{Tor}_n(A_\alpha, B)$.
$(\mathrm{Tor}_n(A, \bigoplus B_\alpha) = \bigoplus \mathrm{Tor}_n(A, B_\alpha))$
(3) $R/a * B = \mathrm{Tor}(R/a, B) \cong ker(B \xrightarrow{\times a} B)$ and $R/a \otimes B \cong B/aB$.
(4) $R:$ PID, $\mathrm{Tor}(R/a, R/b) = R/a * R/b \cong R/d$, $d = (a,b)$.

증명 (1)

$$\cdots \to F \to B \to 0 \qquad : \text{a free resolution of } B$$

and let $K_p = ker\partial = im\partial \subset F_p$ so that

$$0 \to K_p \to F_p \xrightarrow{\partial} K_{p-1} \to 0 \qquad : \text{s.e.s.}$$

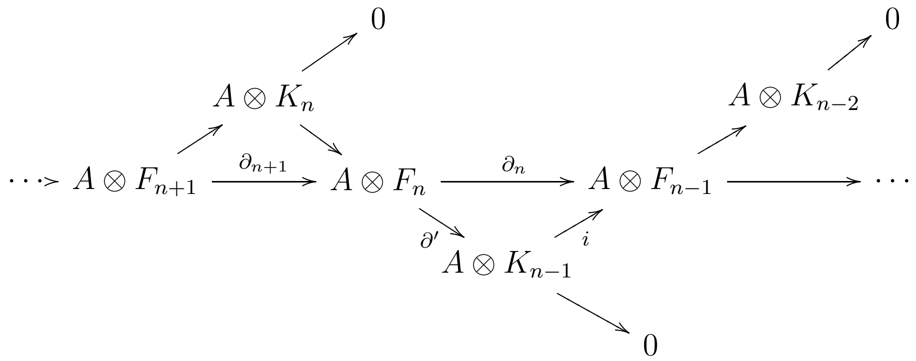
By 7.,

$$\cdots \to \mathrm{Tor}_n(A, K_0) \to \underset{\substack{\| \\ 0}}{\mathrm{Tor}_n(A, F_0)} \to \mathrm{Tor}_n(A, B) \to \mathrm{Tor}_{n-1}(A, K_0) \to \cdots$$

So, $\mathrm{Tor}_n(A,B) \cong \mathrm{Tor}_{n-1}(A, K_0) \cong \cdots \cong \mathrm{Tor}_1(A, K_{n-2})$.
Now consider

$$0 \to \mathrm{Tor}_1(A, K_{n-2}) \to A \otimes K_{n-1} \xrightarrow{i} A \otimes F_{n-1} \to A \otimes K_{n-2} \to 0$$

Note that



$$\begin{aligned} \mathrm{Tor}_1(A, K_{n-2}) &= ker i \cong ker\partial_n / ker\partial' = ker\partial_n / im\partial_{n+1} \\ &= H_n(A \otimes F) = H_n(F \otimes A) = \mathrm{Tor}_n(B, A) \end{aligned}$$

(2) $\mathrm{Tor}_n(A, \bigoplus B_\alpha) := H_n(F \otimes (\bigoplus B_\alpha)) = \bigoplus H_n(F \otimes B_\alpha) = \bigoplus \mathrm{Tor}_n(A, B_\alpha)$
The other follows from (1).

(3) Clear from the following.

$$0 \to R \overset{\times a}{\to} R \to R/a \to 0 \qquad \text{free resolution of } R/a$$

$\Rightarrow$

$$\begin{array}{ccccccccc} 0 \to R/a * B \to & R \otimes B & \to & R \otimes B & \to R/a \otimes B \to 0 \\ & \| & & \| & \\ & B & \xrightarrow{\times a} & B & \end{array}$$

(4) **숙제 27.**
$R/a * R/b = ker(R/b \xrightarrow{\times a} R/b) = R/d,\ d = (a, b)$
$R/a \otimes R/b = coker(R/b \xrightarrow{\times a} R/b) = R/d.$ $\square$