VII.3 The Künneth formula for cohomology

16. (Künneth formula for cohomology)
(1) Algebraic Künneth formula
If $\mathcal{C}^*$(or $\mathcal{D}^*$) is free cochain complex over PID R, then there exists a natural s.e.s

$$0 \to \bigoplus_{p+q=n} H^p(\mathcal{C}^*) \otimes H^q(\mathcal{D}^*) \to H^n(\mathcal{C}^* \otimes \mathcal{D}^*) \to \bigoplus_{p+q=n+1} H^p(\mathcal{C}^*) * H^q(\mathcal{D}^*) \to 0$$
$$\{a\} \otimes \{b\} \quad \mapsto \quad \{a \otimes b\}$$

which splits if both are free.
(This follows from algebraic Künneth formula since $\mathcal{C}^*$ can be viewed as a chain complex by letting $C^n = C_{-n}$ and homology of this is the cohomology of $\mathcal{C}^*$, i.e., $H_{-n}(\mathcal{C}_{-*}) = H^n(\mathcal{C}^*)$.)

(2) Note.
(i) $\mathcal{C}$ free $\not\Rightarrow \mathcal{C}^*$ free in general.
$(\mathrm{Hom}(\oplus A_\alpha, R) = \prod \mathrm{Hom}(A_\alpha, R))$
(ii) $S^*(X \times Y) \underset{EZ}{\simeq} (S(X) \otimes S(Y))^* \ncong S(X)^* \otimes S(Y)^*$

We get around these difficulties by finite approximation of $\mathcal{C}$.

(iii) δ in $\mathcal{C}^* \otimes \mathcal{D}^*$ is defined as before:

$$\delta|_{C^p \otimes D^{n-p}} = \delta \otimes 1 + (-1)^p (1 \otimes \delta)$$

(3) $\alpha \overset{D}{\simeq} \alpha' : \mathcal{C} \to \mathcal{D}$, chain homotopic $\Rightarrow \alpha \otimes 1 \overset{D \otimes 1}{\simeq} \alpha' \otimes 1 : \mathcal{C} \otimes \mathcal{A} \to \mathcal{D} \otimes \mathcal{A}$ for all chain complex $\mathcal{A}$
(Exercise. Check this)

(4) $\mathcal{C} \overset{\alpha}{\simeq} \mathcal{D} \Rightarrow \mathcal{C}^* \overset{\tilde{\alpha}}{\simeq} \mathcal{D}^*$ and $\mathcal{C} \otimes \mathcal{A} \overset{\alpha \otimes 1}{\simeq} \mathcal{D} \otimes \mathcal{A}$.

(5) $\mathcal{C}$(or $\mathcal{D}$) : free and finite type, i.e., each C_p is finitely generated. Then

$$\mathcal{C}^* \otimes \mathcal{D}^* \cong (\mathcal{C} \otimes \mathcal{D})^*$$
$$\alpha \otimes \beta \mapsto \text{"}\alpha \times \beta\text{"}, \text{ where } \alpha \times \beta(c \otimes d) = \alpha(c)\beta(d)$$

증명 Show for C, D : R-module first.
If $D = R$, $C^* \otimes R^* = C^* \otimes R = C^* = (C \otimes R)^*$.
If D is finitely generated and free, $D = \bigoplus_{\text{finite}} R$

$$\Rightarrow C^* \otimes D^* = C^* \otimes (\bigoplus R)^* = C^* \otimes (\bigoplus R^*) = \bigoplus C^* \otimes R^* = \bigoplus (C \otimes R)^*$$
$$= (\bigoplus C \otimes R)^* = (C \otimes \bigoplus R)^* = (C \otimes D)^*$$

Show for chain complex :

$$(\mathcal{C}^* \otimes \mathcal{D}^*)_n = \bigoplus_{p+q=n} (C^p \otimes D^q) \cong \bigoplus_{p+q=n} (C_p \otimes D_q)^* = (\bigoplus_{p+q=n} C_p \otimes D_q)^*$$
$$= ((\mathcal{C} \otimes \mathcal{D})_n)^* = (\mathcal{C} \otimes \mathcal{D})^n = ((\mathcal{C} \otimes \mathcal{D})^*)_n \qquad \square$$

Remark. (i) In general,

$$\begin{aligned} \mathcal{C}^* \otimes \mathcal{D}^* &\to (\mathcal{C} \otimes \mathcal{D})^* \\ a \otimes b &\mapsto a \times b \end{aligned}$$

is a chain map and $\delta(a \times b) = \delta a \times b + (-1)^p a \times \delta b$, $a \in C^p$.

$$\begin{aligned} \because\ \delta(a \times b)(x \otimes y) &= a \times b(\partial(x \otimes y)) \\ &= a \times b(\partial x \otimes y + (-1)^{|x|} x \otimes \partial y) \\ &= a(\partial x) b(y) + (-1)^{|x|} a(x) b(\partial y) \\ &= \delta a(x) b(y) + (-1)^{|x|} a(x) \delta b(y) \\ &= (\delta a \times b + (-1)^{|x|} a \times \delta b)(x \otimes y) \end{aligned}$$

and $(-1)^{|x|} a \times \delta b(x \otimes y)$ is nonzero only when $|x| = p$.
Now

$$\begin{array}{ccc} a \otimes b & \longmapsto & a \times b \\ \downarrow \delta & \circlearrowleft & \downarrow \delta \\ \delta a \otimes b + (-1)^p a \otimes \delta b & \longmapsto & \delta a \times b + (-1)^p a \times \delta b \end{array}$$

(ii) α, β : cocycles $\Rightarrow \alpha \times \delta\gamma = \pm\delta(\alpha \times \gamma)$ and similarly $\delta\gamma \times \beta = \pm\delta(\gamma \times \beta)$.
Therefore the cohomology cross product

$$\begin{aligned} H^*(\mathcal{C}) \otimes H^*(\mathcal{D}) &\xrightarrow{\times} H^*(\mathcal{C} \otimes \mathcal{D}) \\ \{a\} \otimes \{b\} = \alpha \otimes \beta &\longmapsto \alpha \times \beta = \{a \times b\} \end{aligned}$$

is well-defined.

(6) (finite approximation) $\mathcal{C}$: free over PID. $H(\mathcal{C})$: finite type.
$\Rightarrow \exists\ \overline{\mathcal{C}}$: free and finite type such that $H(\overline{\mathcal{C}}) \xrightarrow{\cong} H(\mathcal{C})$.
증명 Consider the exact sequences

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H(\mathcal{C}) \longrightarrow 0$$
$$0 \longrightarrow Z \longrightarrow C \longrightarrow \overline{B} \longrightarrow 0\,, \qquad \overline{B}_p = B_{p-1}.$$

Since $H(\mathcal{C})$ is of finite type, there exist finitely generated Z' and $B' = kerp$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \longrightarrow & Z' & \xrightarrow{p} & H(\mathcal{C}) & \longrightarrow & 0 \\ & & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & Z & \xrightarrow{p} & H(\mathcal{C}) & \longrightarrow & 0 \end{array}$$

and let $\overline{C} = Z' \oplus \overline{B}'$ and $\phi = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z' & \dashrightarrow & \overline{C} & \dashrightarrow & \overline{B}' & \longrightarrow & 0 \\ & & \downarrow & \circlearrowleft & \vdots & \circlearrowleft & \downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & C & \xrightarrow{\partial} & \overline{B} & \longrightarrow & 0 \end{array}$$

$\Rightarrow \phi : \overline{\mathcal{C}} \to \mathcal{C}$ is a chain map and $\phi_* : H(\overline{\mathcal{C}}) \cong H(\mathcal{C})$ from the construction. □

(7) Let $\mathcal{C}$(or $\mathcal{D}$) be free over PID and $H(\mathcal{C})$(or $H(\mathcal{D})$) be of finite type.
$\Rightarrow$ There exists a finite approximation $\overline{\mathcal{C}}$(or $\overline{\mathcal{D}}$) : free
$\Rightarrow \overline{\mathcal{C}}^*$(or $\overline{\mathcal{D}}^*$) : free
By (1),

$$0 \to \bigoplus_{p+q=n} H^p(\overline{\mathcal{C}}^*) \otimes H^q(\mathcal{D}^*) \to H^n(\overline{\mathcal{C}}^* \otimes \mathcal{D}^*) \to \bigoplus_{p+q=n+1} H^p(\overline{\mathcal{C}}^*) * H^q(\mathcal{D}^*) \to 0$$

Since $\overline{\mathcal{C}}$ and $\mathcal{C}$ has the same homology, $\overline{\mathcal{C}} \simeq \mathcal{C}$.[1]
Therefore by (4) and (5), algebraic Künneth formula for cohomology is

If $\mathcal{C}$ (or $\mathcal{D}$) is free chain complex over PID R and $H(\mathcal{C})$ (or $H(\mathcal{D})$) is of finite type, then there exists a natural s.e.s

$$0 \to \bigoplus_{p+q=n} H^p(\mathcal{C}^*) \otimes H^q(\mathcal{D}^*) \to H^n((\mathcal{C} \otimes \mathcal{D})^*) \to \bigoplus_{p+q=n+1} H^p(\mathcal{C}^*) * H^q(\mathcal{D}^*) \to 0$$

which splits if $\mathcal{C}$ and $\mathcal{D}$ are free and $H(\mathcal{C})$ and $H(\mathcal{D})$ are of finite type.

[1] see earlier theorem about mapping cone construction in the chapter of cohomology(IV.1) or Munkres, Elements of algebraic topology, p.279, theorem 46.2

(8) Let $C = S(X, A)$ and $D = S(Y, B)$.
Then

$$S(X, A) \otimes S(Y, B) \underset{\text{rel. EZ}}{\simeq} S(X \times Y)/S(X \times B \cup A \times Y) =: S((X, A) \times (Y, B)).$$

By (7), we have relative Künneth formula for cohomology over PID:

$$0 \to \bigoplus_{p+q=n} H^p(X, A) \otimes H^q(Y, B) \xrightarrow{\times} H^n((X, A) \times (Y, B)) \to \bigoplus_{p+q=n+1} H^p(X, A) * H^q(Y, B) \to 0$$

natural if $H(X, A)$(or $H(Y, B)$) is of finite type, and splits if both are of finite type.[2]

17. (Cross and Cup product)
(1) $S(X \times Y) \xrightarrow[\simeq]{\phi} S(X) \otimes S(Y)$ any chain homotopy equivalence.

$$\Rightarrow \quad \begin{array}{ccccc} S^*(X \times Y) & \xleftarrow[\simeq]{\tilde{\phi}} & (S(X) \otimes S(Y))^* & \xleftarrow{r} & S^*(X) \otimes S^*(Y) \\ a \times b & \longleftarrow & r(a \otimes b) = \text{"}a \times b\text{"} & \longleftarrow & a \otimes b \end{array}$$

Let
$$\zeta = \{x\}, x \in Z_p(X) \quad \alpha = \{a\}, a \in Z^r(X)$$
$$\eta = \{y\}, y \in Z_q(Y) \quad \beta = \{b\}, b \in Z^s(Y).$$

Then $\langle \zeta \times \eta, \alpha \times \beta \rangle = \langle \zeta, \alpha \rangle \langle \eta, \beta \rangle$.

증명

$$\begin{aligned} \langle \zeta \times \eta, \alpha \times \beta \rangle &= \langle \phi_*^{-1}\{x \otimes y\}, \phi^*\{r(a \otimes b)\} \rangle \\ &= \langle \{x \otimes y\}, \{r(a \otimes b)\} \rangle \\ &= \langle x \otimes y, r(a \otimes b) \rangle \\ &= a(x)b(y) \\ &= \langle x, a \rangle \langle y, b \rangle \\ &= \langle \zeta, \alpha \rangle \langle \eta, \beta \rangle \end{aligned}$$

□

(2) Take $\phi = A:$ AW-diagonal approximation

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

[2] 여기서 실제로는 natural에 대한 가정은 반드시 필요한 가정이지만, split에 대한 가정은 생략하여도 무방하다.

Then $\alpha \times \beta = p_1^*\alpha \cup p_2^*\beta$.
In fact, this holds on chain level : $a \times b = p_1^\sharp a \cup p_2^\sharp b$.
증명 For any $\omega \in S_n(X \times Y)$, let $\sigma = p_{1\sharp}\omega$ and $\tau = p_{2\sharp}\omega$.

$$\begin{aligned}
\langle(\sigma,\tau), a \times b\rangle &= \langle(\sigma,\tau), \widetilde{\phi}(r(a \times b))\rangle \\
&= \langle \phi(\sigma,\tau), r(a \times b)\rangle \\
&\overset{\phi=A}{=} \langle \sum_{k+l=n} \sigma\lambda_k \otimes \tau\rho_l, r(a \times b)\rangle \\
&= \sum_{k+l=n} \langle \sigma\lambda_k, a\rangle\langle \tau\rho_l, b\rangle \\
&= \langle \sigma\lambda_p, a\rangle\langle \tau\rho_q, b\rangle \qquad \text{if } a \in S^q(X),\ b \in S^q(Y) \\
&= \langle p_{1\sharp}(\sigma,\tau)\lambda_p, a\rangle\langle p_{2\sharp}(\sigma,\tau)\rho_q, b\rangle \\
&= \langle(\sigma,\tau)\lambda_p, p_1^\sharp a\rangle\langle(\sigma,\tau)\rho_q, p_2^\sharp b\rangle \\
&= \langle(\sigma,\tau), p_1^\sharp a \cup p_2^\sharp b\rangle
\end{aligned}$$

□

(3) Let $\Delta : X \to X \times X$ be the diagonal map.
Then $\Delta^\sharp(a \times b) = a \cup b$. So that $\Delta^*(\alpha \times \beta) = \alpha \cup \beta$.
증명

$$\begin{aligned}
\Delta^\sharp(a \times b) &= \Delta^\sharp(p_1^\sharp a \cup p_2^\sharp b) \\
&= \Delta^\sharp p_1^\sharp a \cup \Delta^\sharp p_2^\sharp b \\
&= a \cup b \qquad (\because \Delta^\sharp p_1^\sharp = (p_1\Delta)^\sharp = id)
\end{aligned}$$

□

(4) $(\alpha \cup \beta) \times (\gamma \cup \delta) = (-1)^{|\beta||\gamma|}(\alpha \times \gamma) \cup (\beta \times \delta)$
증명

$$\begin{aligned}
(\alpha \cup \beta) \times (\gamma \cup \delta) &= p_1^*(\alpha \cup \beta) \cup p_2^*(\gamma \cup \delta) \\
&= p_1^*\alpha \cup p_1^*\beta \cup p_2^*\gamma \cup p_2^*\delta \\
&= (-1)^{|\beta||\gamma|} p_1^*\alpha \cup p_2^*\gamma \cup p_1^*\beta \cup p_2^*\delta \\
&= (-1)^{|\beta||\gamma|}(\alpha \times \gamma) \cup (\beta \times \delta)
\end{aligned}$$

□

(5) Slant product
$c \in (S(X) \otimes S(Y))^{p+q}$, $x \in S_p(X)$.
Define $c/x \in S^q(Y)$ by the formula

$$\langle y, c/x\rangle := \langle x \otimes y, c\rangle$$

Then $\delta(c/x) = (-1)^p(\delta c/x - c/\partial x)$.
증명

$$\begin{aligned}\langle y, \delta(c/x)\rangle &= \langle \partial y, c/x\rangle \\ &= \langle x \otimes \partial y, c\rangle \\ &= (-1)^p\{\langle \partial(x \otimes y), c\rangle - \langle \partial x \otimes y, c\rangle\} \\ &= (-1)^p\{\langle y, \delta c/x\rangle - \langle y, c/\partial x\rangle\}\end{aligned}$$

□

This implies that we have a well-defined slant product on homology :

$$\begin{array}{ccc} H^{p+q}(X \times Y) \otimes H_p(X) & \to & H^q(Y) \\ \gamma \otimes \xi & \mapsto & \gamma/\xi \end{array}$$

숙제 33. 다음을 증명하라.
(6) $(\xi \times \eta) \cap (\alpha \times \beta) = (-1)^{|\beta|(|\xi|-|\alpha|)}(\xi \cap \alpha) \times (\eta \cap \beta)$
(Use slant product and Δ.)

(7) $\{(\alpha \times \beta) \cup \gamma\}/\xi = (-1)^{|\beta|(|\alpha|-|\xi|)}\beta \cup (\gamma/\xi \cap \alpha)$
(Use AMT as in 13. note.)