1 A brief introduction to Volume conjecture

2 Linear Fractional Transformation and 2-dimensional hyperbolic geometry

2.1 Linear Fractional Transformation (LFT)

A linear fractional transformation (or Möbius transformation) is of the form

$$f(z) = \frac{az+b}{cz+d} : \quad \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

where $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$.
Let M^+ be the set of LFT's and define $\phi : \mathrm{GL}(2, \mathbb{C}) \longrightarrow M^+$ by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f : z \mapsto \frac{az+b}{cz+d}$$

Since $\ker \phi = \left\{ \lambda \mathrm{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda \in \mathbb{C} \right\}$, $\mathrm{M}^+ \cong \mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\{\lambda \mathrm{I}\}$.

Remark f is a projective transformation:
$(\phi, p) : (\mathrm{GL}(2, \mathbb{C}), \mathbb{C}^2) \longrightarrow (\mathrm{M}^+ = \mathrm{PGL}(2, \mathbb{C}), \mathbb{C}P^1 = \hat{\mathbb{C}})$ is an equivariant map, i.e., for $A \in \mathrm{GL}(2, \mathbb{C}), p \circ A = \phi(A) \circ p$, and $\phi(A) = f$ is a projective transformation induced by the linear map A.

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Fig.1

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In an affine chart of $\mathbb{C}P^1 = \hat{\mathbb{C}}$ given by $z_2 = 1$ for $(z_1, z_2) \in \mathbb{C}^2$, we see that
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$
Alternatively, we can use $\mathrm{SL}(2, \mathbb{C})$, i.e., if we define $\phi : \mathrm{SL}(2, \mathbb{C}) \longrightarrow M^+$ in the same way, then $\ker \phi = \{\pm \mathrm{I}\}$ and $\mathrm{M}^+ \cong \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{I}\}$

2.2 Geometry

M^+ is generated by
① $z \mapsto z + a$ translation
② $z \mapsto \lambda z$ homothety (rotation, when $|\lambda| = 1$)
③ $z \mapsto \frac{1}{z}$ inversion (orientation preserving)

Note that $z^* = \frac{1}{\bar{z}}$ is a symmetric point of z with respect to the unit circle $|z| = 1$.

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FIG.2

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More generally, $J_{s(a,r)}\colon\ z \mapsto \dfrac{r^2}{\overline{z - a}} + a$

Exercise. Show that $g \in \mathrm{M}^+$ maps circles to circles.

2.3 Cross Ratio

Let $[z_1, z_2, z_3, z_4] := \dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$. Then we have

1) $[1, 0, \infty, z] = z$.
2) $g(z) := [z_1, z_2, z_3, z] = \dfrac{(z_1 - z_3)(z_2 - z)}{(z_1 - z_2)(z_3 - z)}$
$\Rightarrow g(z_1) = 1,\ g(z_2) = 0,\ g(z_3) = \infty$.
3) $\forall g \in \mathrm{M}^+$ preserves cross ratio:
$\because g(z) - g(w) = \dfrac{az + b}{cz + d} - \dfrac{aw + b}{cw + d} = \dfrac{(ad - bc)(z - w)}{(cz + d)(cw + d)}$

$$[g(z_1), g(z_2), g(z_3), g(z_4)] = \dfrac{\dfrac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)}\dfrac{(ad - bc)(z_2 - z_4)}{(cz_2 + d)(cz_4 + d)}}{\dfrac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}\dfrac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}} = [z_1, z_2, z_3, z_4].$$

4) For each distinct points z_1, z_2, z_3, and w_1, w_2, w_3 respectively, there is a unique $g \in \mathrm{M}^+$ such that $g(z_i) = w_i$:
$\because$ By 2), $\exists\, g_1, g_2 \in \mathrm{M}^+$ such that $g_1(z_1) = g_2(w_1) = 1$, $g_1(z_2) = g_2(w_2) = 0$, and $g_1(z_3) = g_2(w_3) = \infty$. Then take $g_2^{-1} \circ g_1$.
5) Other possibilities of defining cross ratio:
This problem essentially reduces to a permutation problem. And under permutation, there are 6 different cross ratios up to sign, namely, λ, $1-\lambda$, $\dfrac{\lambda}{\lambda - 1}$, $\dfrac{1}{\lambda}$, $\dfrac{1}{1 - \lambda}$, $\dfrac{\lambda - 1}{\lambda}$, and hence 3 ($\lambda, \lambda' = \dfrac{1}{1 - \lambda}, \lambda'' = \dfrac{\lambda - 1}{\lambda}$)up to sign and their inverses.(Exercise)
Later we will use Neumann's convention of cross ratio given by $[z_1, z_2, z_3, z_4] := \dfrac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)} =: \lambda$. In this case we have
$[2, 1, 3, 4] = [1, 2, 4, 3] = \frac{1}{\lambda}$
$[3, 2, 1, 4] = [1, 4, 3, 2] = \frac{1}{\lambda'}$
$[4, 2, 3, 1] = [1, 3, 2, 4] = \frac{1}{\lambda''}$
Hence $[1, 2, 3, 4] = [2, 1, 4, 3] = [3, 4, 1, 2] = [4, 3, 2, 1]$, and we have 6 different permutation values out of $4! = 24$ permutations.

Proposition 2.3.1. $\mathrm{M}^+ = \mathrm{Aut}(\hat{\mathbb{C}})$

Proof. ($\subset$) Trivial
($\supset$) For $g \in \mathrm{Aut}(\hat{\mathbb{C}})$, we may assume $g(0) = 0$ and $g(\infty) = \infty$ by composing a suitable LFT. Then $h(z) = \dfrac{g(z)}{z}$ is a holomorphic function with $h(0) \neq \infty$ and $h(\infty) \neq \infty$. Since $\hat{\mathbb{C}}$ is compact, $h : \hat{\mathbb{C}} \longrightarrow \mathbb{C}$ is bounded and hence constant by Liouville's theorem. □

2.4 Poincaré Upper Half Plane and Disk

We shall first find the automorphism group of the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ and the unit disk $\mathrm{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$.

Proposition 2.4.1. $f \in \mathrm{M}^+$ acts on $\mathbb{H}^2$ if and only if $f \in \mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm \mathrm{I}\}$

Proof. ($\Rightarrow$) Suppose that f maps $p, q, r \in \mathbb{R}$ to $1, 0, \infty$ respectively. Then $f(z) = [1, 0, \infty, f(z)] = [p, q, r, z]$ and hence f has a real representative.
($\Leftarrow$) Obviously f sends $\mathbb{R}$ to $\mathbb{R}$ and hence a half plane to a half plane. By direct computation, $f(z) - \overline{f(z)} = \dfrac{z - \bar{z}}{|cz + d|^2}$. (We shall use this result later again.) Therefore, f maps $\mathbb{H}^2$ to itself. □

Proposition 2.4.2. $\mathbb{H}^2 \cong \mathrm{D}$

Proof. $\phi(z) = -i\dfrac{z + i}{z - i}$ maps D onto $\mathbb{H}^2$, which is called a Cayley transformation. Note that ϕ maps $-i, 0, i, and 1$ to $0, i, \infty, and 1$ respectively. □

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FIG.3

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Proposition 2.4.3. $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R})$

Proof. ($\supset$) Propsition 2.4.1.
($\subset$) Let $g \in \mathrm{Aut}(\mathbb{H}^2)$ and may assume $g(i) = i$ using a suitable homothety and a translation in $\mathrm{PSL}(2, \mathbb{R})$. Then $\tilde{g} = \phi^{-1} \circ g \circ \phi$ maps D to itself and $\tilde{g}(0) = 0$. By the Schwarz lemma, $|\tilde{g}(z)| \leq |z|$. Actually, $|\tilde{g}(z)| = |z|$ since $\tilde{g}^{-1}$ satisfies the same condition. Thus $\tilde{g}$ is a rotation, which is an LFT, and so is g. □

Corollary 2.4.1. $\mathrm{Aut}(\mathrm{D}) = \left\{ \dfrac{az+b}{\bar{b}z+\bar{a}} \,\middle|\, a, b \in \mathbb{C} \text{ with } |a|^2 - |b|^2 = 1 \right\}$
$= \left\{ e^{i\theta} \dfrac{z-a}{1-\bar{a}z} \,\middle|\, a \in \mathrm{D} \right\}$.

Proof. Excercise. (Use for instance $|f(z)| = 1$ for $|z| = 1$, and let $\dfrac{a}{\bar{a}} = e^{i\theta}$ for the second equality.) □

Remark

① From the corollary we see that the isotropy subgroup of Aut(D) at 0 is isomorphic to $\mathrm{SO}(2) \cong S^1$.

② $\mathrm{PSL}(2, \mathbb{R})$ is a three dimensional Lie group.

Poincaré metric

If $w = f(z) = \dfrac{az+b}{cz+d} \in \mathrm{PSL}(2, \mathbb{R})$, then $\mathrm{Im}w = \dfrac{\mathrm{Im}z}{|cz+d|^2}$ and $\dfrac{dw}{dz} = \dfrac{1}{(cz+d)^2}$. Hence $\dfrac{|dw|}{\mathrm{Im}w} = \dfrac{|dz|}{\mathrm{Im}z}$ is an invariant metric, which is called the Poincaré metric. If we write $z = x + iy$ and $|dz| = \sqrt{dz^2 + dy^2} = ds_0$, then the Poincaré metric can be expressed as $ds := \dfrac{ds_0}{y}$. Hence the length of a curve γ, $l(\gamma) := \int_\gamma ds = \displaystyle\int_\gamma \frac{|z'(t)|}{\mathrm{Im}z} dt$ is invariant under $g \in \mathrm{PSL}(2, \mathbb{R})$.

Remark The invariance of the Poincaré metric can also be derived from the cross ratio $[z, \bar{z}, w, \bar{w}] = \dfrac{|z-w|^2}{-4\mathrm{Im}z\mathrm{Im}w}$ by considering $w = z + dz$.

Exercise. Show that, on D, the Poincaré metric is given by $\phi^* ds = \dfrac{2|dz|}{1-|z|^2}$ both by computing a pull back metric and by using cross ratio.

Isometry Group

Proposition 2.4.4. $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$

Proof. ($\subset$) Clear.
($\supset$) An isometry is a conformal map, and an orientation preserving conformal map is complex analytic. □

If J is an orientation reversing isometry, e.g., a reflection with respect to the imaginary axis, then $J\mathrm{Isom}^-(\mathbb{H}^2) = \mathrm{Isom}(\mathbb{H}^2)$, and

$$\mathrm{Isom}(\mathbb{H}^2) = \mathrm{Isom}^+(\mathbb{H}^2) \coprod J\mathrm{Isom}^+(\mathbb{H}^2).$$

Exercise. Show that the sectional curvature of $\mathbb{H}^2$ is constant -1.

Note By virtue of prop 2.4.4, we can view a complex analysis problem as a geometry problem and vice versa.