1 A brief introduction to Volume conjecture

2 Linear Fractional Transformation and 2-dimensional hyperbolic geometry

2.1 Linear Fractional Transformation (LFT)

2.2 Geometry

2.3 Cross Ratio

2.4 Poincaré Upper Half Plane and Disk

2.5 2D Hyperbolic Geometry

2.6 Special Polygons

Lambert Quadrilateral

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Fig 16.
$(\phi < \frac{\pi}{2})$.
Then the following rules hold.

$$\cos\phi = \sinh a \sinh b \tag{2.6.1}$$

$$\sin\phi = \frac{\cosh a}{\cosh c} \left(= \frac{\sinh a \cosh b}{\sinh c} \right) \tag{2.6.2}$$

$$\tanh c = \tanh a \cosh b \tag{2.6.3}$$

(Note that a and b determine the quadrilateral!)

Proof. Fig 17.

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$$\begin{aligned} \cos\phi &= -\cos\gamma\cos\delta + \sin\gamma\sin\delta\cosh f & \text{(by Cosine Rule II)} \\ &= -\sin\alpha\sin\beta + \cos\beta\cos\alpha\cosh f & (\alpha+\delta=\beta+\gamma=\frac{\pi}{2}) \end{aligned}$$

Also, $\cos\beta = \sin\alpha\cosh b$ (Right Triangle) and $\cos\alpha = \sin\beta\cosh a$.
$\therefore$ The equation above becomes

$$\begin{aligned}\sin\alpha\sin\beta(\cosh a\cosh b\cosh f-1) &= \sin\alpha\sin\beta(\cosh^2 f-1)\\ &= \sin\alpha\sin\beta\sinh^2 f=\sinh a\sinh b \text{ (by Sine Rule)}\end{aligned}$$

2nd Proof for 2.5.10.

Fig 18.

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$$\begin{aligned}\cos^2\frac{\phi}{2} &= [e^{-i\theta}, e^{i\theta}, ie^{-\tau i}, ie^{\tau i}]\\ &= \frac{1}{2}(\cot\theta\cot\tau+1)\end{aligned}$$

$\therefore \sinh a\sinh b=\cot\theta\cot\tau=2\cos^2\frac{\phi}{2}-1=\cos\phi$

See Fig 17.

Proof for (2.5.11) and (2.5.12).

$$\sin\phi=\frac{\sin\gamma}{\sinh c}\sinh f=\frac{\sinh f}{\sinh c}\cos\beta=\frac{\sinh a\cosh b}{\sinh c}$$

The last equality holds since $\cos\beta=\cosh b\sin\alpha=\cosh b\dfrac{\sinh a}{\sinh f}$ from the identities for right triangle.

From $\sin^2\phi+\cos^2\phi=1$,

$$\begin{aligned}1 &= \frac{\sinh^2 a\cosh^2 b}{\sinh^2 c}+\sinh^2 a\sinh^2 b\\ &= \sinh^2 a(\frac{\cosh^2 b}{\sinh^2 c}+\cosh^2 b)-\sinh^2 a\\ &= \sinh^2 a\cosh^2 b(\frac{1+\sinh^2 c}{\sinh^2 c})-\sinh^2 a\\ &= \frac{\sinh^2 a\cosh^2 b\cosh^2 c}{\sinh^2 c}-\sinh^2 a\end{aligned}$$

Hence $\cosh^2 a=\dfrac{\sinh^2 a\cosh^2 b\cosh^2 c}{\sinh^2 c}$. and it follows that $\tanh c=\tanh a\cdot\cosh b$ □

Saccheri Quadrilateral

Fig 19.

If $\phi = 0$, then $\sinh a \sinh b = 1$

Remark.

Fig 20.

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$\sin\phi = \dfrac{\cos(-ia)}{\cosh c}$ from (2.5.11). We observe that these rules also follows from the corresponding rules of a right triangle with Formal Substitution as in the picture (using "extended model").