1 A brief introduction to Volume conjecture

2 Linear Fractional Transformation and 2-dimensional hyperbolic geometry

3 Inversive geometry and hyperbolic geometry

3.1 Inversion(or reflection) and Möbius transformation

3.1.1 Definition

Definition 3.1.1 (sphere)**.** We define a sphere in $\widehat{\mathbb{R}^n}$ by

$$S(a,r) := \{x \in \mathbb{R}^n \,|\, |x-a| = r\}$$
$$P(a,t) := \{x \in \mathbb{R}^n \,|\, a \cdot x = t,\ |a| = 1\} \ : \text{sphere at } \infty.$$

Unifying equation of sphere is given by

$$\begin{cases} |x|^2 - 2a \cdot x + |a|^2 - r^2 = 0 \\ -2a \cdot x + 2t = 0 \end{cases} \implies a_0|x|^2 - 2a \cdot x + a_{n+1} = 0,$$

where $a = (a_1, \cdots, a_n)$.

Definition 3.1.2 (inversion)**.** $\sigma := J_{s(a,r)}$ = inversion w.r.t. $S(a,r)$ is given by

$$y = \sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a),$$

$\sigma := J_{P(a,r)}$ = inversion w.r.t. $P(a,r)$ is given by

$$y = \sigma(x) = x + 2(t - a \cdot x)a.$$

Definition 3.1.3 (Möbius transformation)**.** A Möbius transformation of $\widehat{\mathbb{R}^n}$ is a finite composition of inversions.

$$\begin{aligned} e.g. \quad & \text{translation} : x \mapsto x + a \\ & \text{homothety} : x \mapsto \lambda x \\ & \text{orthogonal transformation} : x \mapsto Ax,\ A \in O(n) \end{aligned}$$

$\implies$ similarity : $x \mapsto \lambda Ax + a$ is a Möbius transformation (fixing ∞)

3.2 Möbius transformations as conformal maps

Theorem 3.2.1. Möbius transformation is a conformal map.

Proof. It suffices to show for

$$\sigma = J_{S(0,r)} = \sigma_r, \quad \sigma(x) = r^2 \frac{x}{|x|^2}.$$

Now observe

$$\frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|^2}\right) = \frac{\delta_{ij}|x|^2 - 2x_i x_j}{|x|^4} \quad \Longrightarrow \quad d\sigma(x) = \frac{r^2}{|x|^2}(I - 2A)$$

and

$$X := \frac{x}{|x|} = () \, , acolumnvector \quad \Longrightarrow \quad A = \frac{1}{|x|^2} x_i x_j = X \cdot X^t.$$

Hence we have $(I-2A)^t(I-2A) = I$, so that $I - 2A \in O(n)$. □

Fact (Liuville's Thm)**.** $n \geq 3$. $f : U \subseteq \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ conformal $\Rightarrow f \in M(\widehat{\mathbb{R}^n})$.

Note that the equivalence also holds for $n = 2$ for automorphism case :

$$Conf^+(\widehat{\mathbb{C}}) = Aut(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C})$$

3.3 Möbius transformation as a cross-ratio preserving maps

Definition 3.3.1 (cross ratio)**.**

$$[u, v, x, y] := \frac{|u-x||v-y|}{|u-v||x-y|} \quad \left(\Rightarrow [\infty, v, x, y] = \frac{|v-y|}{|x-y|}\right)$$

Theorem 3.3.1.

i) $\phi : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ is a Möbius transformation $\Longleftrightarrow \phi$ preserves cross ratio

ii) $\phi \in M(\widehat{\mathbb{R}^n})$, $\phi(\infty) = \infty \Longleftrightarrow \phi$ is a similarity

iii) $\phi \in M(\widehat{\mathbb{R}^n})$, $\phi(\infty) \neq \infty \Longleftrightarrow \phi : x \mapsto \lambda A \sigma_1(x - b) + a$ for $b = \phi^{-1}(\infty)$

Proof.

i) $(\Rightarrow)$: $\phi = J_{S(a,r)} \Rightarrow |\phi(x) - \phi(y)| = \frac{r^2|x-y|}{|x-a||y-a|} \Rightarrow [\phi(u), \phi(v), \phi(x), \phi(y)] = [u, v, x, y]$

$(\Leftarrow)$: We may assume $\phi(\infty) = \infty$ by composing with a Möbius transformation. Then

$$[\phi(u), \infty, \phi(x), \phi(y)] = [u, \infty, x, y] \Rightarrow \frac{|\phi(a) - \phi(x)|}{|\phi(x) - \phi(y)|} = \frac{|u-x|}{|x-y|}$$

$$[\phi(u), \phi(v), x, \infty] = [u, v, x, \infty] \Rightarrow \frac{|\phi(u) - \phi(x)|}{|\phi(u) - \phi(v)|} = \frac{|u-x|}{|u-v|}$$

$$\Rightarrow \frac{|\phi(u) - \phi(v)|}{|u-v|} = \frac{|\phi(u) - \phi(x)|}{|u-x|} = \frac{|\phi(x) - \phi(y)|}{|x-y|} = \lambda : \text{constant}$$

$$\Rightarrow |\phi(x) - \phi(y)| = \lambda|x-y| \Rightarrow \frac{\phi}{\lambda} \text{ is an isometry and } \phi \text{ is a similarity.}$$

ii) Follows from the proof of i).

iii) Let $h(x) = \sigma_1(x - b)$. Then $\phi h^{-1}(\infty) = \infty$.

$\square$

$\therefore$ Möbius $\Leftrightarrow$ conformal $\Leftrightarrow$ cross-ratio preserving.

Remark 3.3.1. For $n = 2$, identify $\mathbb{R}^2 = \mathbb{C}$. Then ϕ is an orientation-preserving similarity $\Leftrightarrow \phi(z) = az + b$, $a, b \in \mathbb{C}$ and also note that $J_{S(0,1)} : z \mapsto 1/\bar{z}$.

3.4 Möbius transformation as a sphere preserving map

Proposition 3.4.1.

i) $\phi \in M(\widehat{\mathbb{R}^n}) \Rightarrow \phi$ is a sphere preserving map

ii) $\forall$ sphere S, S' $\exists \phi \in M(\widehat{\mathbb{R}^n})$ s.t. $S' = \phi(S)$.

Proof. i) If ϕ is a similarity, the first assertion is clear. It suffices to show for $\phi = \sigma_1 : x \mapsto x/|x|^2$. Observe that the substitution $\sigma_1(x) = x/|x|^2$ in the equation of sphere yields

$$a_0|x|^2 - 2a \cdot x + a_{n+1} = 0 \Rightarrow a_0 - 2a \cdot x + a_{n+1}|x|^2 = 0.$$

For the second assertion, it suffices to show $\exists \phi : S(0,1) \to \widehat{\mathbb{R}^{n-1}} \subset \widehat{\mathbb{R}^n}$. Let $\phi = J_{S(e_n,\sqrt{2})}$, which is in fact is a stereographic projection. $\square$

Remark 3.4.1. Stereo graphic projection is a conformal map.

Remark 3.4.2. $\eta = J_{\widehat{\mathbb{R}^n}} \cdot J_{S(e_n,\sqrt{2})} : \mathbb{B}^n \to \mathbb{H}^n$ (Cayley transformation)

Proposition 3.4.2. $\phi \in M(\widehat{\mathbb{R}^n})$, $\phi|_{U \subset S} = id$ for an open subset U of a sphere $S \Rightarrow \phi = id$ or J_S.

Proof. According to the above proposition, we may assume that $S = \widehat{\mathbb{R}^{n-1}}$ and $\phi|_U = id_U$ for some open subset $U \subset S$. Then since ϕ is sphere and cross-ratio preserving, $\phi(\infty) = \infty$. Hence ϕ is a similarity. $\square$

Proposition 3.4.3. $\phi \in M(\widehat{\mathbb{R}^n}) \Rightarrow \phi$ preserves inversion. *i.e.*, $\phi \circ J_S \circ \phi^{-1} = J_{\phi(S)}$.

Proof. $\phi \circ J_S \circ \phi^{-1}|_{\phi(S)} = id \Rightarrow \phi \circ J_S \circ \phi^{-1} = J_{\phi(S)}$ by the previous proposition. $\square$

Fact. Conversely, sphere preserving map is a Möbius transformation for $n \geq 2$.