Local flow theorem.

In the proof of Fundamental theorem, $K = Lipschitz\ constant\ of\ f$,
$M = upper\ bound\ for\ f\ on\ \overline{B_b}(x_0)$. $\Rightarrow \exists!$ integral curve α_{x_0} defined at least on $J = [-a, a]$ if $a < min\{\frac{b}{M}\frac{1}{K}\}$, and $\alpha_{x_0}(J) \subset \overline{B}$.
따라서 $\overline{B_{\frac{b}{2}}}(x_0)$안의 모든 점 x에 대해 $\overline{B_{\frac{b}{2}}}(x) \subset \overline{B_b}(x_0)$ 가 성립한다. 또한 $\epsilon < min\{\frac{b/2}{M}, \frac{1}{K}\}$ 에 대해 $\exists!$ integral curve α_x defined at least on $[-\epsilon, \epsilon]$, and $\alpha_x[-\epsilon, \epsilon] \subset \overline{B_{\frac{b}{2}}}(x) \subset \overline{B_b}(x_0)$.
Now define $\alpha : [-\epsilon, \epsilon] \times \overline{B_{\frac{b}{2}}}(x_0) \to \overline{B_b}(x_0) \subset W$ by $\alpha(t, x) = \alpha_x(t)$.
α is called a **local flow** *of f.*

Claim : α is continuous.

정리 1 *(Continuous dependency on initial conditions)*
$f : W(\subset \mathbb{R}^{n+1}) \to \mathbb{R}^n$, Lipschitz. Let $y(t), z(t)$ be solutions of $x' = f(t, x)$ on $[t_0, t_1]$. Then $|y(t) - z(t)| \leq |y(t_0) - z(t_0)|e^{K(t-t_0)}$, $\forall t \in [t_0, t_1]$.
(K=Lipschitz constant)

위 정리의 증명을 위해 다음 보조정리를 먼저 보이자.

보조정리 2 *(Gronwall's inequality) Let $u : [t_0, t_1] \to \mathbb{R}$ be continuous, non-negative function. Suppose $u(t) \leq C + \int_{t_0}^{t} Ku(s)ds, \forall t \in [t_0, t_1]$ for some constants $C, K \geq 0$. Then $u(t) \leq Ce^{K(t-t_0)}$.*

증명 1.($C > 0$) $v(t) := C + \int_{t_0}^{t} Ku(s)ds$라 놓자. $u(t) \leq v(t)$ 이고
$v'(t) = Ku(t)$, $\frac{d}{dt}\log v(t) = \frac{v'}{v} = K\frac{u}{v} \leq K$ 이므로 양변을 적분하면,
$\log v(t) \leq K(t - t_0) + \log C \Rightarrow v(t) \leq Ce^{K(t-t_0)}$.
$\therefore u(t) \leq Ce^{K(t-t_0)}$.
2. ($C = 0$) Choose $C_i > 0$ such that $C_i \to 0$ and apply 1 to C_i. Then $u(t) \leq C_i e^{K(t-t_0)}$ and take limit.

$\square$

(proof of the theorem 1.)
Let $u(t) = |y(t) - z(t)|$.
$y(t) - z(t) = y(t_0) - z(t_0) + \int_{t_0}^{t} f(s, y(s)) - f(s, z(s))ds$.
$u(t) = |y(t) - z(t)| \leq u(t_0) + \int_{t_0}^{t} |f(s, y(s)) - f(s, z(s))|ds$.
$u(t) \leq u(t_0) + \int_{t_0}^{t} Ku(s)ds$ (Lipschitz condition.)
$u(t) \leq u(t_0)e^{K(t-t_0)}$ by the *Gronwall's inequality.*
$\therefore |y(t) - z(t)| \leq |y(t_0) - z(t_0)|e^{K(t-t_0)}$. $\square$

proof of the claim : α is continuous.

$$|\alpha(t,x)-\alpha(t_1,x_1)| \le |\alpha(t,x)-\alpha(t,x_1)| + |\alpha(t,x_1)-\alpha(t_1,x_1)|$$
$$\le |\alpha_x(t)-\alpha_{x_1}(t)| + |\alpha_{x_1}(t)-\alpha_{x_1}(t_1)|$$

$\alpha(t,x_1) = x_1 + \int_0^t f(\alpha(s,x_1))ds.$
$\alpha(t_1,x_1) = x_1 + \int_0^{t_1} f(\alpha(s,x_1))ds.$
and by theorem 1.

$$\le |\alpha_x(0)-\alpha_{x_1}(0)|e^{K|t|} + |\int_{t_1}^t f(x(s))ds|$$
$$\le |x-x_1|e^{K|t|} + |\int_{t_1}^t f(x(s))ds|$$
$$\le |x-x_1|e^{K\epsilon} + M|t-t_1| \to 0 \text{ as } (t,x)\to(t_1,x_1)$$

$\therefore \alpha$ is continuous. □

Fact. α is $\mathcal{C}^\infty$ if f is $\mathcal{C}^\infty$.

Reference. Hirsch and Smale. Differential equations, Dynamical systems and Linear algebra. Ch. 15.