Lie Bracket.

정리 1 *Let X be a $\mathcal{C}^\infty$ vector field on M with $X(p) \neq 0$. Then $\exists$coordinate chart (U, x) around p such that $X = \frac{\partial}{\partial x_1}$ on U.*

증명 Since this is a local problem, we may assume X is a $\mathcal{C}^\infty$ vector field on a neighborhood of $0 \in \mathbb{R}^n$ and $X(0) = \frac{\partial}{\partial u_1}(0)$.
Define $\varphi : \mathbb{R} \to \mathbb{R}^n$ by $\varphi(a_1, \cdots, a_n) = \alpha(a_1, p)$ where $p = (0, a_2, \cdots, a_n)$ and $\alpha(t, p)$ is the flow of X so that $\varphi_*(\frac{\partial}{\partial u_1}(a)) = X(\varphi(a))$.
In particular $\varphi(\frac{\partial}{\partial u_1}(0)) = X(\varphi(0)) = X(0) = \frac{\partial}{\partial u_1}(0)$.
Note that $\varphi|_{\{0\}\times\mathbb{R}^{n-1}} = id$ and hence $\varphi_*(\frac{\partial}{\partial u_i}(0)) = \frac{\partial}{\partial u_i}(0)$, $i = 1, 2, \cdots n$.
$\therefore\ \varphi_*|_0 = id$ and locally φ^{-1} gives a desired coordinate chart. □

정의 1 *Let X, Y be $\mathcal{C}^\infty$ vector fields on M. The Lie bracket of X and Y, denoted by $[X, Y]$ is a $\mathcal{C}^\infty$ vector field on M defined by*
$[X, Y]_p f = X_p(Yf) - Y_p(Xf)\ \forall p \in \mathcal{C}^\infty(p)$.

사실 $[X, Y]$가 잘 정의됨을 보이기 위해서는 다음내용을 체크해야 한다.

Check. (1) $[X, Y]_p$ is a linear derivation:

$$\begin{aligned}
[X,Y]_p(f+cg) &= X_p(Y(f+cg)) - Y_p(X(f+cg)) \\
&= X_p(Yf + cYg) - Y_p(Xf + cXg) \\
&= X_p(Yf) + cX_p(Yg) - Y_p(Xf) - cY_p(Xg) \\
&= [X,Y]_p(f) + [X,Y]_p(g).
\end{aligned}$$

$$\begin{aligned}
[X,Y]_p(fg) &= X_p(Y(fg)) - Y_p(X(fg)) \\
&= X_p((Yf)g + f(Yg)) - Y_p((Xf)g + f(Xg)) \\
&= g(p)X_p(Yf) - g(p)Y_p(Xf) + f(p)X_p(Yg) - f(p)Y_p(Xg) \\
&= g(p)[X,Y]_p(f) + f(p)[X,Y]_p(g)
\end{aligned}$$

(2) $[X, Y]$ is a $\mathcal{C}^\infty$ vector field :
각 f에 대해 $[X, Y](f) = X(Yf) - Y(Xf)$ 에서 f, X, Y 가 모두 $\mathcal{C}^\infty$이므로 $[X, Y](f)$ 역시 $\mathcal{C}^\infty$이다. 모든 $\mathcal{C}^\infty f$에 대해 $\mathcal{C}^\infty$ 이므로 $[X, Y]$는 $\mathcal{C}^\infty$ vector field이다.

명제 2 *(1)* $[X, Y + cZ] = [X, Y] + c[X, Z], c \in \mathbb{R}$.
(2) $[X, Y] = -[Y, X]$.
(3) $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$.

(4) (*Jacobi identity*) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$.

(증명) "$[X,Y]=XY-YX$"로부터 직접 계산에 의해 네가지 모두 쉽게 증명할 수 있다.

Note. In $\mathbb{R}^n$, $[\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}]=0$.