1. Tensor product.

Let $V_1, V_2, \cdots, V_k$ be vector spaces over $\mathbb{R}$ or $\mathbb{C}$.
$V_1 \otimes \cdots \otimes V_k := M/N$, where $M =$ vector space generated by the set $V_1 \times \cdots \times V_k$, $N =$ subspace generated by elements of type $(x_1, \cdot\cdot, x_i + x_i', \cdot\cdot, x_k) - (x_1, \cdot\cdot, x_i, \cdot\cdot, x_k) - (x_1, \cdot\cdot, x_i', \cdot\cdot, x_k)$ and $(x_1, \cdot\cdot, ax_i, \cdot\cdot, x_k) - a(x_1, \cdot\cdot, x_i, \cdot\cdot, x_k)$.
이 때 $(x_1, \cdots, x_n)$의 equivalence class를 $[(x_1, \cdot\cdot, x_k)] = x_1 \otimes \cdots, \otimes x_k$ 라고 쓴다.

<Universal property>
아래 그림에서와 같이 multilinear f가 주어지면 diagram을 commute하는 linear transformation $\overline{f}$가 유일하게 존재한다.

$$\begin{array}{ccc} V_1 \times \cdots \times V_k & \xrightarrow{\pi} & \bigotimes V_i \\ \textit{multilinear } f \searrow & & \swarrow \exists! \overline{f} \\ & W & \end{array} \qquad \overline{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1, \cdots, v_k).$$

Let V be a vector space with basis $\{e_1, \cdots, e_n\}$.
$T^r(V) := \bigotimes_{i=1}^r V$, $T^0(V) := \mathbb{R}$ and $\{e_{i_1} \otimes \cdot\cdot \otimes e_{i_r}\}$ becomes a basis of T^r.

Let V, W be vector spaces with basis $\{e_1, \cdot\cdot, e_n\}, \{b_1, \cdot\cdot, b_m\}$. Then
$V \otimes W$=vector space with basis $\{e_i \otimes b_j \,|\, i = 1, \cdot\cdot, n, j = 1, \cdot\cdot, m\}$
(증명은 $V \bigotimes (\bigoplus_i W_i) \cong \bigoplus_i (V \bigotimes W_i)$, $\mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$ 을 이용해서 보일 수 있다.)

이제 graded algebra구조를 갖는 tensor algebra $T(V)$를 정의하자.
$T(V) = \bigoplus_{r=0}^{\infty} T^r(V)$: tensor algebra with obvious associative product.
$T(V) \cong P$: noncommutative polynomial algebra generated by $\{e_1, \cdot\cdot, e_n\}$
T is a functor : $\{\textit{vector space}\} \to \{\textit{graded algebra}\}$

숙제 14-1. Show $v \otimes \cdot\cdot \otimes v_k = 0 \Leftrightarrow v_1 = 0$, $\cdots$, or $v_k = 0$.

2. Alternating algebra.

$\Lambda^r(V) = T^r(V)/\mathbf{a}_r$, $\mathbf{a}_r =$ subspace generated by elements of type $x_1 \otimes \cdots \otimes x_r$ where $x_i = x_j$ for some $i \neq j$. Denote the equivalence class of $x_1 \otimes \cdots \otimes x_r$ by $[x_1 \otimes \cdots \otimes x_r] = x_1 \wedge \cdots \wedge x_r$. Then $x_1 \wedge \cdot\cdot \wedge x_i \wedge \cdot\cdot \wedge x_j \wedge \cdot\cdot \wedge x_r = -x_1 \wedge \cdot\cdot \wedge x_j \wedge \cdot\cdot x_i \wedge \cdot\cdot x_r$.

< Universal property >
아래 그림에서와 같이 multilinear alternating f가 주어지면 diagram을 commute하는 $\overline{f}$가 유일하게 존재한다.

$$\begin{array}{ccc} V\times\cdots\times V & \xrightarrow{\pi} & \Lambda^r(V) \\ f\searrow & & \swarrow \exists!\overline{f} \\ & W & \end{array} \qquad \overline{f}(v_1\wedge\cdots\wedge v_k)=f(v_1,\cdot\cdot,v_k).$$

$\Lambda(V)=\bigoplus_{r=0}^{\infty}\Lambda^r(V)$: Exterior(Alternating) algebra with obvious associative product $\wedge$.
Λ is a functor : $\{vector\ spaces\}\to\{graded\ algebra\}$
$\Lambda^r(V)$ has a basis $\{e_{i_1}\wedge\cdots\wedge e_{i_r}, i_1<\cdot\cdot<i_r\}$, and $dim\Lambda^r(V)={}_nC_r$.

숙제 14-2. Show $v_1\wedge\cdots\wedge v_k\neq 0\Leftrightarrow\{v_1,\cdots,v_k\}$ is linearly independent .

3. Symmetric algebra.

$S^r(V)=T^r(V)/\mathbf{b}_r$, $\mathbf{b}_r$ = subspace generated by elements of type
$x_1\otimes\cdots\otimes x_r - x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}$, $\sigma\in Perm(r)=S_r$.

< Universal property >
아래 그림에서와 같이 multilinear symmetric f가 주어지면 diagram을 commute하는 linear $\overline{f}$가 유일하게 존재한다.

$$\begin{array}{ccc} V\times\cdots\times V & \xrightarrow{\pi} & S^r(V) \\ f\searrow & & \swarrow \exists!\overline{f} \\ & W & \end{array} \qquad \overline{f}(v_1\cdots v_k)=f(v_1,\cdot\cdot,v_k).$$

$S^r(V)$ has a basis $\{e_{i_1}\cdots e_{i_r}, i_1\leq\cdot\cdot\leq i_r\}$, $dimS^r(V)={}_nH_r$.
$S(V)=\bigoplus_{r=0}^{\infty}S^r(V)$: Symmetric algebra with obvious product $\cdot$.
$\cong$ commutative polynomial algebra in $n-$variables, $\mathbb{R}[x_1,\cdot\cdot,x_n]$.
S is a functor : $\{vector\ spaces\}\to\{graded\ algebra\}$

숙제 14-3. Show $v\cdots v_k=0\Leftrightarrow v_1=0,\cdots,$ or $v_k=0$.

4. T,Λ,S as functors.

For $f:V\to V'$, $g:W\to W'$, linear, the tensor product of f and g ,
$f\otimes g:V\otimes W\to V'\otimes W'$ is induced and given by $(f\otimes g)(v\otimes w)=f(v)\otimes g(w)$.
Let $f=(f_{ij})$ with respect to basis $e=\{e_1,\cdot\cdot,e_m\}$ and $e'=\{e'_1,\cdot\cdot,e'_n\}$,
$g=(g_{ij})$ with respect to basis $b=\{b_1,\cdot\cdot,b_p\}$ and $b'=\{b'_1,\cdot\cdot,b'_q\}$. Then
the matrix of $f\otimes g$ with respect to $\{e_i\otimes b_j\}$ and $\{e'_i\otimes b'_j\}$ is given as the following :

$e_i \otimes b_j \mapsto f(e_i) \otimes g(b_j) = (\sum_k f_{ki} e'_k) \otimes (\sum_l g_{lj} b'_l) = \sum_{k,l} f_{ki} g_{lj} e'_k \otimes b'_l$

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A linear $f : V \to W$ induces $T^r(f), \Lambda^r(f), S^r(f)$ such that
$T^r(f) : T^r(V) \to T^r(V),\ v_1 \otimes \cdots \otimes v_r \mapsto f(v_1) \otimes \cdots \otimes f(v_r)$
$\Lambda^r(f) : \Lambda^r(V) \to \Lambda^r(V),\ v_1 \wedge \cdots \wedge v_r \mapsto f(v_1) \wedge \cdots \wedge f(v_r)$
$S^r(f) : S^r(V) \to S^r(V),\ v_1 \cdots v_r \mapsto f(v_1) \cdots f(v_r)$
Let $e = \{e_1, \cdot\cdot, e_n\}$,$b = \{b_1, \cdot\cdot, b_m\}$ be basis for V, W and $f = (f_{ij})$ with respect to e and b. i.e., $f(e_i) = \sum_i f_{ji} b_j$. Then

$$T^r(f) : e_I = e_{i_1} \otimes \cdot\cdot \otimes e_{i_r} \mapsto f(e_{i_1}) \otimes \cdot\cdot \otimes f(e_{i_r}) = \sum_{j_1} f_{j_1 i_1} b_{j_1} \otimes \cdot\cdot \otimes \sum_{j_r} f_{j_r i_r} b_{j_r}$$
$$= \sum_{j_1, \cdot\cdot, j_r} (f_{j_1 i_1} \cdot\cdot f_{j_r i_r}) \underbrace{b_{j_1} \otimes \cdot\cdot \otimes b_{j_r}}_{b_J}$$

$$\Lambda^r(f) : e_I = e_{i_1} \wedge \cdot\cdot \wedge e_{i_r} \mapsto f(e_{i_1}) \wedge \cdot\cdot \wedge f(e_{i_r}) = \sum_{j_1, \cdot\cdot, j_r} (f_{j_1 i_1} \cdot\cdot f_{j_r i_r})(b_{j_1} \wedge \cdot\cdot \wedge b_{j_r})$$
$$I = (i_1 < \cdot\cdot < i_r)\ ,\ J = (j_1 < \cdot\cdot < j_r)) \qquad = \sum_J \underbrace{(\sum_{\sigma \in S_r} sgn(\sigma) f_{\sigma(j_1) i_1} \cdot\cdot f_{\sigma(j_r) i_r})}_{det\, f_{IJ}} b_J$$

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