Effect of Mappings

$\varphi : M \to N,\ \mathcal{C}^\infty \ \Rightarrow\ \varphi_* : TM \to TN \ \ bundle\ map.$
$And\ \ \varphi_*|_p \ :\ T_pM \to T_{\varphi(p)}N,\ \ linear\ \Rightarrow (\varphi_*|_p)^* \ :\ T^*_{\varphi(p)}N \to T^*_pM$, dual map.
$This\ induces\ a\ linear\ map\ \ \varphi^*_p \ :=\ \bigwedge^r(\varphi_*|^*_p)\ :\ \bigwedge^r(T^*_{\varphi(p)}N) \to \bigwedge^r(T^*_pM)$

$For\ \alpha \in \mathcal{E}^1(N),\ \varphi^*\alpha \in \mathcal{E}^1(M)\ is\ defined\ \ as\ \varphi^*\alpha|_p = \varphi^*_p(\alpha|_{\varphi(p)}).$
Note that $\varphi^*\alpha(X_p) \ =\ \alpha(\varphi_* X_p)$

Similarly, for $\alpha \in \mathcal{E}^r(N),\ \varphi^*\alpha \in \mathcal{E}^r(M)$ and

$$\begin{aligned}
\varphi^*\alpha(X_1,\cdots,X_r)(p) &= \varphi^*\alpha(X_1(p),\cdots,X_r(p)) \\
&= \alpha((\varphi_*X_1)(\varphi(p)),\cdots,(\varphi_*X_r)(\varphi(p))) \\
&= \alpha(\varphi_*X_1,\cdots,\varphi_*X_r)(\varphi(p))
\end{aligned}$$

$$\Rightarrow\ \varphi^*\ :\ \mathcal{E}(N)\ \to\ \mathcal{E}(M) = \bigoplus_{r=0}^{\infty}\mathcal{E}^r(M).$$

이 경우 differential d에 관하여 $\mathcal{E}(M)$은 "differential graded algebra"가 된다.

명제 1 *Let* $\varphi\ :\ M\ \to\ N\ ,\ \ \mathcal{C}^\infty$. *Then* $\varphi^*\ :\ \mathcal{E}(N)\ \to\ \mathcal{E}(N)$ *is a differential graded algebra homomorphism, i.e.,*

1. $\varphi^*(\alpha+\beta)\ =\ \varphi^*\alpha\ +\ \varphi^*\beta$
2. $\varphi^*(\alpha\wedge\beta)\ =\ \varphi^*\alpha\ \wedge\ \varphi^*\beta$
3. $\varphi^* d\ =\ d\varphi^*$

증명 1 and 2 are clear.
$(\because)\ f\ :\ V\ \to\ W\ \Rightarrow\ f^*\ :\ W^*\ \to\ V^*$
$\Rightarrow\ \bigwedge(f^*)\ :\ \bigwedge(W^*)\ \to\ \bigwedge(V^*)$ is an algebra homomorphism.
And note that $+, \wedge$ are pointwise operations.

Show $\varphi^*\alpha$ is $\mathcal{C}^\infty$ if $\alpha\ \in \mathcal{E}^r(N)$:
1st $\varphi^*(df)$ case :

$$\begin{aligned}
\varphi^*(df)(X)(p) &= \varphi^*(df)(X_p)\ =\ df(\varphi_*X_p) \\
&= (\varphi_*X_p)f = X_p(f\circ\varphi) = X(f\circ\varphi)(p) \quad : \mathcal{C}^\infty - function\ \ of\ \ p. \\
\text{Moreover,}\quad &= d(f\circ\varphi)(X)(p)
\end{aligned}$$

$\therefore\ \varphi^*(df)\ =\ d(f\circ\varphi)\ =\ d(\varphi^* f)$ and $\varphi^* d\ =\ d\varphi^*\ on\ \mathcal{F}$

$2nd\ \ \alpha \in \mathcal{E}^r(N)$ case:

Locally $\varphi \ : V \to \ (U, x)$

$\qquad\qquad p \ \mapsto \ \varphi(p)$

Let $\alpha = \sum f_I dx_I$. Then $\varphi^*\alpha = \sum \varphi^*(f_I)\varphi^*(dx_{i_1}) \wedge \cdots \wedge \varphi^*(dx_{i_r}) \quad : \mathcal{C}^\infty$.

Now we can prove 3 as follows.

$d(\varphi^*\alpha) = d(\sum(\varphi^* f_I) d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_r}))$

$= \sum d(\varphi^* f_I) \wedge d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_r})$

$= \varphi^*(\sum df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = \varphi^* d\alpha$ $\square$