Orientation.

정의 1 V : n-dimensional vector space.
Two ordered basis $b = (b_1, \cdots, b_n)$ and $e = (e_1, \cdots, e_n)$ are equivalently oriented ($b \sim e$) if $detA > 0$, where $A = (a_{ij}), e_j = \sum_{i=1}^{n} a_{ij} b_i$.
Or equivalently, if $e_1 \wedge \cdots \wedge e_n = \alpha b_1 \wedge \cdots \wedge b_n$ for some α >0.

Note $\sim$ is an equivalence relation and there exists two equivalent classes in the set of all ordered basis. A choice of one of the two equivalent classes is called an orientation.

Note $f : (V^n, \nu) \overset{\simeq}{\rightarrow} (W^n, \mu)$ is orientation preserving(or orientation reserving, resp.) , if $(v_1, \cdots, v_n) \in \nu \Rightarrow (fv_1, \cdots, fv_n) \in \mu(\notin, resp.)$.(Well-definedness is clear)

정의 2 M : a $\mathcal{C}^\infty$-manifold (with boundary)
$\mu = \{\mu_p | p \in M, \mu_p$ is an orientation of $T_pM\}$
μ is a (smooth) orientation of M if $\forall p \in M$, $\exists$ a coordinate chart (U, φ) of p s.t. $(\varphi_*)_q$ is orientation-preserving $\forall q \in U$ (w.r.t. the standard orientation of $\mathbb{R}^n$)

정의 3 A connected M is orientable if M admits a (smooth) orientation.

Note M is orientable iff each component of M is orientable.

정리 1 *M : connected $\mathcal{C}^\infty$-manifold of dimension n. TFAE.*
(1) M is orientable.
(2) $\exists$ a collection of coordinate charts Φ=$\{(U, x)\}$ on M s.t. M=$\bigcup_{U \in \Phi} U$ and det($\frac{\partial x}{\partial y}$)> 0 on $U \cap V$ for (U, x),$(V, y) \in \Phi$.
(3) $\bigwedge^n (T^*M) \backslash 0 - section$ has exactly two components.
(4) $\exists$ non-vanishing n-form on M.(globally)
(5) $\forall$ loop in M, one can cover it by a finite positively connected chain of coordinate charts.

증명
(1)⇒(4)⇒(3)⇒(2)⇒(1)

(1)⇒(4):
We can cover M by a coordinate charts (U_α, x^α) where x^α_* is orientation-preserving.
Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$.
Let $\omega(p) := \sum_\alpha \varphi_\alpha(p) dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$. Then ω is clearly non-vanishing.

(4)⇒(3):
Let ω be a non-vanishing n-form.
Define $\bigwedge^+ = \bigcup_{p\in M}\{c\omega_p | c > 0\}$ and $\bigwedge^- = \bigcup_{p\in M}\{c\omega_p | c < 0\}$.
Then $\bigwedge^+$ is connected, because M is connected and hence the graph of ω is connected.
Note also that $\bigwedge^+$ and $\bigwedge^-$ are disjoint open sets.

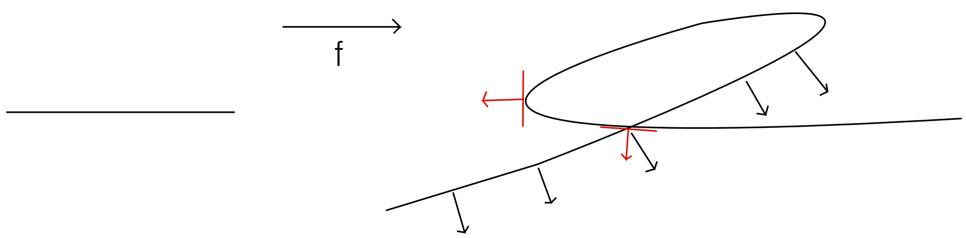
(3)⇒(2):
Suppose that $\bigwedge^n \backslash 0$ has two components C_1 and C_2.
Let $\Phi = \{(U,x) | dx^1 \wedge \cdots \wedge dx^n \in C_1\}$. Then clearly Φ covers M.
$(\because (U,x) = (x_1, \cdots, x_n) \notin \Phi \Rightarrow (U,\tilde{x}) = (-x_1, x_2, \cdots, x_n) \in \Phi)$
And for $(U,x),(V,y) \in \Phi$, $dx^1 \wedge \cdots \wedge dx^n = det(\frac{\partial x}{\partial y}) dy^1 \wedge \cdots \wedge dy^n \Rightarrow det(\frac{\partial x}{\partial y}) > 0$.

(2)⇒(1):
Define $\mu_p = \{(\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p))\}$ for $(U,x) \in \Phi$. This is well-defined by (2).

(2)⇔(5): **exercise.**

□

숙제 21. $f : M^n \to \mathbb{R}^{n+1}$ is an immersion.
M is orientable ⇔ ∃ smooth non-vanishing normal vector field along (M,f)



Induced orientation on ∂M

정의 4 Let M be orientable with an orientation μ.
The induced orientation $\partial\mu$ on ∂M is defined as follows:
$\forall p \in \partial M$, let (U, φ) be a coordinate chart and let $out = \varphi_*^{-1}(\frac{\partial}{\partial u_1})$.
Then $(X_2, \cdots, X_n) \in \partial\mu_p \Leftrightarrow (out, X_2, \cdots, X_n) \in \mu_p$