4. Integration of a function on M.

이 장에서는 manifold위에서의 form의 적분뿐만이 아니라 함수의 적분을 다루기로 한다.

1. Volume element α and $\int_M f\alpha$

Let α be a function given on M^n such that $\forall p \in M$,
$\alpha(p) = |\omega_p|$ for some $\omega_p \in \Lambda^n(T_p^*M)$
where $|\omega_p|(X_1, \cdots, X_n) := |\omega_p(X_1, \cdots X_n)|,\ X_i \in T_pM$.

Such a function α is called a *volume element* - a way of measuring n-dimensional volume(without sign).
α is *smooth* if it can be written locally as $\alpha = g|dx_1 \wedge \cdots \wedge dx_n|$ for $g \in \mathcal{C}^\infty,\ g \geq 0$.
Let α be a smooth volume element on M.
M 이 $\mathbb{R}^n$일 때 대표적인 volume element로는 $|du_1 \wedge \cdots \wedge du_n|$이 있다.
($\mathbb{R}^n$에서는 $\int_{\mathbb{R}^n} f|du_1 \wedge \cdots \wedge du_n| := \int_{\mathbb{R}^n} f du_1 \cdots du_n$ 으로 정의된다.)

$<$ Change of variable formula. $>$
Domain $D \subset \mathbb{R}^n$ 상에서 diffeomorphism $\varphi : D \to \varphi(D)$에 대해
$\int_D (f \circ \varphi)|det\varphi_*|dt_1 \cdots dt_n = \int_{\varphi(D)} f du_1 \cdots du_n$ (변수변환공식)

$$\Rightarrow \qquad \int_D \varphi^*(f|du_1 \wedge \cdots \wedge du_n|) = \int_{\varphi(D)} f|du_1 \wedge \cdots \wedge du_n|$$

(증명) $\varphi^*|du_1 \wedge \cdots du_n|(X_1, \cdots X_n) = |du_1 \wedge \cdots \wedge du_n|(\varphi_* X_1, \cdots, \varphi_* X_n)$
$= |du_1 \wedge \cdots \wedge du_n(\varphi_* X_1, \cdots, \varphi_* X_n)|$
$= |\varphi^* du_1 \wedge \cdots \wedge \varphi^* du_n(X_1, \cdots, X_n)|$
$= |d\varphi^* u_1 \wedge \cdots \wedge d\varphi^* u_n(X_1, \cdots, X_n)|$
$= |d(u_1 \circ \varphi) \wedge \cdots \wedge d(u_n \circ \varphi)(X_1, \cdots, X_n)|$
$= |det\varphi_* \cdot dt_1 \wedge \cdots \wedge dt_n(X_1, \cdots, X_n)|$
$= |det\varphi_*||dt_1 \wedge \cdots \wedge dt_n|(X_1, \cdots, X_n)$
$\therefore \varphi^*|du_1 \wedge \cdots \wedge du_n| = |det\varphi_*||dt_1 \wedge \cdots \wedge dt_n|$ $\square$

For f with $supp\, f \subset (U, x)$, let $\varphi = x^{-1}$, $\widetilde{U} = x(U)$. Then

$$\int_M f\alpha := \int_{\widetilde{U}} \varphi^*(f\alpha) \text{ is well defined as before.}$$

Note that if M is orientable and ω is a positive n-form, then $\int_M f\omega = \int_M f|\omega|$.

In general, for f with compact support on (M, α), choose a finite cover $\{U_1, \cdots, U_k\}$ of $supp\, f$ and a partition of unity subordinate to this cover and define $\displaystyle\int_M f\alpha := \sum_{i=1}^k \int_M \rho_i f\alpha$.

This is well-defined as before independent of choice of a cover and a partition of unity.

2. Classical examples.

(1) $n = 1$:
$\alpha = ds$ =length element는 $\alpha = |"ds"|$ 로 정의된다. (여기서 "ds" 는 Riemannian oriented volume form이다.) i.e., $ds(v_p) = |v_p|$. 그러면 a parametrization $\varphi : [a, b] \to M \subset \mathbb{R}^n$에 대해

$$\int_M f ds = \int_a^b f(\varphi(t)) \frac{ds}{dt} dt = \int_{[a,b]} \varphi^*(f ds) \text{ 가 된다.}$$

실제로 $\varphi^*(f ds) = (\varphi^* f)(\varphi^* ds)$이고 $\varphi^* ds$를 구하기 위해 $\varphi^* ds = g|dt|$라 두자. 그러면

$\varphi^* ds(\frac{d}{dt}) = g|dt|(\frac{d}{dt}) = g|dt(\frac{d}{dt})| = g|1| = g.$

$$\begin{aligned}\therefore \varphi^* ds &= \varphi^* ds(\tfrac{d}{dt})|dt| \\ &= ds(\varphi_* \tfrac{d}{dt})|dt| \\ &= ds(\tfrac{d\varphi}{dt})|dt| \\ &= |\varphi'(t)||dt|\end{aligned}$$

$\therefore \varphi^*(f ds) = (f \circ \varphi)|\varphi'(t)||dt| = (f \circ \varphi)\frac{ds}{dt}|dt|.$

(2) $n = 2$:
$\alpha = d\sigma$ =area element$=|"d\sigma"|$,
$d\sigma(X_p, Y_p) = |"d\sigma"(X_p, Y_p)| = X_p$와 Y_p가 이루는 평행사변형의 넓이가 된다. 실제로

Choose $N = (a_1, a_2, a_3)$, a unit normal on M such that $[(N, X, Y)] = [(X, Y, N)]$ giving the standard orientation of $\mathbb{R}^3$. Then

$$\begin{aligned}&\text{area of parallelogram } (X, Y) \\ &= \text{volume of parallelepiped } (N, X, Y) \qquad (\because |N| = 1) \\ &= |det(N, X, Y)| \\ &= |dx \wedge dy \wedge dz(N, X, Y)| \\ &= |i_N \mu(X, Y)| \qquad (\text{let } dx \wedge dy \wedge dz = \mu) \\ &= |i_N \mu|(X, Y) \\ &= |i_N(dx \wedge dy \wedge dz)|(X, Y) \\ &= |i_N(dx) \wedge dy \wedge dz - dx \wedge i_N(dy) \wedge dz + dx \wedge dy \wedge i_N(dz)|(X, Y)\end{aligned}$$

Since $a_1 = dx(N), a_2 = dy(N), a_3 = dz(N)$,

$$\begin{aligned}&= |\underbrace{a_1 dy \wedge dz - a_2 dx \wedge dz + a_3 dx \wedge dy}_{"d\sigma"}|(X, Y) \\ &= d\sigma(X, Y)\end{aligned}$$

parametrization이 주어졌을 때 실제 면적분을 계산해 보자.
M에 대한 parametrization 을 $\varphi : D(\subset \mathbb{R}^2) \to M$ 이라 두면
$\int_M f d\sigma = \int_D \varphi^*(f d\sigma) = \int_D (f \circ \varphi)\varphi^*(d\sigma)$ 에서
$\varphi^*(d\sigma)$를 구하기 위해 이를 $f|du \wedge dv|$라 두자. ($\varphi^*(d\sigma)$는 D에서의 measure이므로 이와 같이 둘 수 있다.) 그러면 $\varphi^*(d\sigma)(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = f$ 이고 따라서

$$
\begin{aligned}
\varphi^*(d\sigma) &= \varphi^*(d\sigma)(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial v})|du \wedge dv| \\
&= d\sigma(\varphi_* \tfrac{\partial}{\partial u}, \varphi_* \tfrac{\partial}{\partial v})|du \wedge dv| \\
&= d\sigma(\tfrac{\partial \varphi}{\partial u}, \tfrac{\partial \varphi}{\partial v})|du \wedge dv| \\
&= \text{area of } (\tfrac{\partial \varphi}{\partial u}, \tfrac{\partial \varphi}{\partial v}) \cdot |du \wedge dv| \\
&= |\tfrac{\partial \varphi}{\partial u} \times \tfrac{\partial \varphi}{\partial v}||du \wedge dv|
\end{aligned}
$$

$$
\therefore \int_M f d\sigma = \int_D (f \circ \varphi) | \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} | du\, dv
$$

3. Riemannian volume element.

Let g be a Riemannian metric and let $\{e_1, \cdots, e_n\}$ be an orthonormal basis at $p \in M$ and $\{\epsilon_1, \cdots, \epsilon_n\}$ be its dual basis. The *Rimannian volume element*, dvol at p is given by $|\epsilon_1 \wedge \cdots \wedge \epsilon_n|$. ($\nu = \epsilon_1 \wedge \cdots \wedge \epsilon_n$ is called a *oriented volume form* if $(e_1, \cdots, e_n)$ is positive.)
Locally on (U, x), let $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $G = (g_{ij})$. If we let $\frac{\partial}{\partial x_j} = \sum_{i=1}^n a_{ij} e_i$, then $\epsilon_i = \sum_j a_{ij} dx_j$.
$\therefore \epsilon_1 \wedge \cdots \wedge \epsilon_n = (det\, A) dx_1 \wedge \cdots \wedge dx_n,\ where\ \ A = (a_{ij})$.
한편 $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = g(\sum_k a_{ki} e_k, \sum_l a_{lj} e_l)$
$\quad = \sum_{k,l} a_{ki} a_{lj} \delta_{kl}$
$\quad = \sum_k a_{ki} a_{kj} = \sum_k {}^t a_{ik} a_{kj}$
$\therefore G = {}^t AA$ and $det\, G = (det\, A)^2 (> 0)$.
$\therefore dvol = |\epsilon_1 \wedge \cdots \wedge \epsilon_n| = |det\, A||dx_1 \wedge \cdots \wedge dx_n| = \sqrt{det\, G}|dx_1 \wedge \cdots \wedge dx_n|$

예. $M^{n-1} \subset \mathbb{R}^n$ with induced Riemannian metric 에 대해 먼저 M이 orientable일 경우, positive orthonormal frame $e_2, \cdots, e_n$과 unit normal N을 $(N, e_2, \cdots, e_n)$이 $\mathbb{R}^n$의 positive orthonormal frame이 되도록 잡는다. 그리고 $e_1 = N$이라 두고 $\{e_1, \cdots, e_n\}$의 dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$에 대해 ν를 M의 Riemannian oriented volume form이라 두자. 그러면 $\nu = \epsilon_2 \wedge \cdots \wedge \epsilon_n = i_N(\epsilon_1 \wedge \cdots \wedge \epsilon_n)$ 이고
($\because\ \ i_N(\epsilon_1 \wedge \cdots \wedge \epsilon_n) = i_N(\epsilon_1) \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n - \epsilon_1 \wedge i_N(\epsilon_2) \wedge \cdots \wedge \epsilon_n + \cdots$ 에서 첫항을 제외한 나머지는 모두 0이 된다.) 또한 $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$도 $\mathbb{R}^n$의 orthonormal basis이므로 $\epsilon_1 \wedge \cdots \wedge \epsilon_n = dx_1 \wedge \cdots \wedge dx_n$임을 알 수 있다. 따라서
$\nu = \epsilon_2 \wedge \cdots \wedge \epsilon_n = i_N(\epsilon_1 \wedge \cdots \wedge \epsilon_n) = i_N(dx_1 \wedge \cdots \wedge dx_n)$.

$\therefore\ |\nu| = |\sum(-1)^{i-1}a_i dx_1 \wedge \cdots \wedge \overset{\wedge}{dx_i} \wedge \cdots \wedge dx_n| \quad for\ \ N = (a_1, \cdots, a_n)$

M이 orientable이 아닌 경우 앞의 orientable의 경우를 locally(in fact pointwise) 적용하면 $|\nu|$는 orientation에 관계없이 잘 정의된다.

예. $M = \mathbb{S}_r^{n-1}(0)$ = sphere of radius r 인 경우 $N = \frac{1}{r}(x_1, \cdots, x_n)$이고 따라서 구의 Riemannian volume form은 다음과 같다.
$\nu = \frac{1}{r}\sum(-1)^{i-1}x_i dx_i \wedge \cdots \wedge \overset{\wedge}{dx_i} \wedge \cdots \wedge dx_n.$

$\mathbb{R}^n$에서 중요하게 쓰이는 또다른 form으로는 solid angle form "$d\theta$"가 있다. (통상적으로 $d\theta$로 쓰나 $\mathbb{R}^n \backslash \{0\}$상에서 exact form은 아니다.)
$d\theta := \dfrac{1}{r^{n-1}}\nu = \dfrac{1}{r^{n-1}}\sum(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \overset{\wedge}{dx_i} \wedge \cdots \wedge dx_n$

Exercise. $d\theta$ 가 closed임을 보여라.

Exercise.(See. 김홍종,윤옥경, "미분다양체론 입문")

1. (1) Let μ be an m-form on M^m and ν be an n-form on N^n.
Let "$\mu \wedge \nu$" $= \pi_1^*\mu \wedge \pi_2^*\nu$ where π_i is the canonical projection of $M \times N$ onto M or N. Then

$$\int_{M\times N} f(x,y)\mu \wedge \nu = \int_M (\int_N f(x,y)\nu)\mu.$$

(2) For $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{S}^{n-1}} (\int_0^\infty f(r,\theta) r^{n-1} dr) d\theta$

2. Volume of $\mathbb{B}^n$ and $\mathbb{S}^n$:

$$\text{vol}(\mathbb{S}^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \qquad where\ \ \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \quad \text{and}$$
$$\text{vol}(\mathbb{B}^n) = \frac{1}{n}\text{vol}(\mathbb{S}^{n-1}) \ .$$

(proof of 2) $\displaystyle\int_{\mathbb{R}^n} e^{-r^2} = (\int_{\mathbb{R}} e^{-x^2}dx)^n = (\sqrt{\pi})^n$ 에서

위 식의 좌변을 극형식으로 표현하면

$$\int_{\mathbb{S}^{n-1}}\int_0^\infty e^{-r^2}r^{n-1}dr d\theta = \text{vol}(\mathbb{S}^{n-1})\int_0^\infty e^{-r^2}r^{n-1}dr$$
$$= \text{vol}(\mathbb{S}^{n-1})\int_0^\infty e^{-t}t^{\frac{n}{2}-1}\frac{1}{2}dt \qquad (\text{Let } r^2 = t \text{ then } 2rdr = dt)$$

$$=\mathrm{vol}(\mathbb{S}^{n-1})\frac{1}{2}\Gamma(\frac{n}{2})$$

$\therefore\ (\sqrt{\pi})^n = \mathrm{vol}(\mathbb{S}^{n-1})\frac{1}{2}\Gamma(\frac{n}{2})$.

$\mathrm{vol}(\mathbb{B}^n) = \frac{1}{n}\mathrm{vol}(\mathbb{S}^{n-1})$ 는 Stokes 정리를 쓰면 보일 수 있다.