I.2 Topological Space, basis and subbasis

Definition 1 Topological space X is a set with a specific collection $\mathcal{T}$ of subsets called open sets with the following properties.

1. $\emptyset, X \in \mathcal{T}$.
2. A union of open sets is open.
3. A finite intersection of open sets is open.

Examples 1. A metric space with the usual open sets.

2. Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is called the discrete topology and $\mathcal{T} = \{\emptyset, X\}$ the indiscrete topology.
3. $X = \{a, b\}$. Then $\mathcal{T} = \{\emptyset, X, \{a\}\}$ is a topology.
4. Let X be an infinite set. Then $\mathcal{T} = \{U \subset X | U^c \; is \; a \; finite \; set\} \cup \{\emptyset\}$ is called cofinite topology.

Definition 2 Let X and Y be topological spaces. Then $f : X \to Y$ is continuous if $f^{-1}(V)$ is open for all V open in Y.

Definition 3 A collection $\mathcal{B}$ of open sets of a topological space X is called a basis if each open set in X can be represented as a union of elements of $\mathcal{B}$.

Proposition 1 *Suppose that a collection $\mathcal{B}$ of subsets of a set X satisfies the following two properties:*

1. *The elements of $\mathcal{B}$ cover X, i.e., $X = \cup_{B \in \mathcal{B}} B$.*
2. *If x belongs to two elements B_1 and B_2 of $\mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.*

Then the collection $\mathcal{T}(\mathcal{B})$ of all unions of elements of $\mathcal{B}$ defines a topology on a set X (called the topology generated by $\mathcal{B}$).

Proof is easy.

Examples 1. Let $\mathcal{B} = \{(a, b) | a < b, \; a \; and \; b \; real\}$. Then $(\mathbb{R}, \mathcal{T}(\mathcal{B}))$ is called the usual topology of $\mathbb{R}$.

2. Let $\mathcal{B} = \{[a, b) | a < b, \; a \; and \; b \; real\}$. Then $\mathbb{R}_l = (\mathbb{R}, \mathcal{T}(\mathcal{B}))$ is called the real line with half-open topology.

3. Let $\mathcal{B} = \{[a,b] | a \leq b,\ a \text{ and } b \text{ real}\}$. Then $(\mathbb{R}, \mathcal{T}(\mathcal{B}))$ becomes a discrete space since $[a,a] = \{a\}$.

Proposition 2 *Let $\mathcal{B}$ and $\mathcal{B}^{'}$ be basis for the topology $\mathcal{T}$ and $\mathcal{T}^{'}$, respectively on X. Then $\mathcal{T} \subset \mathcal{T}^{'}$ if and only if for each $B \in \mathcal{B}$ and $x \in B$, there is a basis element $B^{'} \in \mathcal{B}^{'}$ such that $x \in B^{'} \subset B$.*
In this case, $\mathcal{T}^{'}$ is said to be finer than $\mathcal{T}$.

Proof ($\Rightarrow$) For each B in $\mathcal{B}$, $B \in \mathcal{B} \subset \mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{B}^{'})$. Therefore $B \in \mathcal{T}(\mathcal{B}^{'})$. Since $\mathcal{T}(\mathcal{B}^{'})$ is generated by $\mathcal{B}^{'}$, for each $x \in B$ there is an element $B^{'} \in \mathcal{B}^{'}$ such that $x \in B^{'} \subset B$.
($\Leftarrow$) Let U be an element of $\mathcal{T}$ and $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Then there is $B^{'}$ such that $x \in B^{'} \subset B \subset U$ and hence $U \in \mathcal{T}^{'}$.
□

Examples 1. Let $\mathcal{B} = \{(a,b) | a < b,\ a \text{ and } b \text{ real}\}$ and $\mathcal{B}^{'} = \{[a,b) | a < b,\ a \text{ and } b \text{ real}\}$. Then $\mathcal{T}(\mathcal{B}) \subsetneq \mathcal{T}(\mathcal{B}^{'})$

2. Let $\mathcal{B}$ be the collection of all open discs in the plane and $\mathcal{B}^{'}$ is the collection of open squares. Then $\mathcal{T} = \mathcal{T}^{'}$

Homework 3 Let $\mathcal{C}([0,1])$ be the collection of continuous functions on $[0,1]$. Consider the following topologies.
$\mathcal{T}_1$ = the topology induced by L_1 norm, i.e., $\|f\|_1 = \int_0^1 |f|$
$\mathcal{T}_2$ = the topology induced by L_2 norm, i.e., $\|f\|_2 = (\int_0^1 |f|^2)^{1/2}$
$\mathcal{T}_\infty$ = the topology induced by L_∞ norm, i.e., $\|f\|_\infty = sup|f|$
이 topology들의 상호포함관계를 설명하라.

Definition 4 Let X be a topological space. A collection $\mathcal{S}$ of open sets is called a subbasis if each open set in X can be written as a union of finite intersections of elements of $\mathcal{S}$.

Proposition 3 *Let $\mathcal{S}$ be a collection of subsets of a set X whose union is X. Then the collection of all unions of finite intersections of sets in $\mathcal{S}$ form a topology for X. This topology $\mathcal{T}(\mathcal{S})$ will be called the topology generated by $\mathcal{S}$, and $\mathcal{T}(\mathcal{S})$ is the smallest topology containing $\mathcal{S}$.*

(Proof is easy using Prop 1.)

Examples 1. $\mathcal{S} = \{(a,\infty), (-\infty, b)\}$ is a subbasis for the standard topology of $\mathbb{R}$.

2. $\mathcal{S} = \{[a, \infty), (-\infty, b)\}$ is a subbasis for $\mathbb{R}_l$.

3. $\mathcal{S} = \{\mathbb{R} - \{p\} | p \in \mathbb{R}\}$ is a subbasis for the cofinite topology.