II.1 Separation Axioms

정의 1 A topological space X is called a Hausdorff space ($T_2 - space$) if each two disjoint points have non-intersecting neighborhoods, i.e., for each x, y, there exist O_x, O_y which are open sets with $x \in O_x$ and $y \in O_y$ such that $O_x \bigcap O_y = \emptyset$.

정의 2 A topological space X is said to be T_1, if for each pair of distinct point, each has a neighborhood which does not contain the other.
A space X is said to be regular, if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. (T_3)
A space X is said to be normal, if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively. (T_4)

Example A discrete space is Hausdorff.
A metric space is Hausdorff.
A indiscrete space is not Hausdorff.
A space with cofinite topology is not Hausdorff but is T_1.

명제 1 *(1) Each subspace of a Hausdorff space is Hausdorff.*
(2) $\prod X_\alpha$ is Hausdorff if and only if each X_α is Hausdorff.

증명 (1) Let X be a Hausdorff space and Y be a subspace of X. Let $a, b \in Y \subset X$ with $a \neq b$. Since X is Hausdorff, there are disjoint open neighborhoods U and V, containing a and b, respectively. By definition of subspace, $Y \cap U$ and $Y \cap V$ are disjoint open neighborhoods in Y containing a and b, respectively.
(2) ($\Leftarrow$) Let $X = \prod X_\alpha$. Let $x = (x_\alpha)$, $y = (y_\alpha)$ with $x_\alpha \neq y_\alpha$ for some α. Since X_α is Hausdorff, there are separating open neighborhoods O_{x_α} and O_{y_α}. Then $p^{-1}(O_{x_\alpha})$ and $p^{-1}(O_{y_\alpha})$ are separating open neighborhoods in X.
($\Rightarrow$) Since X_α can be embedded as a subspace of $\prod X_\alpha$ which is Hausdorff, X_α

is also Hausdorff by (1).

(exercise) For each $\beta \neq \alpha$, fix a point $a_\beta \in X_\beta$. Then $s : X_\alpha \to \prod X_\alpha$ given by

$$s(x_\alpha)_\beta = \begin{cases} a_\beta & \beta \neq \alpha \\ x_\alpha & \beta = \alpha \end{cases}$$

is an embedding. □

명제 2 *X is a Hausdorff space if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.*

증명 X is Hausdorff.
$\Leftrightarrow \forall (x, y) \in \triangle^c$, $\exists$ Open neighborhoods U_x, U_y of x and y s.t. $U_x \times U_y \subset \triangle^c$.
$\Leftrightarrow \triangle^c$ is open in $X \times X$.
$\Leftrightarrow \triangle$ is closed in $X \times X$. □

명제 3 *Suppose that X is Hausdorff, then the followings hold.*
(1) Each point in X is closed
(2) If x is an accumulation point of A in X, then each neighborhood of x contains infinitely many points of A

증명 (1) Clear by definition.
(2) Suppose U is an open set containing x and only finite number of points of A different from x. Since $B := U \cap A - \{x\}$ is a finite subset of a Hausdorff space, it is closed and hence $V := U - B$ is open. Then V is a neighborhoods of x containing no points of A different from x. Thus x is not an accumulation point, which is a contradiction. □

명제 4 *Let $f, g : X \longrightarrow Y$ be continuous maps from a topological space X to a Hausdorff space Y. Then*
(1) $\{x \mid f(x) = g(x)\}$ is closed
(2) If $D \subset X$ is dense, i.e., $\overline{D} = X$ and $f\mid_D = g\mid_D$, then $f = g$ on X
(3)The graph of f is closed in $X \times Y$

증명 (1)Define $\varphi : X \longrightarrow Y \times Y$ by $\varphi : x \longmapsto (f(x), g(x))$, then $\{x \mid f(x) = g(x)\} = \varphi^{-1}(\Delta)$. Since Y is Hausdorff, Thus Δ is closed. Since φ is continuous, $\varphi^{-1}(\Delta)$ is closed.

(2) Since $f \mid_D = g \mid_D$, $D \subset \{x : f(x) = g(x)\}$. Since $\{x : f(x) = g(x)\}$ is closed, $X = \overline{D} \subset \{x : f(x) = g(x)\} \subset X$. Thus $f = g$ on X.

(3) Define $\psi : X \times Y \longrightarrow Y \times Y$ by $\psi : (x, y) \longmapsto (f(x), y)$. Then the graph of $f = \{(x, y) : f(x) = y\}$ is equal to $\psi^{-1}(\Delta)$. Thus the graph of f is closed. $\square$

Homework 1 Suppose Y is not Hausdorff in the preceding proposition. Find counter examples to (1) and (2) above.