IV.2 Basic topological properties

4. Connectedness

Definition 1 A topological space X is disconnected if there is nonempty open sets A and B in X s.t. $X = A \cup B$ and $A \cap B = \phi$.
In this case $\{A, B\}$ is called a disconnection or a separation of X.
A topological space X is connected if it is not disconnected.

Example $\mathbb{Q}$ is disconnected by $(-\infty, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

Proposition 1 *$\mathbb{R}$ is connected.*

proof. Suppose not. Then $\mathbb{R} = A \cup B, A \cap B = \phi$ where A and B are open. Choose $a \in A$, $b \in B$ and we may assume that $a < b$. Let $S = A \cap [a, b] \neq \phi$. S is bounded and hence S has supremum $s \in (a, b)$ since $A \cap [a, b], B \cap [a, b]$ are open neighborhoods of a and b respectively.
(1) $s \notin A$:
If $s \in A, \exists$ open neighborhood $(s - \epsilon, s + \epsilon) \subset A \cap (a, b) \subset S$, and hence $s \neq sup(S)$.
(2) $s \notin B$:
If $s \in B, \exists$ open neighborhood $(s - \epsilon, s + \epsilon) \subset B \cap (a, b)$ and hence $s \neq sup(S)$ since $sup(S) \leq s - \epsilon$.
Hence we obtain a contradiction. □

Theorem 2 *The product of connected spaces is connected.*

proof. Let $X = \prod_i X_i$ fix a point $a \in X$

Claim $D := \{x \in X |\ x$ and a differ in at most finitely many coordinates. $\}$ is a dense subset of X.
(Proof of Claim) We want to show that each basic open set $U = p_{i_1}^{-1}(O_{i_1}) \cap \cdots \cap p_{i_n}^{-1}(O_{i_n})$, where O_{i_k} is open subset of X_{i_k}, intersects D.
Choose a point $b_{i_k} \in O_{i_k} \subset X_{i_k}$ for k=1,2,...,n. Let $x = (x_i)$ be a point in X given by $x_i = b_{i_k}$ if $i = i_k$, and $x_i = a_i$ if $i \neq i_k$ for k=1,2,...n. Then clearly $x = (x_i) \in U \cap D$.

Suppose that $X = A \cup B$ is a disconnection of X. Let's define an equivalence relation $\sim$ such that $x \sim y$ if both x and y belong to the same open set A or B.

Let's show that $x \in D \Rightarrow x \sim a$:
Suppose that x differs from a in only one coordinate say $x_i \neq a_i$. Then x and a are in $s(X)$ where s is a slice map from X_i to X defined by $s(x_i)_j = x_i$ if $j = i$, and $s(x_i)_j = a_j$ if $j \neq i$. Since $s(X_i)$ is connected, $x \sim a$. Otherwise $A \cap s(X_i)$ and $B \cap s(X_i)$ give a disconnection of $s(X_i)$. Apply the above argument repeatedly to each coordinate in which x and a differ.

Hence we conclude that either $D \subset A$ or $D \subset B$ exclusively, which is a contradiction to the fact that D is dense and hence intersects every non-empty open set. □

Proposition 3 *The followings are equivalent.*
(1) X is connected.
(2) The only open and closed sets in X are X and ϕ.
(3) If $f : X \to \{0, 1\}$ is continuous, then f is not onto, i.e., f is constant.

proof. Clear. □

Proposition 4 *A continuous image of connected space is connected.*

proof. $f : X \to Y$ be a continuous function and X is connected.
If $f(X) = A \cup B$ is a disconnection, then $f^{-1}(A) \cup f^{-1}(B)$ will be a disconnection of X. □

Remark (Intermediate value property)
Let $f : X \to \mathbb{R}$ be a continuous function where X is connected and $f(a) \leq p \leq f(b)$. Then there exists $x \in X$ such that $p = f(x)$.

Proposition 5 *X is a topological space.*
(1) Let $A_\alpha \subset X$ be a connected subset for all α. Then $\cap_\alpha A_\alpha \neq \phi \Rightarrow \cup_\alpha A_\alpha$ is connected.
(2) A: a connected subset of X
$A \subset B \subset \overline{A} \Rightarrow B$ is connected.
In particular, $\overline{A}$ is connected.

proof. (1)If $f : \cup_\alpha A_\alpha \to \{0, 1\}$ is continuous, then $f|_{A_\alpha}$ is continuous and hence it is constant. If $a \in \cap_\alpha A_\alpha$ then $f|_{A_\alpha} \equiv f(a)$ for all α.
Therefore $f \equiv f(a)$.
(2)Let $f : B \to \{0, 1\}$ be a continuous function.

$\Rightarrow f|_A$ is continuous
$\Rightarrow f|_A$ is constant c since A is connected.
$\Rightarrow f \equiv c$ is the unique extension of $f|_A$ on $\overline{A}$ and hence on B. □

Example (1) $\mathbb{R} \cong (0,1) \subset (0,1] \subset [0,1]$ all connected.
(2) The union of the graph of $y = sin(1/x)$, $(x > 0)$ (topologist's sine curve) and $\{0\} \times [-1,1]$ is the closure of the graph and hence it is connected.

6. Path-connectedness

Definition 2 Let X be a topological space. A continuous map $\gamma : I = [0,1] \to X$ is called a path joining $\gamma(0)$ and $\gamma(1)$.
A space X is path-connected if each pair of points can be joined by a path.

Proposition 6 *Path-connected $\Rightarrow$ connected. (not $\Leftarrow$)*

proof. Let $f : X \to \{0,1\}$, where X is path-connected, be a continuous function. If there exists x and y s.t. $f(x) = 0$ and $f(y) = 1$, then there is $\gamma : I \to X$ s.t. $\gamma(0) = x, \gamma(1) = y$
$\Rightarrow f \circ \gamma : I \to \{0,1\}$ continuous and onto.
$\Rightarrow I$ is not connected. (A contradiction!)

A counterexample of $(\Leftarrow)$: the closure of the topologist's sine curve □

Remark (1) Let A_α be path connected for all $\alpha \in I$. Then $\cap_{\alpha \in I} A_\alpha \neq \phi \Rightarrow \cup_{\alpha \in I} A_\alpha$ is path connected.
(2) The closure of a path connected space is not necessarily path connected.(A counterexample is the topologist's sine curve.)

Definition 3 A maximal (path-) connected subspace of a topological space is called a (path-) component of the space.

[Figure describing component and path-component on real line, topologist' sine curve, rational number, and Cantor set]

Definition 4 A topological space is said to be totally disconnected if every component is a point.

Proposition 7 *X is a topological space.*
(1) Each point in X is contained in exactly one (path-) component of X.
(2) X is a disjoint union of (path-) components.
(3) Each component is closed. (not necessarily for path-component)

proof. (1) The union of all (path-)connected sets containing $x \in X$ is a (path-)component.
(2) If they intersect, their union will be connected, so it is a contradiction to the maximality of (path-)component. Therefore components are disjoint.
(3) If C is a component. Then C is connected and so is $\overline{C}$. By maximality of component, $C \supset \overline{C}$. Therefore $C = \overline{C}$ is closed.
(3) does not hold for path component. (A counterexample is the topologist's sine curve). □

Proposition 8 *(1) $\forall x \in X$ has a (path-) connected neighborhood*
$\Leftrightarrow \forall$ each (path-)component is open(and hence closed).
(2) X is path-connected
$\Leftrightarrow X$ is connected and $\forall x \in X$ has a path-connected neighborhood.

proof. (1)($\Rightarrow$) Let C be a component. Then by maximality, $U_x \subset C$ (U_x is a connected neighborhood of x).
($\Leftarrow$) Trivial.
By the same argument this is true for path component, too.
(2)($\Rightarrow$) Clear.
($\Leftarrow$) By (1), each path-component is open. So they are disjoint and open. Therefore each path-component is closed since its complement is open being a disjoint union of open sets. Hence it is both open and closed. Since X is connected, a path-component becomes X itself. □

Corollary 9 *An open set in $\mathbb{R}^n$ is connected $\Leftrightarrow$ it is path-connected.*

Definition 5 A space X is locally (path-) connected if for all x in X, each neighborhood of x contains a (path) connected neighborhood.(i.e., each point has a basis consisting of connected open sets).

Remark Concepts of connectedness and local connectedness are independent, i.e., one does not necessarily imply the other.

Indeed an example of locally connected but not connected space is $(0,1) \cup (2,3)$. An example of connected but not locally connected is the closure of topologist sine curve.

Proposition 10 *(1) X is locally (path-)connected*
$\Leftrightarrow$ the (path-)components of each open set are open.
(In particular, each (path-) component is open for a locally (path-)connected space).

(2) X is locally path-connected
$\Rightarrow$ the components and the path-components of X are the same.

Proof is a Homework.