II Basic topological properties

5. Axioms of countability

Definition 1 A space X satisfies the **first axiom of countability(first countable)** if for all $x \in X$, there exists a countable collection of open sets $\mathcal{O} = \{O_n\}$ which satisfies the condition that for all open neighborhood O of x, there exists an $O_n \in \mathcal{O}$ s.t. $x \in O_n \subset O$.
This $\mathcal{O}$ is called **a basis at** x.

Example A metric space is 1st countable.

proof. {balls with rational radius} forms a basis at each point. □

Definition 2 A space X satisfies the **second axiom of countability(second countable)** if X has a countable basis.

Note Second countable ⇒ First countable.

Example

1. $\mathbb{R}^n$ is second countable: $\{B_r(q) |\, q \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ is a countable basis of $\mathbb{R}^n$.

2. A discrete space is first countable. An uncountable discrete space is not second countable.

3. $\mathbb{R}_l$ is $\mathbb{R}$ with half open interval topology, i.e., the topology is generated by $\{(a, b] |\, a < b\}$. Then $\mathbb{R}_l$ is 1st countable ($(r, x], r \in \mathbb{Q}$ is a countable basis at x) but not 2nd countable.
proof. Suppose $\mathbb{R}_l$ is 2nd countable, then there exists a basis of the form $\{(a_n, b_n] | n \in \mathbb{N}\}$ by Lemma 1. Choose b with $b \neq b_n$ for all $n \in \mathbb{N}$. Then $(a, b]$ is not a union of the intervals $(a_n, b_n]$ and this is contradiction. □

Lemma 1 *If X is 2nd countable, each basis for X has a countable subcollection which also forms a basis for X.*

(Proof is a homework.)

Proposition 2 *A subspace of 1st countable space(2nd countable respectively) is 1st countable(2nd countable respectively).*

Proof is clear.

Proposition 3 *X, Y: 1st countable* ⇒ $X \times Y$ *:1st countable.*
X, Y: 2nd countable ⇒ $X \times Y$ *:2nd countable.*

proof. $\mathcal{O}_X, \mathcal{O}_Y$: countable basis of X,Y, respectively.
$\Rightarrow \mathcal{O}_X \times \mathcal{O}_Y = \{O_X \times O_Y | O_X \in \mathcal{O}_X, O_Y \in \mathcal{O}_Y\}$ is a countable basis for $X \times Y$
Similarly for the 1st countability. □

Example In general, not all product preserves the property.
Let $X = \prod_{\alpha \in A} I_\alpha$, where A is an uncountable index set and $I_\alpha = [0,1]$. Now we claim that X is not 2nd countable:
If $X = \{x : A \to I = [0,1]\}$ is 2nd countable, the standard basis for the product topology has a countable subcollection $\mathcal{U} = p_{\alpha_1}^{-1}(O_{\alpha_1}) \bigcap \cdots \bigcap p_{\alpha_n}^{-1}(O_{\alpha_n})$ which is also a basis by the previous lemma. Choose an index $\alpha \in A$ which does not appear in any basic open set in $\mathcal{U}$. Such an index exists since A is uncountable. Then for $x \in P_\alpha^{-1}(0, 1/2)$, there is no $U \in \mathcal{U}$ s.t. $x \in U \subset P_\alpha^{-1}(0, 1/2)$ since $P_\alpha(U) = I$.

Homework 1 Is the above example first countable?

Definition 3 A sequence in a space X is a function $x : \mathbb{N} \to X$ usually written as $(x_n)_{n=1}^\infty$ where $x_n = x(n)$.

Theorem 4 *When X is 1st countable, the following statements hold.*
(1) When $A \subset X$, $x \in \overline{A} \Leftrightarrow \exists$ a sequence (a_n) in A s.t. $a_n \to x$.
(2) $A \subset X$ is closed $\Leftrightarrow \exists a_n \to x$ with $a_n \in A$ implies $x \in A$.
(3) $f : X \to Y$ is continuous $\Leftrightarrow x_n \to x$ implies $f(x_n) \to f(x)$.

proof. (1) ($\Rightarrow$) Since X is 1st countable, $\forall x \in X$, we can construct a decreasing sequence of basic open neighborhoods of x. Indeed if we let $\mathcal{U} = \{U_n\}$ be a countable basis at x, then $\mathcal{V} = \{V_n | V_n = U_1 \cap \cdots \cap U_n, n = 1, 2, \cdots\}$ is clearly a decreasing sequence of open neighborhoods of x. If $x \in A$, let $a_n = x$. If $x \notin A$, then $x \in A'$. Choose a point $a_n \in V_n \cap A$ and then (a_n) is a sequence converging to x.
($\Leftarrow$) $a_n \to x \Rightarrow$ for any neighborhood of x it contains a_n's for large n. Then either $x \in A$ or $x \in A'$.
Hence $x \in \overline{A}$. (We do not need 1st countability of X.)
(2) ($\Rightarrow$) $a_n \to x$ with $a_n \in A \Rightarrow x \in \overline{A} = A$. (We do not need 1st countability of X.)
($\Leftarrow$) Show $\overline{A} \subset A$:
$x \in \overline{A}$
$\Rightarrow \exists\, (a_n)$ in A s.t. $a_n \to x$ by (1)
$\Rightarrow x \in A$.

(3) ($\Rightarrow$) For any open neighborhood U of $f(x)$, $f^{-1}(U)$ is an open neighborhood of x
$\Rightarrow x_n \in f^{-1}(U)$ for large n
$\Rightarrow f(x_n) \in U$ for large n.
Hence $f(x_n) \to f(x)$. (We do not need 1st countability of X.)
($\Leftarrow$) Show $f^{-1}(\mathit{closed\,set})$ is closed:
$x \in \overline{f^{-1}(C)}$ where C is closed set.
$\Rightarrow \exists$ a sequence (a_n) in $f^{-1}(C)$ s.t. $a_n \to x$ by (1)
$\Rightarrow f(a_n) \to f(x)$ by the hypothesis and $f(a_n) \in C$
$\Rightarrow f(x) \in C$ by (2)
$\Rightarrow x \in f^{-1}(C)$.
Hence $f^{-1}(C)$ is closed. $\square$