III.3 Completion of a metric space

Let (X, d) be a metric space. For two subsets $A, B \subset X$, define distance between A and B by $d(A, B) := \inf\{d(a, b)|a \in A, b \in B\}$.

명제 1 (a) $|d(x, A) - d(y, A)| \leq d(x, y)$.
(b) $f(x) := d(x, A)$ *is a continuous funtion* : $X \to \mathbb{R}$.

증명 $\forall x, y \in X, d(x, a) \leq d(x, y) + d(y, a)$.
$d(x, A) = inf_{a\in A} d(x, a) \leq d(x, y) + inf_{a\in A} d(y, a) = d(x, y) + d(y, A)$
$\Rightarrow d(x, A) - d(y, A) \leq d(x, y)$
$\Rightarrow |f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y)$. □

명제 2 (a) *C: closed and $x \notin C \Rightarrow d(x, C) > 0$.*
(b) *C: closed, A: compact, and $A \cap C = \emptyset \Rightarrow d(A, C) > 0$.*

증명 (a) Suppose $d(x, C) = 0$
$\Rightarrow \exists (c_n) \in C$ s.t. $d(x, c_n) \to 0$
$\Rightarrow c_n \to x$
$\Rightarrow x \in \overline{C} = C$: a contradiction.
(b) Suppose $d(A, C) = 0$. Then $\exists (a_n)$ in A such that $d(a_n, C) \to 0$. May assume $a_n \to x$ by passing to a subsequence. Then $x \in A$ since A is compact, and also $d(x, C) = 0$. Thus $x \in C$ and again a contradiction. □

정의 1 If $f : X \to Y$ is an isometric embedding of X into a complete metric space Y, then the space $\overline{f(X)}$ of Y is a complete metric space. It is called the *completion* of X.

정리 3 *(Existence of Completion) Let (X, d) be a metric space. Then there exists an isometric embedding of X into a complete metric space.*

증명 Fix a point $x_0 \in X$, and for $a \in X$ define $f_a : X \to \mathbb{R}$ by $f_a(x) = d(x, a) - d(x, x_0)$. Then f_a is a bounded function since $|f_a(x)| \leq d(a, x_0)$ by Proposition

1(a). Now $f : X \to \mathcal{B}(X, \mathbb{R})$ defined by $f(a) = f_a$ is an isometric embedding : Indeed $|f_a(x) - f_b(x)| = |d(x,a) - d(x,b)| \leq d(a,b)$ and $|f_a(a) - f_b(a)| = d(a,b)$ $\Rightarrow \|f_a - f_b\| = d(a,b)$. $\square$

숙제 1 *(Uniqueness of Completion)* Let $f_i : X \to Y_i$, $i = 1, 2$ be an isometric embedding. Then $\exists$ an isometry : $\overline{f_1(X)} \to \overline{f_2(X)}$ which extends $f_2 \circ f_1^{-1}$: $f_1(X) \to f_2(X)$.