IV.3 Hilbert cube and Hillbert space

Hilbert cube: $I^\infty = \prod_{n=1}^{\infty} I_n$, where $I_n = [0,1]$.
Hilbert space: $H = \{x \in \mathbb{R}^\infty | \sum x_i^2 < \infty\}$, where $\|x\|^2 = \sum x_i^2$.

명제 1 *$(H, \|\cdot\|)$ is a complete metric space.*

증명 $\|\cdot\|$ is a norm, i.e., $\|x+y\| \le \|x\| + \|y\|$:
Since we know this inequality in $\mathbb{R}^n$, we have
$\sqrt{(x_1+y_1)^2 + \cdots + (x_n+y_n)^2} \le \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}$.
Now the right hand side is clearly $\le \|x\| + \|y\|$ for any n, and so is the left hand side, and the desired inequality follows.
Show that H is complete:
Let (x_n) be a Cauchy sequence, i.e., $\forall \epsilon > 0$, $\exists N$ such that $m, k \ge N \Rightarrow \|x_m - x_k\| < \epsilon$. Let $x_n(i)$ denote $x_n's\ i^{th}$ coordinate. Then $\{x_n(i)\}_{n=1,2,\cdots}$ is a Cauchy sequence in $\mathbb{R}$, and $x(i) := \lim_{n\to\infty} x_n(i)$ exists by the completeness of $\mathbb{R}$. In the inequality $\sum_{i=1}^{n}(x_m(i) - x_k(i))^2 \le \|x_m - x_k\|^2 < \epsilon$, if we let $k \to \infty$, then we obtain $\sum_{i=1}^{n}(x_m(i) - x_(i))^2 \le \epsilon^2$, and hence $\|x_m - x\|^2 = \sum_{i=1}^{\infty}(x_m(i) - x(i))^2 \le \epsilon^2$. Now $x \in H$ since $\|x\| = \|x - x_m + x_m\| \le \|x - x_m\| + \|x_m\| < \infty$. □

Remark $(H, \|\cdot\|) \ne H \subset \mathbb{R}^\infty$.

증명
There is no basic open set w.r.t. the product topology contained in $\|\cdot\|$-ball about the origin. □

명제 2 *I^∞ is metrizable with $d(x,y) = sup\{\frac{d_n(x_n,y_n)}{n}\}$, where $d_n(x_n, y_n)$ is the standard metric on $I_n = I, n = 1, 2, \cdots$.*

증명
(Step 1) Show d is a metric:
$d(x,z) = sup_n \frac{d_n(x_n,z_n)}{n} \le sup_n(\frac{d_n(x_n,y_n)}{n} + \frac{d_n(y_n,z_n)}{n}) \le sup_n \frac{d_n(x_n,y_n)}{n} + sup_n \frac{d_n(y_n,z_n)}{n}$
$= d(x,y) + d(y,z)$.
(Step 2) d-topology = product topology:
$(\supset) : \mathfrak{U} = \{U = p_{i_1}^{-1}(O_{i_1}) \bigcap \cdots \bigcap p_{i_k}^{-1}(O_{i_k})\}$ is a basis for the product topology and U is a typical basic open set. Let $B_\epsilon(x)$ be the ball in I^∞ with radius ϵ and centered at x.
$\forall x \in U, \exists \delta > 0$ s.t. $V = p_{i_1}^{-1}(x_{i_1} - \delta, x_{i_1} + \delta) \bigcap \cdots \bigcap p_{i_k}^{-1}(x_{i_k} - \delta, x_{i_k} + \delta) \subset U$.
Choose $\epsilon = \frac{\delta}{i_k}$. Then $x \in B_\epsilon(x) \subset V \subset U$:

$y \in B_\epsilon(x) \Rightarrow \frac{d_n(x_n,y_n)}{n} < \epsilon$, $\forall n \Rightarrow y \in V$.
Therefore d-topology $\supset$ product top.
$(\subset)$: Conversely for given $B_\epsilon(x)$, $\exists n_0$ s.t. $n > n_0$ implies $n\epsilon > 1$. Then $x \in p_1^{-1}(x_1-\epsilon, x_1+\epsilon) \bigcap p_1^{-1}(x_2-2\epsilon, x_2+2\epsilon) \bigcap \cdots \bigcap p_{n_o}^{-1}(x_{n_0}-n_0\epsilon, x_{n_0}+n_0\epsilon) \subset B_\epsilon(x)$.
Therefore product topology $\supset$ d-topology. □

Remark 1. We can replace d by d_p in the above proposition, where $d_p(x,y) := sup\frac{d_n(x_n,y_n)}{n^p}$, $p > 0$.

2. By the same proof, we can show that $\prod_{n=1}^{\infty} X_n$ is metrizable if each X_n is metrizable with $diam(X_n) < M$, $\forall n$.

명제 3 *I^∞ can be embedded into H.*

증명
Show $f : I^\infty \to \prod_{n=1}^{\infty}[0,1/n] \subset H$ given by $f(x) = (x_1, x_2/2, \cdots)$ is an embedding:
1. f is obviously bijective.
2. $f^{-1} : y \mapsto (y_1, 2y_2, \cdots)$ is continuous since $p_n : (H, \|\cdot\|) \to \mathbb{R}$ is continuous.
3. f is continuous, i.e., show that $\forall\epsilon > 0, \exists\delta > 0$ such that $d_p(x,y) = sup\frac{|x_n-y_n|}{n^p} < \delta \Rightarrow \|f(x)-f(y)\| = \sqrt{\sum_{n=1}^{\infty}\frac{(x_n-y_n)^2}{n^2}} < \epsilon$:
Now let $p = 1/4$. Then

$$\frac{(x_n-y_n)^2}{\sqrt{n}} < \delta^2 \Rightarrow \sum_{n=1}^{\infty}\frac{(x_n-y_n)^2}{n^2} < \delta^2\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

Let $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} =: \alpha$ and choose $\delta = \epsilon/\sqrt{\alpha}$. □