IV.4 Urysohn Embedding and Metrization theorem

Theorem 1 ***(Urysohn Embedding Theorem)***
X:normal and 2nd countable $\Rightarrow \exists$an embedding $f : X \hookrightarrow I^\infty \hookrightarrow H$.

Theorem 2 ***(Urysohn Metrization Theorem)***
X:normal and 2nd countable $\Rightarrow$ X is metrizable.

Proof Urysohn Metrization Theorem follows from Urysohn Embedding Theorem immediately, and let's show the embedding theorem.
Let $\mathfrak{U} = \{O_i\}$ be a countable basis. $\forall O_i \in \mathfrak{U}$ and $\forall x \in O_i$, $\exists O_j \in \mathfrak{U}$ such that $x \in O_j \subset \overline{O_j} \subset O_i$.
All such pairs (O_i, O_j) form a countable collection $\{P_1, P_2, \cdots\}$.
$\forall P_n = (O_i, O_j)$, by Urysohn lemma, $\exists f_n : X \to [0,1]$ such that $f_n(\overline{O}_j) = 0, f_n(O_i^c) = 1$.
Define $f : X \to I^\infty$ by $f(x) = (f_1(x), f_2(x), \cdots)$. Then
1. f is 1-1:
$x \neq y$
$\Rightarrow \exists O_k, O_l$: disjoint basic open neighborhoods of x,y respectively in $\mathfrak{U}$
$\Rightarrow f_m(x) \neq f_m(y)$.
i.e., "*$\{f_n\}$ is a family of enough functions separating points of X*" in the sense that $x \neq y \Rightarrow \exists f_m$ s.t. $f_m(x) \neq f_m(y)$.
2. f is continuous: obvious.
3.$f^{-1} : f(X) \to X$is continuous:
Show $\forall x \in X$ and O_i containing x, $\exists \delta$ s.t. $d(f(x), f(y)) < \delta \Rightarrow y \in O_i$:
Let $P_{n_0} = (O_i, O_j)$ s.t. $x \in O_j \subset \overline{O_j} \subset O_i$.
If $d(f(x), f(y)) < \delta < 1/n_0$
$\Rightarrow \frac{|f_n(x) - f_n(y)|}{n} < \delta, \forall n$
$\Rightarrow \frac{|f_{n_0}(x) - f_{n_0}(y)|}{n_0} < \delta \Rightarrow \frac{f_{n_0}(y)}{n_0} < \delta \Rightarrow f_{n_0}(y) < n_0\delta < 1$.
Since $f_{n_0}(y) = 1$ if $y \in O_i^c$, $f_{n_0}(y) < 1 \Rightarrow y \in O_i$ □

Remark We can change normality to regularity in Urysohn Embedding and Metrization Theorem since "regular+2nd countable $\Rightarrow$ normal". (See Munkres Theorem 32.1.)

Homework 1. Is uncountable product of $I = [0,1]$ normal? metrizable?
2. Read or prove by yourself Munkres' Theorem 32.1.