IV.5 Tietze extension theorem

Theorem 1 ***(Tietze Extension Theorem)*** *Let X be a normal space and A be a closed subset in X. If $f : A \to [a,b]$ is a continuous function, then f has a continuous extension $\overline{f} : X \to [a,b]$, i.e., $\overline{f}$ is continuous and $\overline{f}|_A = f$.*

Proof. We may assume that $[a,b] = [-1,1]$.
Step 1.
Let $g : A \to [-c,c]$ be a continuous function. Then $C = g^{-1}([-c,-c/3])$ and $D = g^{-1}([c/3,c])$ are closed in A, and hence closed in X. By Urysohn Lemma, $\exists \hat{g} : X \to [-c/3,c/3]$ such that $\hat{g}(C) = -c/3$ and $\hat{g}(D) = c/3$.
Note that $\|\hat{g}\| \le c/3$ and $\|g - \hat{g}\|_A \le 2c/3$.

Step 2.
Start with $f : A \to [-1,1]$. (c=1 in Step 1.)
Let $\overline{f_1} := \hat{f}$. Then $\|f - \overline{f_1}\|_A \le 2/3$ and $\|\overline{f_1}\| \le 1/3$.
$f - \overline{f_1} = f - \hat{f} : A \to [-2/3, 2/3]$.
Now apply Step 1 to this function $f - \overline{f_1}$ with $c = 2/3$:
Let $\overline{f_2} := (f - \overline{f_1})\hat{}$. Then $\|f - \overline{f_1} - \overline{f_2}\|_A \le (2/3)^2$ and $\|\overline{f_2}\| \le 1/3 \cdot 2/3 = 2/9$.
In this way, we can obtain a sequence $(\overline{f_n})$ with the property that
(1)$\|f - \overline{f_1} - \overline{f_2} \cdots - \overline{f_n}\|_A \le (2/3)^n$
(2)$\|\overline{f_n}\| \le 1/3 \cdot (2/3)^{n-1}$.

Step 3.
Let $s_n = \overline{f_1} + \overline{f_2} \cdots + \overline{f_n}$.
Then (s_n) is a Cauchy sequence in $\mathcal{C}(X,\mathbb{R})$ since
$\|s_n - s_m\| = \|\overline{f_{n+1}} + \cdots + \overline{f_m}\| \le \|\overline{f_{n+1}}\| + \cdots + \|\overline{f_m}\|$
$\le (1/3)((2/3)^n + \cdots + (2/3)^{m-1}) < (1/3)(2/3)^n(1 + 2/3 + (2/3)^2 + \cdots) = (2/3)^n$
By the completeness of $\mathcal{C}(X,\mathbb{R})$, $s_n \to \overline{f}$ uniformly and $\overline{f} \in \mathcal{C}(X,\mathbb{R})$.
Now we claim that $\overline{f}$ is a desired extension of f:
Step 2(1) $\Rightarrow \|f - s_n\|_A \le (2/3)^n \Rightarrow s_n \to f$ uniformly on A$\Rightarrow \overline{f} = f$ on A.
Note that $\|s_n\| \le \|\overline{f_1}\| + \|\overline{f_2}\| \cdots + \|\overline{f_n}\| \le 1/3 \cdot (1 + 2/3 + (2/3)^2 + \cdots) \le 1/3 \cdot 3 = 1$ for all n, and hence $\|\overline{f}\| \le 1$. □

Remark $sin(1/x)$ on $(0,1]$ can not be extended to [0,1].

Remark The followings are equivalent.
(i) X is normal
(ii)$\forall$ disjoint and closed $A, B \subset X$, $\exists f : X \to [0,1]$ s.t. $f(X) = 0, f(B) = 1$.

(iii) $\forall A^{closed} \subset X$ and$f : A \to [0,1]$, $\exists \overline{f} : X \to [0,1]$, an extension of f.

Proof. "(i)⇒(iii)" is the Tietze extension theorem.
(iii)⇒(ii): Define $g : A \bigcup B \to [0,1]$ by $g(A) = 0$ and $g(B) = 1$. Then g is continuous on a closed set $A \bigcup B$ and has an extension $f : X \to [0,1]$.
(ii)⇒(i): $f^{-1}([0,\epsilon))$ and $f^{-1}((1-\epsilon, 1])$ are disjoint open neighborhoods of A and B respectively. □

Remark In Tietze extension theorem. $[a,b]$ can be replaced by (a,b) or $\mathbb{R}$.

Proof. Let A be a closed subset of a normal space X, and let $f : A \to (a,b) \subset [a,b]$. Here we may assume $[a,b] = [-1,1]$. Then by Tietze extension theorem, f has an extension $\overline{f} : X \to [-1,1]$ and let $C_1 = \overline{f}^{-1}(-1)$ and $C_2 = \overline{f}^{-1}(1)$. Then C_1, C_2, and A are disjoint subsets of X and hence there exist separating open neighborhoods V_1, V_2 and W respectively. Choose open sets U_i(i=1,2) such that $C_i \subset \overline{U_i} \subset V_i$, and bump functions φ_i such that $\varphi_i(\overline{U_i}) = 0$ and $\varphi_i({V_i}^c) = 1$. Then $\varphi_1\varphi_2\overline{f}$ is an extension of f with the range in $(-1,1)$. □

Theorem 2 *Let X be a normal space and let A be a closed subset of X. Then a continuous function $f : A \to I^n = [0,1]^n$ has an extension defined on X.*

Proof. By Tietze extension theorem $f_i = p_i \circ f : A \to I$ has an extension $\overline{f_i}$ defined on X. Now $\overline{f} = (\overline{f_1}, \cdots, \overline{f_n})$ is an extension of f on X. □