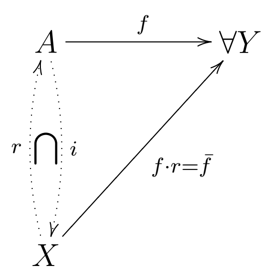
IV.5 AR and ANR

질문 1 In the Tietze extension theorem, which space Y can replace I? Such Y is said to have the **universal extension property**.

Example Let $Y = S^1$ and $f : S^1 \to S^1$. Then does the extension $\bar{f} : D^2 \to S^1$ exist?
존재하지 않으나 증명이 어려움. Algebraic topology의 개념이 필요하다. 그러나 $D^2 - \{0\}$으로의 extension은 존재한다.
그림 삽입

질문 2 For what pair (X, A), does the Tietze extension theorem hold for all Y?

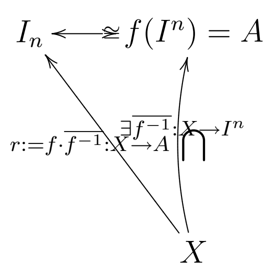


If A is a **retract** of X, i.e., there exists a **retraction** $r : X \to A$ such that $r \cdot i : A \hookrightarrow^i X \to^r A$ is an identity map, then question2 holds.

정의 1 A normal space A is called an **absolute retract(AR)** if for all $f : A \to X$ which is an embedding of A into a normal space X with $f(A)$ closed, $f(A)$ is a **retract** of X, i.e., $\exists$ a retraction $r : X \to f(A)$, namely $r \circ i = id$.

정리 1 *I^n is an absolute retract.*

증명
For any embedding $f : I^n \to X$ such that $f(I^n) = A \subset X$ is closed, let's show that A is a retract of X. Consider the following map :



Check that $r \cdot i = f \cdot \overline{f^{-1}} \cdot i = f \cdot f^{-1}$ =identity map. □

Remark In the proof we use only Theorem 2 of the previous section, and hence any space Y that can replace I^n in Theorem 2 becomes an AR. Hence a space with UEP is an AR. In fact the converse also holds.

Homework Prove that AR ⇔ UEP.

정의 2 A normal space A is called an **absolute neighborhood retract(ANR)** if for all $f : A \to X$ which is an embedding of A into a normal space X with $f(A)$ closed, $f(A)$ is a **neighborhood retract** of X, i.e., $\exists U$ an open neighborhood of $f(A)$ and a retraction $r : U \to f(A)$.

정리 2 *S^n is an absolute neighborhood retract.*

증명
For any embedding $f : S^n \to X$ such that $f(S^n) = A \subset X$ is closed, let's show that A is a neighborhood retract of X. Since $S^n \subset D^{n+1}$ and D^{n+1} has UEP, $\phi := i \circ f^{-1} : A \to D^{n+1}$ has an extension $\bar{\phi} : X \to D^{n+1}$. Now a retraction $r : D - 0 \to S^n$ induces a retraction $f \circ r \circ \bar{\phi} : \bar{\phi}^{-1}(D - 0) \to A$. □

Remark As for the AR case, the property of being an ANR is equivalent to having the corresponding extension property for some neighborhood.

Homework 1. A product is AR iff each factor is AR.
2. Same for a finite product of ANR's.