V.1 Paracompactness

Definition 1 Let X be a topological space and $\mathcal{U}$ be a collection of subsets of X. $\mathcal{U}$ is **locally finite** if for every $x \in X$ there exists a neighborhood V_x such that $\{U \in \mathcal{U} \mid U \bigcap V_x \neq \emptyset\}$ is a finite set.

Proposition 1 *If $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ is a locally finite collection of open sets, $\{\overline{U_\alpha} \mid \alpha \in J\}$ is also locally finite.*

Proof Proof follows from that $U_\alpha \bigcap V_x = \emptyset \Leftrightarrow \overline{U_\alpha} \bigcap V_x = \emptyset$. □

Proposition 2 *If $\{F_\alpha \mid \alpha \in J\}$ is a locally finite collection of closed sets in X, $\bigcup_{\alpha \in J} F_\alpha$ is closed in X*

Proof We prove that $(\bigcup_{\alpha \in J} F_\alpha)^c$ is open. Suppose $x \notin \bigcup_{\alpha \in J} F_\alpha$ and let V_x be an open neighborhood of x such that $\mathcal{C} = \{F_\alpha \mid F_\alpha \bigcap V_x \neq \emptyset\}$ is a finite collection. Then $F = \bigcup \{F_\alpha \mid F_\alpha \in \mathcal{C}\}$ is closed. Thus $W_x = V_x - F$ is an open neighborhood of x and W_x is contained in $(\bigcup_{\alpha \in J} F_\alpha)^c$. Hence $(\bigcup_{\alpha \in J} F_\alpha)^c$ is open. □

Proposition 3 *If $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ is locally finite, $\overline{(\bigcup_{\alpha \in J} U_\alpha)} = \bigcup_{\alpha \in J} \overline{U_\alpha}$*

Proof Let $\mathcal{F} := \{\overline{U_\alpha} \mid U_\alpha \in \mathcal{U}\}$ and $F := \bigcup_{\alpha \in J} \overline{U_\alpha}$. Since the collection $\mathcal{F}$ is locally finite, F is closed. Thus $\overline{(\bigcup_{\alpha \in J} U_\alpha)}$ is contained in F. Conversely it is clear that $\overline{(\bigcup_{\alpha \in J} U_\alpha)} \supset F$ □

Definition 2 Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X. $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

Remark When we consider an indexed collection of sets we allow the identical set to be indexed repeatedly. Actually an indexed collection is a family of pairs (U, α).

Definition 3 X is **paracompact** if every open covering of X has a locally finite open refinement that covers X.

Example 1. compact $\Rightarrow$ paracompact.
2. $\mathbb{R}^n$ is paracompact:
Let $X = \mathbb{R}^n$ and $\mathcal{A}$ be an open covering of X. Let $B_0 = \emptyset$ and B_m be an open ball of radius m centered at the origin for each $m = 1, 2, \cdots$. For B_m choose

$$A_1, \ldots, A_{k_m} \in \mathcal{A}$$

which covers $\overline{B_m}$. Let

$$A'_i = A_i \bigcap (X - \overline{B_{m-2}})$$

and

$$\mathcal{C}_m = \{A'_1, \dots, A'_{k_m}\}.$$

Now $\mathcal{C} = \bigcup \mathcal{C}_m$ is a refinement of $\mathcal{A}$ and a locally finite open covering of X.

Proposition 4 *Suppose that X is a paracompact space. For all open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ of X, there exists a locally finite precise open refinement $\mathcal{W} = \{W_\alpha \mid \alpha \in J\}$ of $\mathcal{U}$ that covers X.("precise" means that $\mathcal{U}$ and $\mathcal{W}$ have the same index set J and $W_\alpha \subset U_\alpha, \forall \alpha \in J$.)*

Proof Since X is paracompact, there exists a locally finite open refinement $\mathcal{V} = \{V_\beta \mid \beta \in K\}$ of $\mathcal{U}$. For each $\beta \in K$, there exists $\alpha \in J$ such that V_β is contained in U_α. Define

$$\varphi : K \to J \quad \text{as } \alpha = \varphi(\beta).$$

Also let

$$W_\alpha := \bigcup \{V_\beta \mid \alpha = \varphi(\beta)\}.$$

Then $W_\alpha \subset U_\alpha$ and $\mathcal{W} := \{W_\alpha \mid \alpha \in J\}$ is locally finite since $\mathcal{V}$ is locally finite.

□

Proposition 5 *If a topological space X is paracompact Hausdorff, X is normal.*

Proof

Step 1 : X is regular. Let A be a closed subset of X and x be any point which is not in A. Since X is a Hausdorff space, $x \in A^c$ and $a \in A$ can be separated by two disjoint open sets U_a and U_x so that $x \notin \overline{U_a}$. Define

$$\mathcal{U} = \{U_a \mid a \in A\}.$$

Then $\mathcal{U}$ is an open covering of A. Thus

$$\mathcal{U} \bigcup \{A^c\}$$

is an open covering of X. There exists a precise locally finite open refinement $\mathcal{V} \bigcup \{G\}$ of $\mathcal{U} \bigcup \{A^c\}$ that covers X, where

$$\mathcal{V} = \{V_a \mid a \in A\} \quad \text{and} \quad G \subset A^c.$$

Let V be the union of all the elements in $\mathcal{V}$. Now A is contained in V and thus in $\overline{V}$. Recall that $\overline{V} = \bigcup\{\overline{V_a} \mid a \in A\}$ since $\mathcal{V}$ is locally finite. Since each $\overline{V_a}$ is in $\overline{U_a}$,

$$A \subset V \subset \overline{V} \subset \bigcup \overline{U_a}.$$

So $\overline{V}$ is disjoint from x, namely we can separate x and A using V and $\overline{V}^c$.

Step2 : X is normal. Suppose A and B are two disjoint closed sets in X. Since X is regular, a point a of A and B can be separated by two open sets. Paracompactness of X enables us to construct a locally finite open covering of A which is disjoint from B. Repeat exactly the same procedure in *Step 1* to obtain two disjoint open neighborhoods of A and B. □

Proposition 6 *(Shrinking lemma) Suppose X is paracompact Hausdorff; Then for any collection $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ of open subsets of X which covers X, there exists a locally finite precise open refinement $\mathcal{V} = \{V_\alpha \mid \alpha \in J\}$ which covers X such that $V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in J$.*

Proof

For each x there exists U_α containing x and an open neighborhood O_x of x such that

$$x \in O_x \subset \overline{O_x} \subset U_\alpha.$$

Let

$$\varphi : X \to J \quad \text{as} \quad \alpha = \varphi(x).$$

Using Proposition 4, we can construct a precise locally finite open refinement $\mathcal{W} = \{W_x \mid x \in X\}$ of $\{O_x \mid x \in X\}$ which covers X. Let

$$V_\alpha = \bigcup\{W_x \mid \varphi(x) = \alpha\}$$

for each $\alpha \in J$. Note that

$$W_x \subset O_x \subset \overline{O_x} \subset U_\alpha.$$

Thus $V_\alpha \subset U_\alpha$. Now $\mathcal{V} = \{V_\alpha \mid \alpha \in J\}$ is a locally finite precise open refinement of $\mathcal{U}$ which covers X and

$$\overline{V_\alpha} = \bigcup_{\alpha=\varphi(x)} \overline{W_x} \subset \bigcup_{\alpha=\varphi(x)} \overline{O_x} \subset U_\alpha.$$

□

Definition 4 Let $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ be an open covering of X. An indexed family of continuous functions

$$\phi_\alpha : X \to [0, 1]$$

is said to be a **partition of unity** on X subordinate to $\{U_\alpha\}$ if

1. *support*ϕ_α is contained in U_α.
2. $\{$*support*$\phi_\alpha \mid \alpha \in J\}$ is locally finite.
3. $\Sigma_\alpha \phi_\alpha(x) = 1$ for each x.

Remark *support* f is the closure of $\{x \in X \mid f(x) \neq 0\}$

Theorem 7 *(Existence of partition of unity) If X is a paracompact Hausdorff space, any open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ has a partition of unity $\{f_\alpha\}$ subordinate to $\mathcal{U}$.*

Proof Shrink $\mathcal{U}$ to get a precise locally finite open refinement $\mathcal{V} = \{V_\alpha\}$ that covers X. Shrink $\mathcal{V}$ once more to get $\mathcal{W} = \{W_\alpha\}$ using the shrinking lemma. Thus

$$\overline{W_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha \text{ for each } \alpha \in J.$$

By Urysohn's lemma, there exists $g_\alpha : X \to [0, 1]$ such that $g_\alpha(\overline{W_\alpha}) = \{1\}$ and $g_\alpha(V_\alpha^c) = \{0\}$. If $W_\alpha = \emptyset$, $g_\alpha \equiv 0$. Since *support* $g_\alpha \subset \overline{V_\alpha} \subset U_\alpha$, $\{$*support* $g_\alpha \mid \alpha \in J\}$ is locally finite. Thus $\Sigma_\alpha g_\alpha$ is a well-defined continuous function such that $\Sigma_\alpha g_\alpha \geq 1$ since $\mathcal{W} = \{W_\alpha\}$ is a covering and $g_\alpha(\overline{W_\alpha}) = 1$. Define

$$f_\alpha := \frac{g_\alpha}{\Sigma g_\alpha}$$

then

$$\textit{support } f_\alpha = \textit{support } g_\alpha \text{ and } \Sigma f_\alpha \equiv 1.$$

□

Remark 1. A product of paracompact spaces need not to be paracompact.($\mathbb{R}^J$) Also a subspace of paracompact space need not to be paracompact, but a closed subspace is paracompact obviously.
2. See Munkres for ⌜ Stone's theorem ⌟ and ⌜ Smirnov metrization theorem ⌟.

Homework Show that the product of paracompact space and a compact space is paracompact.