V.2 Local compactness

Definition 1 A topological space X is **locally compact** if for each $x \in X$, there exists an open neighborhood U of x such that $\overline{U}$ is compact(i.e., $\exists$a relatively compact open neighborhood).

Example $\mathbb{R}^n$ is locally compact since any open ball is relatively compact.

Example $\mathbb{Q}$ is not locally compact.

Definition 2 Suppose X is a locally compact Hausdorff space and ∞ is a point which is not in X. Let $Y = X \bigcup \{\infty\}$ with a topology consisting of the sets of the following 2 types :

1. U : open subset of X
2. $Y - C$: C is a compact subset of X

That is, $\mathcal{T}(Y) = \mathcal{T}(X) \bigcup \{Y - C \mid C \text{ is compact in } X\}$. Such a topological space $(Y, \mathcal{T}(Y))$ is called an **one point compactification** of X.

Example $X = \mathbb{R}^n$, $Y = X \bigcup$ North Pole $\cong \mathbb{S}^n$

[Figure]

Lemma 1 $\mathcal{T}(Y)$ *in the above definition is indeed a topology.*

Proof
1. $\emptyset$ and Y are clearly in $\mathcal{T}(Y)$.
2. The union of the elements of any subcollection of $\mathcal{T}(Y)$ is in $\mathcal{T}(Y)$: Suppose that $\{U_\alpha\}$ is a collection of open sets in X and $\{C_\beta\}$ is a collection of compact sets in X.

a. Type I sets : $\bigcup U_\alpha \subset \mathcal{T}(X) \subset \mathcal{T}(Y)$

b. Type II sets : $\bigcup(Y - C_\beta) = \bigcup C_\beta^c = (\bigcap C_\beta)^c = Y - \bigcap C_\beta$. Since $\bigcap C_\beta$ is compact, $Y - \bigcap C_\beta \in \mathcal{T}(Y)$. (Note that since X is a Hausdorff space, any compact set of X is closed.)

c. Mixed type sets : $U \bigcup (Y - C) = (C - U)^c \in \mathcal{T}(Y)$ since $C - U$ is compact.

3. The intersection of two elements of $\mathcal{T}(Y)$ is in $\mathcal{T}(Y)$: This follows easily using similar arguments as above. $\square$

Theorem 2 *Suppose X is locally compact Hausdorff and $\hat{X}$ is an one-point compactification of X. Then*

1. *$\hat{X}$ is compact Hausdorff*

2. *The subspace topology for X is equal to the original topology of X, and X is open in $\hat{X}$.*

3. *$\bar{X} = \hat{X}$ if X is NOT compact.*

Proof
1. $\hat{X}$ is compact Hausdorff :

   a. $\hat{X}$ is compact : Let $\mathcal{U}$ be an open covering of $\hat{X}$, then there exists $U \in \mathcal{U}$ which contains ∞. Since U is a type II open set, we can find a compact set $C \subset X$ such that $U = \hat{X} - C$. Let $\mathcal{V} \subset \mathcal{U}$ be a finite subcovering for C, then $\mathcal{V} \bigcup \{Y - C\}$ is a finite subcover for $\hat{X}$. Hence $\hat{X}$ is compact.

   b. $\hat{X}$ is Hausdorff.
   Since X is locally compact, there exists a relatively compact open neighborhood U_x of a given $x \in X$. U_x and $\hat{X} - \bar{U_x}$ are separating open neighborhoods of x and ∞.

2. It is immediate from the construction.
3. If X is NOT compact, ∞ becomes an accumulation points of X. If X is compact, $\{\infty\}$ is an open set and ∞ is an isolated point. $\square$

Proposition 3 *Let X be locally compact Hausdorff. Suppose we have an embedding into a compact Hausdorff $\hat{Y}$,*

$$f : X \to^{\cong} Y \subset \hat{Y} \quad \textit{with} \quad \hat{Y} - Y = \{y\}.$$

Then there exists a homeomorphism $\bar{f} : \hat{X} \to^{\cong} \hat{Y}$ such that

$$\bar{f}|_X = f \quad \textit{and} \quad \bar{f}(\infty) = y.$$

That is, one-point compactification is unique upto homeomorphism.

Proof
1. $\bar{f}$ is bijective : It is clear from the fact that f is a homeomorphism between X and Y.

2. $\bar{f}$ is continuous : Let U_y be an open neighborhood of y. It is sufficient to show that $\bar{f}^{-1}(U_y)$ is open. U_y^c is compact in Y. $f^{-1}(U_y^c)$ is compact in X since f is a homeomorphism between X and Y. Thus

$$\bar{f}^{-1}(U_y) = \hat{X} - [f^{-1}(U_y^c)]$$

is open in $\hat{X}$.
3. Now $\bar{f}$ is a homeomorphism since $\hat{X}$ is compact and $\hat{Y}$ is Hausdorff. $\square$

Example 1. $\widehat{[0,1]} = [0,1] \bigcup \{\infty\}$
2. $\mathbb{R}^2 \bigcup \{\infty\} \cong \mathbb{S}^2$
3. $\mathbb{R}^n \bigcup \{\infty\} \cong \mathbb{S}^n$
4. $(0,1) \cong \mathbb{R} \Rightarrow \widehat{(0,1)} \cong \hat{\mathbb{R}} \cong \mathbb{S}^1$
5. $\widehat{[0,1)} \cong [0,1]$
6. [**Figure**] 십자선
7. $\mathbb{D}^2 \cong \mathbb{R}^2 \Rightarrow \hat{\mathbb{D}}^2 = \mathbb{S}^2$
8. [**Figure**] Annulus = 가운데 빈 원통 = 극점이 빠진 sphere $\Rightarrow =$ pinched sphere.

Proposition 4 *If X is Hausdorff, the followings are equivalent :*

1. *X is locally compact.*
2. *For given x in X and any neighborhood U_x of x, there is a relatively compact open neighborhood V_x of x such that $x \in V_x \subset \overline{V_x} \subset U_x$.*
3. *For given a compact set $C \subset X$ and any neighborhood U of C, there is a relatively compact open neighborhood V of C such that $C \subset V \subset \bar{V} \subset U$.*
4. *X has a basis consisting of relatively compact open sets.*

Proof We proceed as $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$.
($1 \Rightarrow 3$) Since $\hat{X}$ is compact Hausdorff, $\hat{X}$ is normal. Hence there is an open set V such that $C \subset V \subset \bar{V} \subset U$. For $\bar{V}$ is a closed subset of a compact space $\hat{X}$, $\bar{V}$ is compact.
($3 \Rightarrow 2$) Clear.
($2 \Rightarrow 4$) Clear.
($4 \Rightarrow 1$) Clear. $\square$

Theorem 5 *Suppose X is locally compact Hausdorff. Then X is completely regular.*

Proof Let D be a closed subset of X and $x \notin D$. We want to show the existence of a continuous function $f : X \to [0,1]$ such that $f(D) = 0$ and $f(x) = 1$. Since the one point compactification $\hat{X}$ is normal, we want to apply Urysohn lemma. Note that D may not be closed in $\hat{X}$, but $D \bigcup \{\infty\}$ is closed since $\hat{X} - (D \bigcup \{\infty\}) = X - D$ is an open subset of X and hence of $\hat{X}$. Now apply Urysohn lemma to the disjoint closed sets $D \bigcup \{\infty\}$ and $\{x\}$. $\square$

Remark

$$
\begin{tikzcd}
metric \arrow[d, Rightarrow] & \\
paracompact\, Hausdorff \arrow[d, Rightarrow] & compact\, Hausdorff \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
normal \arrow[d, Rightarrow] \arrow[r, Rightarrow, "/" marking] & locally\, compact\, Hausdorff \arrow[dl, Rightarrow] \\
completely\, regular \arrow[d, Rightarrow] & \\
regular \arrow[d, Rightarrow] & \\
Hausdorff \arrow[d, Rightarrow] & \\
T_1 &
\end{tikzcd}
$$

Homework (1) Open or closed supspace of a locally compact Hausdorff space is locally compact Hausdorff.
(2) X is locally compact Hausdorff iff X is an open subspace of a compact Hausdorff space.
(3) A product of completely regular spaces is completely regular.