V.3 Quotient Space

Suppose we have a function $p : X \to Y$ from a topological space X onto a set Y. we want to give a topology on Y so that p becomes a continuous map.

Remark If we assign the indiscrete topology on Y, any function $p : X \to Y$ would be continuous. But such a topology is too trivial to be useful and the most interesting one would be the finest topology.

Definition 1 (1^{st} definition) Given $p : X \to Y$, a function from a topological space X onto a set Y, **the quotient topology of Y induced by** p is the finest topology which makes p continuous, i.e., $\mathcal{T}(Y) = \{U \subset Y \mid p^{-1}(U) \text{ is open }\}$. Y with $\mathcal{T}(Y)$ is called a quotient space.

Exercise Show that the topology on Y defined in the above definition is indeed a topology.

Definition 2 Suppose that X and Y are topological spaces. Also suppose p is a continuous, surjective map from X onto Y. Then p is called a quotient map provided U is open in Y iff $p^{-1}(U)$ is open in X. In this case topology of Y is a quotient topology.

Exercise 1. p is a quotient map iff $C \subset Y$ is closed $\Leftrightarrow p^{-1}(C)$ is closed.
2. A composite of two quotient maps is quotient.

Remark A quotient map is continuous and 'partially' open. (A continuous and open map is a quotient map, but not vice versa.)

Example A projection map is a quotient map since it is continuous and open.

Definition 3 (2^{nd} definition) Suppose X is a topological space and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the canonical surjective map which sends a point x in X to the element of X^* containing x. X^* with quotient topology is called a quotient space.

Example 1. Projection
2. Consider $\mathbb{R} = (-\infty, 0) \bigcup [0, \infty)$. Let $R^* = \{0, 1\}$. Then $\mathcal{T}(R^*) = \{\{0\}, \emptyset, X^*\}$ is a quotient topology.

Theorem 1 *The following diagram is called a commutative diagram of functions if q is a quotient map.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Z \\ Y \arrow[ur, "h"'] & \end{tikzcd}$$

Then f is continuous iff h is continuous.

Proof
($\Leftarrow$) Clear since q is a quotient map.
($\Rightarrow$) Let U be an open set in Z. Note that $h^{-1}(U)$ is open in $Y \Leftrightarrow g^{-1}(h^{-1}(U))$ is open in X. $g^{-1}(h^{-1}(U)) = f^{-1}(U)$ is open. $\square$

Example Consider $X = [0,1]$ and a partition $X^* = X/\{0,1\}$. Is X^* homeomorphic to S^1?

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\cong"] & S^1 \\ X^* \arrow[ur, "\exists h"'] & \end{tikzcd}$$

Recall the followings :

1. The image of a compact space under a continuous map is compact.
2. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

From the fact that X is compact, X^* is compact. Further from the fact that S^1 is Hausdorff, any continuous map from X^* to S^1 is a homeomorphism.

Example 1. Suppose $\mathbb{R}^{2*} = D^2/R^2 - D^2$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[d, "q"'] & S^2 \\ \mathbb{R}^{2*} \arrow[ur, "\exists h"'] & \end{tikzcd}$$

2. **[Figure]** 사각형, 원통, Moebius band, torus, Klein bottle
3. **[Figure]** 두 개의 D^2 : S^2

Remark $p : X \to Y$ is a quotient map $\not\Rightarrow p|_A : A \to p(A)$ is a quotient map. (Consider $f : I \to S^1$ with $A = [0,1)$.)

Exercise Show that this holds if A is open and p is an open map. (Or if A is closed and p is a closed map.)

Definition 4 (3^{rd} definition) Let X be a space and $\sim$ be an equivalence relation on X. Then $\sim$ gives a partition on X. Let $[x] = \{y \in X | y \sim x\}$ be the equivalence class of x. Define

$$Y := X/\sim = \{\ [x] \mid x \in X\}$$

with canonical $p : X \to X/\sim$. Such Y is called a quotient space of X with respect to $\sim$.

Remark Given a partition X^*, define $x \sim y$ iff x and y belong to the same partition element. Then $\sim$ is an equivalence relation. Conversely $\sim$ gives a partition.

This description is useful when we consider a group action.

Definition 5 Let X be a space; let G be a topological group. An **action** of G on X is a continuous map $\alpha : G \times X \to X$, denoting $\alpha(g \times x)$ by $g \cdot x$, such that

(i) $e \cdot x = x$ for all $x \in X$,

(ii) $g_1 \cdot (g_x \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Define an equivalence relation as $x \sim y$ iff x and y belong to the same **orbit**, namely $y = g \cdot x$ for some $g \in G$. The resulting quotient space is denoted by

$$X/G := \{G \cdot x \mid x \in X\} \text{ where } G \cdot x := \{g \cdot x \mid g \in G\}$$

and called the **orbit space** of the action α.

Example 1. 구면위의 점을 위도선을 따라 회전시키는 group action을 생각해보자. 이 때 각각의 위도선은 orbit을 이루며 이러한 orbit들의 집합, 즉 위도선들의 집합은 orbit space를 이룬다. 또한 위도선들로 이루어진 orbit space는 북극과 남극을 잇는 경도선과 homeomorphic함을 알 수 있다. [**Figure**] **지구 표면**

2. Define $\tau : x \mapsto x+1$ on $X = \mathbb{R}$ and let $G = <\tau> = \{\tau^n : x \mapsto x+n \mid n \in \mathbb{Z}\}$. Then the orbit space $\mathbb{R}/G$ is homeomorphic to S^1. Indeed define $f : \mathbb{R} \to S^1$ by $x \mapsto e^{2\pi i x}$. Then

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d] & S^1 \subset \mathbb{C} \\ \mathbb{R}/G \arrow[ur] & \end{tikzcd}$$

3. On $\mathbb{R}^2$ define $\sigma : (x, y) \mapsto (x + 1, y)$ and $\tau : (x, y) \mapsto (x, y + 1)$. Note that $\sigma \cdot \tau = \tau \cdot \sigma$. Let

$$G :=< \sigma, \tau >\cong \mathbb{Z} \bigoplus \mathbb{Z} = \mathbb{Z}^2$$

and

$$f : (x, y) \mapsto (e^{2\pi ix}, e^{2\pi iy}).$$

Then

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[d] & S^1 \times S^1 \\ \mathbb{R}^2/G \arrow[ur] & \end{tikzcd}$$

Similarly $\mathbb{R}^n/\mathbb{Z}^n = \overbrace{S^1 \times \ldots \times S^1}^{n} = T^n$.

Homework Suppose G acts on X. Show that the orbit map $p : X \to X/G$ is open.

Example Let $X = [0, 1]$ and identify all the rationale points. Or equivalently define an equivalence relation $\sim$ as

$$x \sim y \Leftrightarrow x, y \in \mathbb{Q} \bigcap [0, 1].$$

Then $X^* = X/\sim$ is not Hausdorff. To show this, consider a quotient map q such that

$$q(x) = \begin{cases} * & \text{if } x \in \mathbb{Q} \\ x & \text{otherwise} \end{cases}$$

Then for any open neighborhood U of x in X^*, $q^{-1}(U)$ is an open neighborhood of x and contains a rationale point. Therefore $* \in U = p(p^{-1}(U))$

Proposition 2 *Suppose X is regular and C is a closed subset of X. Then X/C is Hausdorff.*

Proof First suppose $x, y \notin C$. Then there are two open neighborhoods U_x and U_y separating x and y disjoint from C. Thus $q(U_x)$ and $q(U_y)$ give a separation for $q(x)$ and $q(y)$.
Now suppose $q(C) = *$, $x' = q(x)$, $x \notin C$. Then there are open neighborhoods U_x and V separating x and C. Thus $q(U_x)$ and $q(V)$ give disjoint neighborhoods of $*$ and x'. $\square$

Example Identify the interval $[0,1]$ in $\mathbb{R}$ to a single point. Then $\mathbb{R}/[0,1]$ is Hausdorff and in fact $\cong \mathbb{R}$. However $\mathbb{R}/[0,1)$ is not Hausdorff because $* = q\Big([0,1)\Big)$ and $\{1\}$ cannot be separated.

Homework Let X be a normal space and $p : X \to Y$ be a quotient map. Show that Y is normal if p is closed.