S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 3, стр. 67–70 (2006) УДК 514.132, 514.133

Краткие сообщения MSC 57M27, 57M25, 58J28

ON CHERN-SIMONS INVARIANTS OF GEOMETRIC 3-MANIFOLDS

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We consider 3-dimensional cone-manifolds, which admit complete Riemannian metric of constant curvature. 3-dimensional cone-manifolds with singular knots and links are investigated actively in recent years (see e.g. [1]–[10]). Very important geometric invariants of nonsingular manifolds are the volume Vol and the Chern-Simons invariant CS of manifolds. By the Mostow Rigidity Theorem [11] the Vol as well as CS is a topological invariant. It occurs that these invariants are extended on cone-manifolds. This generalization was reached by efforts of Kojima, Montesinos-Amilibia, Mednykh and others.

The CS-invariants of 2-bridge knots cone-manifolds were found by Hilden, Lozano and Montesinos-Amilibia [2] (they presented formulas with implicit function as integrand). Explicit integrand formulas for the volumes of Whitehead link cone-manifolds were obtained by Mednykh and Vesnin [10].

In the present talk we investigate the Chern-Simons invariant of a Whitehead link cone-manifold $W(\alpha, \beta)$ with cone angles α and β along the link components (figure 1).

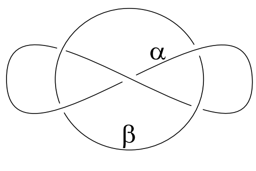

FIGURE 1. Whitehead link cone-manifold $W(\alpha, \beta)$

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This research was partially supported by the Russian Foundation for Basic Research (grant 06–01–00153) and Scientific Schools Support Program (grant SS–311.2003.1).
Communicated by A.D.Mednykh February 27, 2006, published March 1, 2006.

First of all to calculate the $CS(W(\alpha,\,\beta))$ we have to find a generalized Chern-Simons function $I(W(\alpha,\,\beta))$ at first (see [3] for formal definition of number I). The function $I(W(\alpha,\,\beta))$ has important property — it satisfys the analog of classic Schlafli formula for a torsion:

Theorem 1. (The analog of Schläfli formula for a torsion)[3] *Let $C_{\theta(t)}$ be a family of hyperbolic cone-manifold structures of curvature K in $\mathbb{S}^3$, where $\theta(t)$ and $\phi(t)$ are differentiable functions of t. Then the following equation between differential forms holds*

$$KdI(C_{\theta(t)}) = \frac{1}{4\pi^2}\,\phi\,d\theta,$$

where ϕ is imaginary part of the complex length and θ is the cone angle along the singular geodesic.

Let $W(\alpha,\,\beta)$ be a Whitehead link cone-manifold with cone angles α and β along the link components. By the above theorem we have in this case:

$$KdI(W(\alpha,\,\beta)) = \frac{1}{4\pi^2}\,(\phi_\alpha\,d\alpha + \phi_\beta\,d\beta) \tag{1}$$

An imaginary part of complex lengths of singular geodesics ϕ_α and ϕ_β can be found from trigonometrical identyties for Whitehead link, obtained in [9].

Due to this formula (1) we prove the following two theorems, which give an explicit formulas for the generalized Chern-Simons function $I(W(\alpha,\,\beta))$ on the family of cone-manifold structures $W(\alpha,\beta)$ in hyperbolic and spherical cases:

Theorem 2. *Let $W(\alpha,\beta)$ be a hyperbolic Whithead link cone-manifold. Then the number $I(W(\alpha,\beta))$ is given by the formula:*

$$I(W(\alpha,\beta)) = \int_{\zeta_1}^{-1} F(\zeta,A,B)d\zeta + \int_{\zeta_2}^{-1} F(\zeta,A,B)d\zeta - \\ - \left(\frac{\pi-\alpha}{2\pi}\right)^2 - \left(\frac{\pi-\beta}{2\pi}\right)^2 + C,$$

where $F(\zeta,A,B) = \dfrac{1}{2\pi^2(\zeta^2-1)}\log\left[\dfrac{2(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta^2-\zeta^3)}\right]$, $A = \cot\dfrac{\alpha}{2}$, $B = \cot\dfrac{\beta}{2}$, $C = \dfrac{11}{24}$, $\zeta_1 = z$, $\zeta_2 = \overline{z}$, $Im(z) > 0$ *and z is a root of the cubic equation*

$$z^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)z^2 - A^2B^2z + A^2B^2 = 0.$$

Note, that C is the constant, which importance was shown in the paper of Coulson, Goodman, Hodgson and Neumann [1].

The formulation of the second theorem is almost the same. It differs by the word spherical and by properties of the roots.

Theorem 3. *Let $W(\alpha,\beta)$ be a spherical Whithead link cone-manifold. Then the number $I(W(\alpha,\beta))$ is given by the formula:*

$$I(W(\alpha,\beta)) = \int_{\zeta_1}^{-1} F(\zeta,A,B)d\zeta + \int_{\zeta_2}^{-1} F(\zeta,A,B)d\zeta -$$

$$-\left(\frac{\pi-\alpha}{2\pi}\right)^2 - \left(\frac{\pi-\beta}{2\pi}\right)^2 + C,$$

where $F(\zeta,A,B) = \dfrac{1}{2\pi^2(\zeta^2-1)} \log\left[\dfrac{2(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta^2-\zeta^3)}\right]$, $A = \cot\dfrac{\alpha}{2}$, $B = \cot\dfrac{\beta}{2}$, $C = \dfrac{11}{24}$, $\zeta_1 = z_1$, $\zeta_2 = z_2$ *and* z_1, z_2, $0 \leqslant z_1 < z_2$ *are real roots of the cubic equation*

$$z^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)z^2 - A^2B^2z + A^2B^2 = 0.$$

Remark 1. *Let $W\left(\dfrac{2\pi}{n},\dfrac{2\pi}{m}\right)$ be a hyperbolic or a spherical Whithead link orbifold. Then the Chern-Simons invarint of $W\left(\dfrac{2\pi}{n},\dfrac{2\pi}{m}\right)$ is given by the formula*

$$CS(\mathbb{S}^3, W, n, m) = I\left(W\left(\frac{2\pi}{n},\frac{2\pi}{m}\right)\right) \left(mod\, \frac{1}{l.c.m.\{n,m\}}\right)$$

This remark is based on ideas which were shown by Hilden, Lozano and Montesinos-Amilibia [3].

Then the following theorem gives an useful application of orbifolds CS-invariants for calculating of manifolds classical CS-invariants.

Theorem 4. *Let $M_n(W)$ be the n-cyclic cover of $\mathbb{S}^3$ branched over the Whitehead link W. Then the CS-invariant of the manifold $M_n(W)$ can be obtained by the following formula:*

$$CS(M_n(W)) = n\, CS(\mathbb{S}^3, W, n, n)\ (mod\, 1)$$

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