Dehn fillings on 3-manifolds

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References

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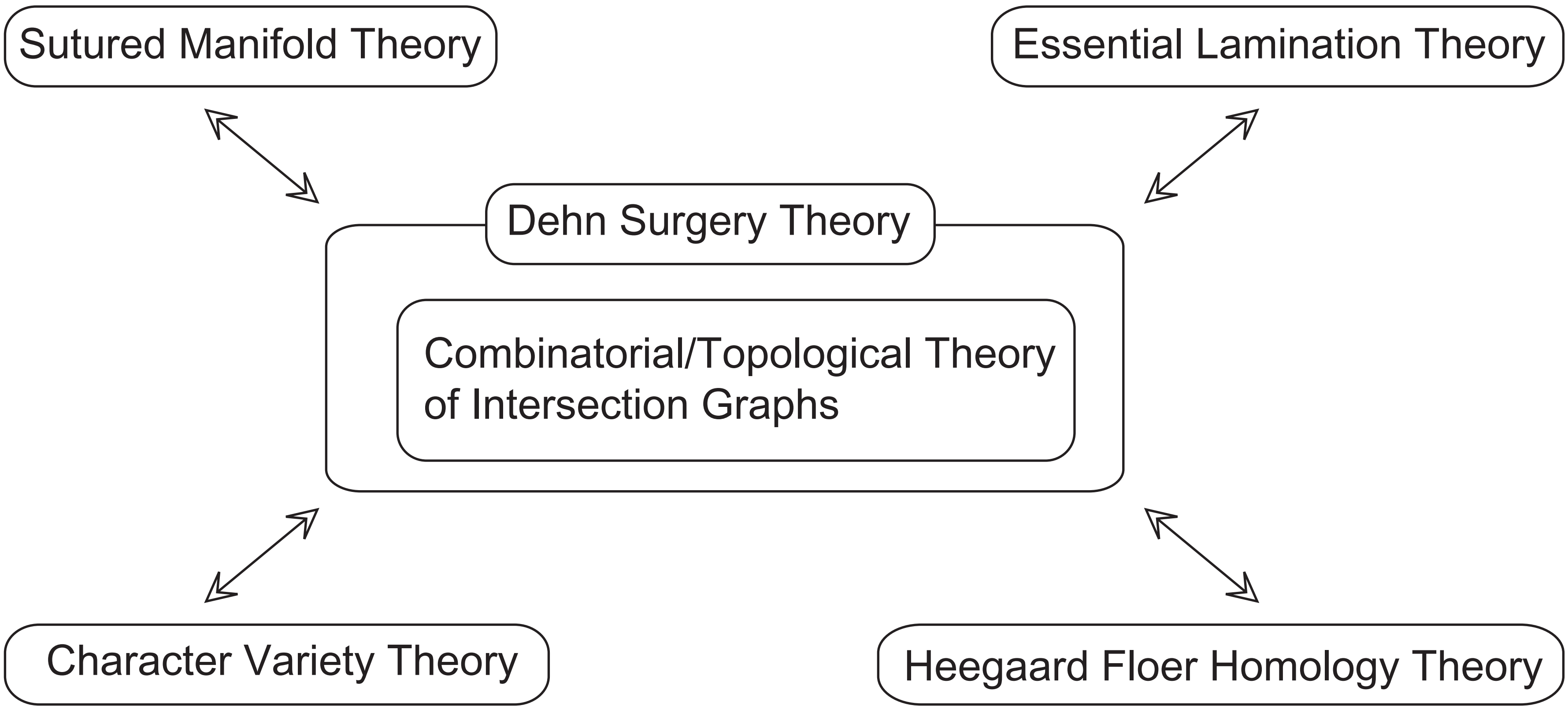
- C. McA. Gordon, *Small surfaces and Dehn filling*, Proceedings of the Kirbyfest, Geom. Topol. Monogr. **2** (1998), 177–199.

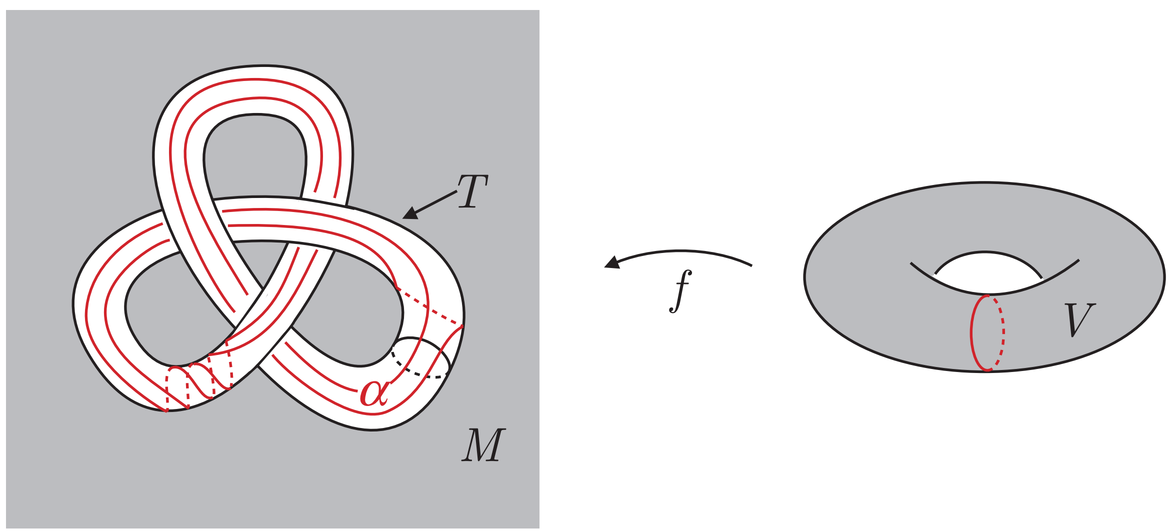
- S. Boyer, Dehn surgery on knots, Chapter 4 of the Handbook of Geometric Topology, R.J. Daverman, R.B. Sher, ed., Amsterdam, Elsevier, 2002.

Machineries for Dehn surgery theory



Dehn fillings

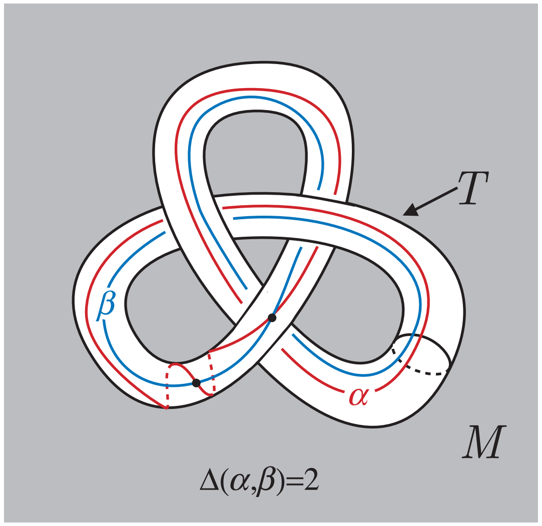
M is a compact, connected, orientable 3-manifold with a torus boundary component T.



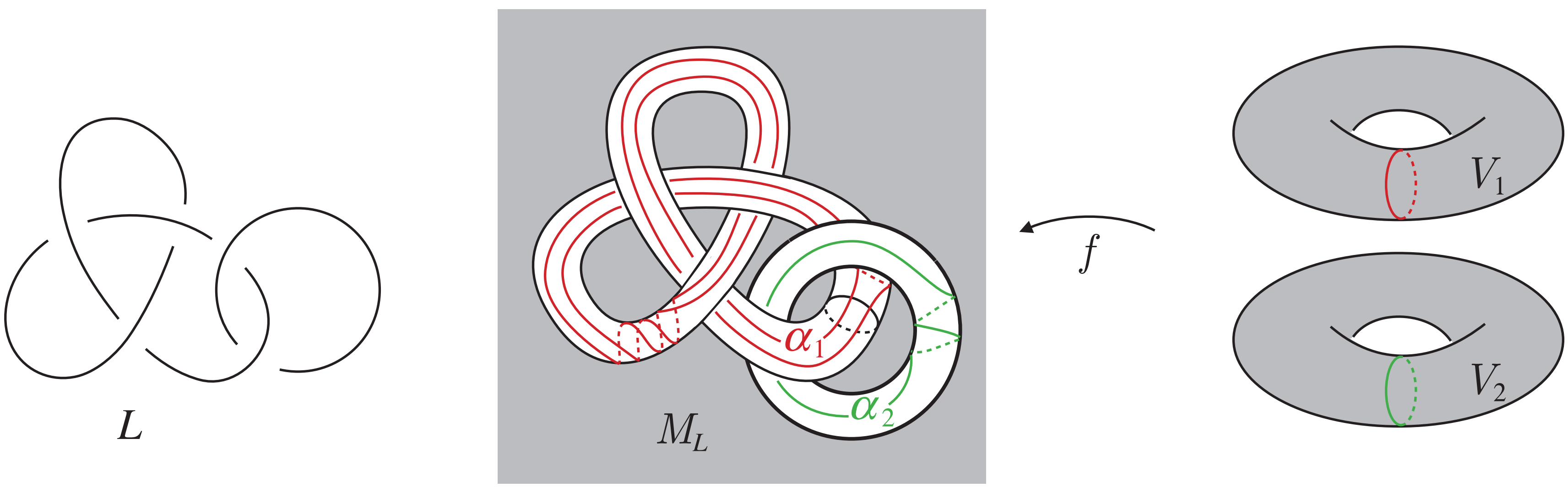
$$M(\alpha) = M \cup_f V$$

Slopes

The *slope* of an essential circle on T is its isotopy class ($T \subset \partial M$).
Let α, β be two slopes on T.
$\Delta(\alpha, \beta)$:= minimal geometric intersection number of α and β.



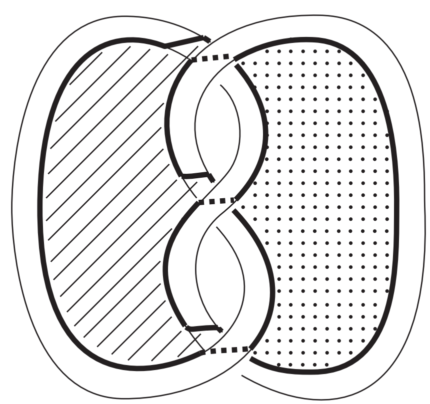
Dehn surgeries = deleting + filling



$$L(\alpha_1, \alpha_2) = M_L \cup_f (V_1 \cup V_2)$$

Parameterizing slopes

K : a knot in S^3, $E(K) = S^3 - \text{int}N(K)$
μ, λ : meridian and longitude $\subset \partial E(K)$
α : an essential simple closed curve in $\partial E(K)$
$\alpha \sim m\mu + l\lambda$ for some coprime integers m, l
{slopes} $\leftrightarrow \mathbb{Q} \cup \{1/0\}$
$\alpha \leftrightarrow m/l$



Realizing 3-manifolds by Dehn surgery

A set of *surgery data* $(L; \alpha_1, \ldots, \alpha_n)$: a link $L = K_1 \cup \ldots \cup K_n$ together with a slope α_i for each component K_i.

$L(\alpha_1, \ldots, \alpha_n) =$ the manifold obtained by performing the Dehn surgeries prescribed the surgery data $(L; \alpha_1, \ldots, \alpha_n)$.

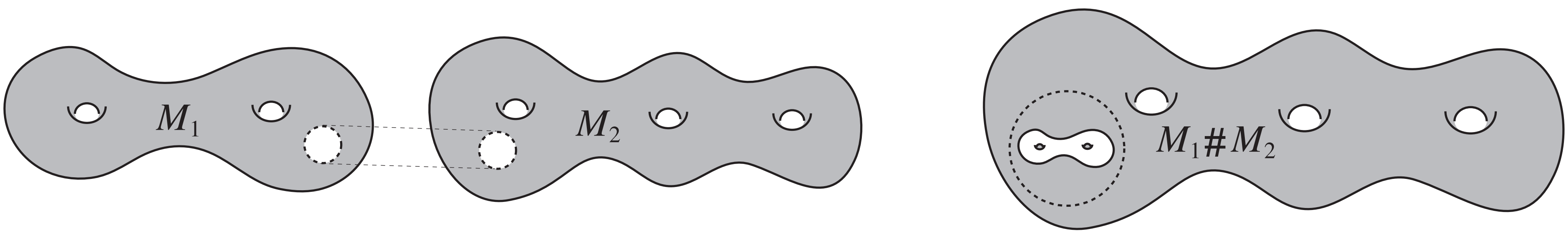
Theorem (Lickorish, Wallace, 1960). *Every closed connected orientable 3-manifold M is homeomorphic to $L(\alpha_1, \ldots, \alpha_n)$ for some n-component link L in S^3.*

Essential surfaces

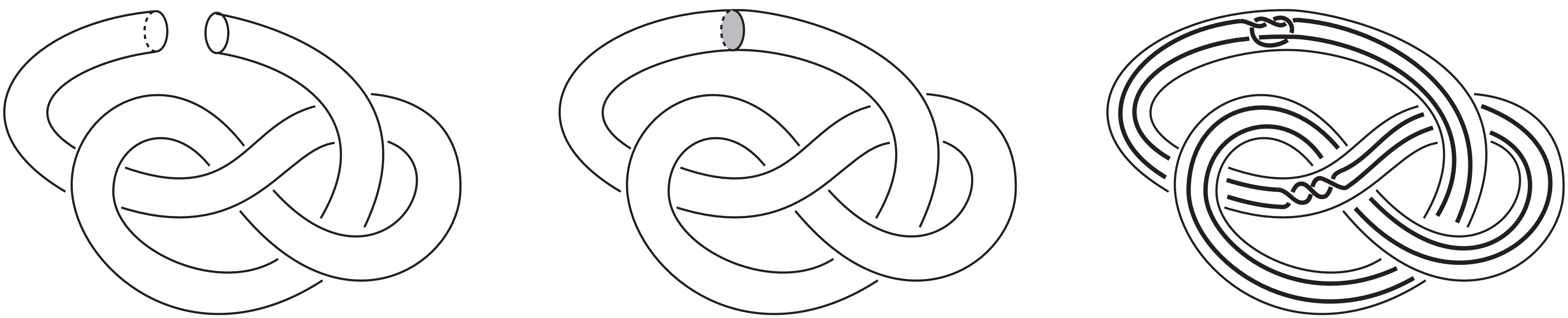
A 2-sphere S in M is *essential* if S does not bound a 3-ball in M (and M is called *reducible*). If M is not reducible, M is called *irreducible*.

e.g.)

- $S = S^2 \times \{\mathrm{pt}\} \subset S^2 \times S^1$
- S : a decomposing sphere in $M_1 \# M_2$



$F(\subset M, \not\cong S^2)$ is *compressible* if $\exists D$ in M such that $D \cap F = \partial D$ is not contractible in F. Otherwise, *incompressible*.



A properly embedded surface $F(\neq S^2)$ in M is *essential* if incompressible and not parallel into ∂M.

A 3-manifold X is said to be *prime* if $X = P\#Q \Rightarrow P = S^3$ or $Q = S^3$.

Prime Decompositon Theorem (Kneser, Milnor). Any compact orientable 3-manifold M has a prime decomposition, i.e. $M = P_1\# \cdots \#P_n$ (P_i's are prime).

Torus Decomposition Theorem (Jaco and Shalen, Johannson). Any irreducible 3-manifold M contains a finite collection of disjoint incompressible tori $T_1, \ldots, T_n$ such that each component of $M - \mathrm{Int}N(T_1 \cup \ldots \cup T_n)$ is either Seifert fibered or atoroidal.

Topological rigidity of Haken 3-manifolds

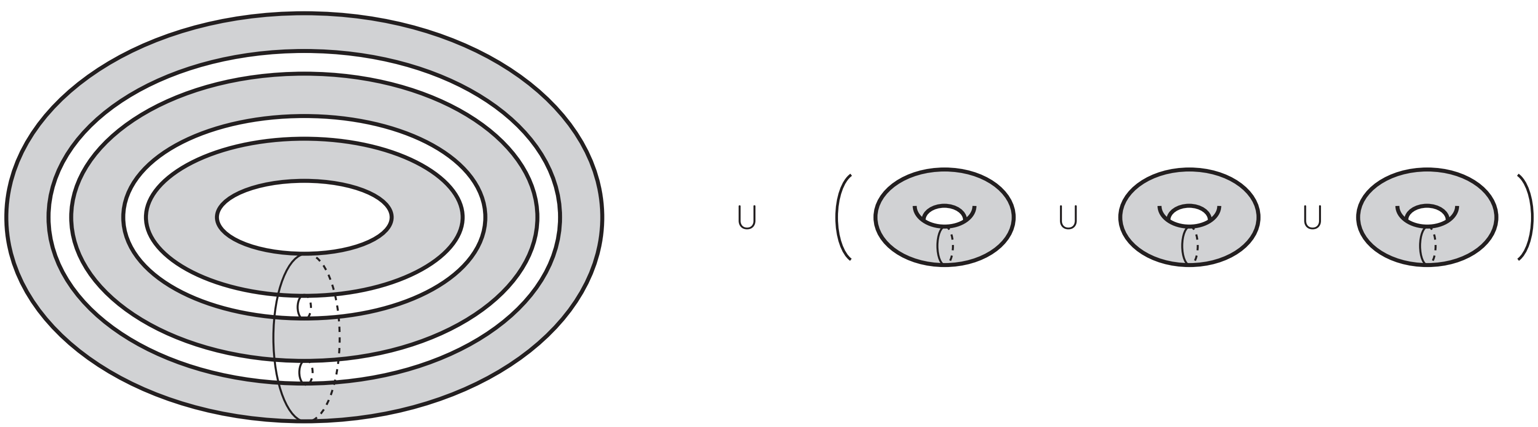
A *Haken 3-manifold* is a compact irreducible 3-manifold that contains an incompressible surface.

Theorem (Waldhausen). *Haken 3-manifolds are determined up to homeomorphism by their fundamental groups.*

cf. $L(5,1) \not\cong L(5,2)$

Small Seifert Fiber Spaces

Every small fiber space can be obtained from $P \times S^1$ by suitably performing Dehn filling three times, where P is a pair of pants.



Background(Thurston's work)

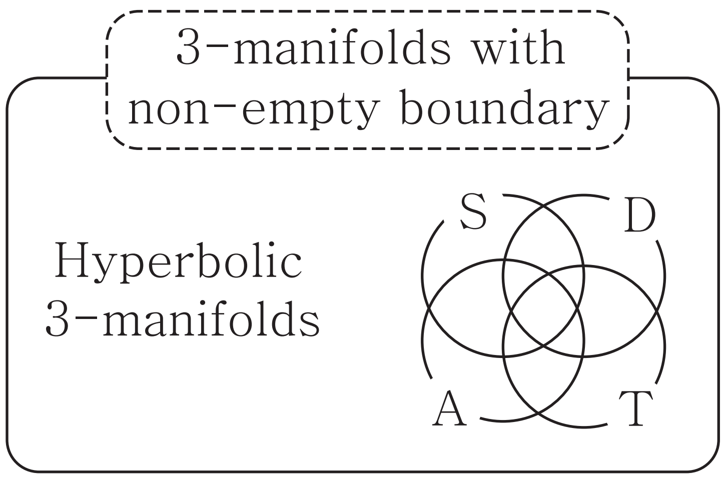
A compact orientable 3-manifold M is *hyperbolic* if M with its boundary tori removed has a finite volume complete hyperbolic structure.

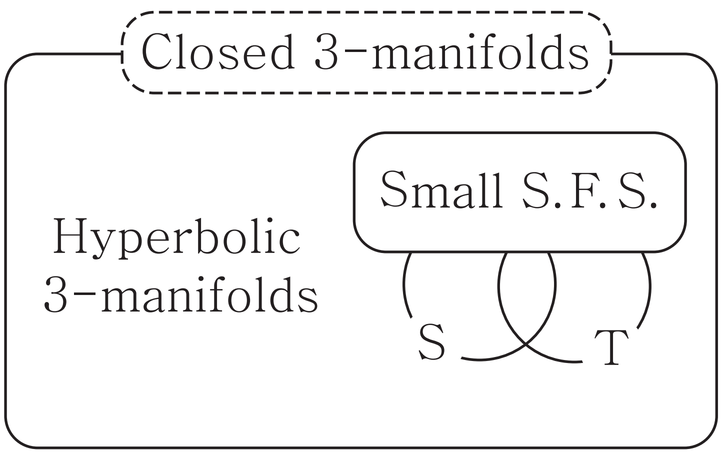
Theorem (Hyperbolic Dehn Surgery Theorem). *If M is a hyperbolic 3-manifold with a torus boundary component T, then $M(\alpha)$ are hyperbolic for all but finitely many slopes α on T.*

Theorem (Geometrization Theorem for Haken manifolds). *A compact 3-manifold with non-empty boundary is not hyperbolic if and only if it is reducible($\mathcal{S}$), boundary-reducible ($\mathcal{D}$), annular ($\mathcal{A}$), or toroidal ($\mathcal{T}$).*

Geometrization Conjecture A closed 3-manifold is not hyperbolic if and only if it is reducible, toroidal, or a small Seifert fiber space.

Geometrization Theorem for Haken manifolds and Geometrization Conjecture





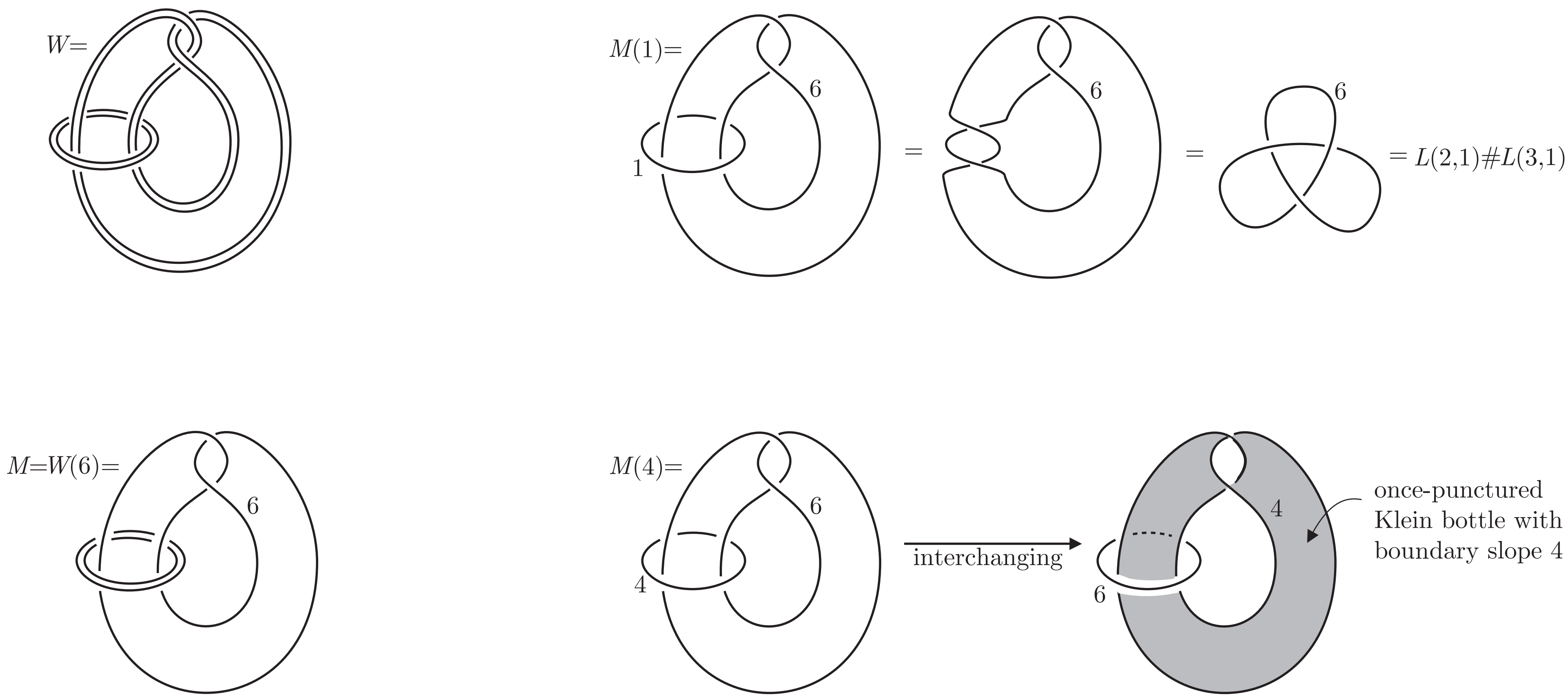
Known results

$\Delta \leq ?$	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Upper bounds for Δ

For example, $\Delta(\mathcal{S}, \mathcal{T}) \leq 3$ means:
Given a hyperbolic manifold M, if $M(\alpha), M(\beta)$ each contain an essential sphere and an essential torus, then $\Delta(\alpha, \beta) \leq 3$ [Oh, Wu].

Boyer and Zhang's example; $\Delta(\mathcal{S},\mathcal{T}) = 3$



Theorem (Gordon and Luecke, 1996). *($\Delta(\mathcal{S}, \mathcal{S}) \leq 1$) Let M be a hyperbolic 3-manifold with a torus boundary component T. If α, β are two slopes on T such that both $M(\alpha)$ and $M(\beta)$ are reducible, then $\Delta(\alpha, \beta) \leq 1$.*

- $\Delta(\mathcal{S}, \mathcal{S}) \leq 5$; Gordon and Litherland, 1984

- $\Delta(\mathcal{S}, \mathcal{S}) \leq 2$; Wu, 1992

- $\Delta(\mathcal{S}, \mathcal{S}) \leq 1$; Gordon and Luecke, 1996

- $\Delta(\mathcal{S}, \mathcal{S}) \leq 1$; Lee, Oh, and Teragaito, 2006, a simple proof

We prove the following theorem.

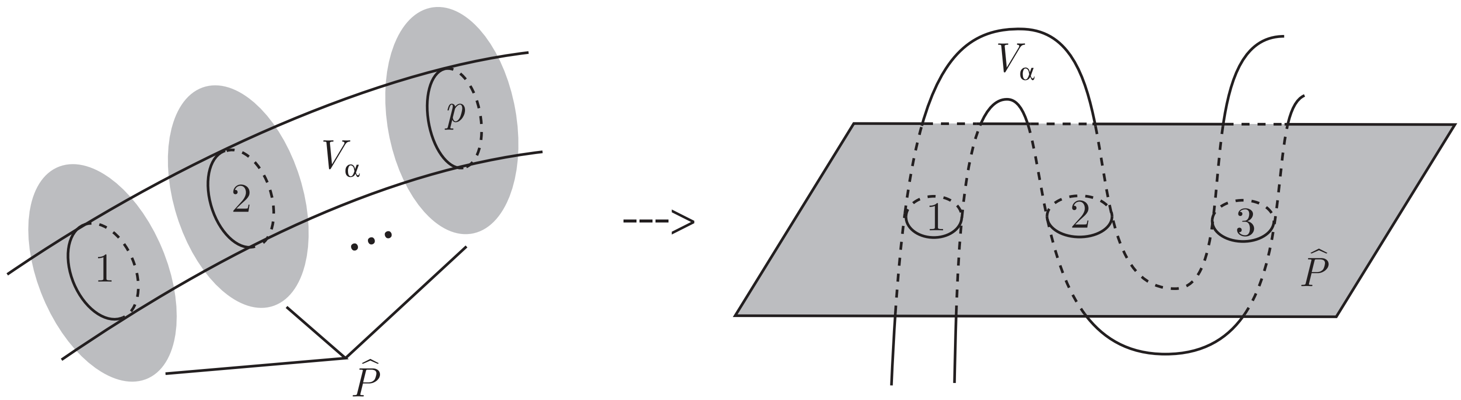
Theorem. $\Delta(\mathcal{S}, \mathcal{S}) \leq 3$.

Assume for contradiction that $\Delta(\alpha, \beta) \geq 4$.

V_α, V_β : attached solid tori in $M(\alpha), M(\beta)$

$\widehat{P} \subset M(\alpha)$, $\widehat{Q} \subset M(\beta)$: essential spheres

We may assume
$\widehat{P} \cap V_\alpha = u_1 \cup \ldots \cup u_p$: meridian disks of V_α
$\widehat{Q} \cap V_\beta = v_1 \cup \ldots \cup v_q$: meridian disks of V_β



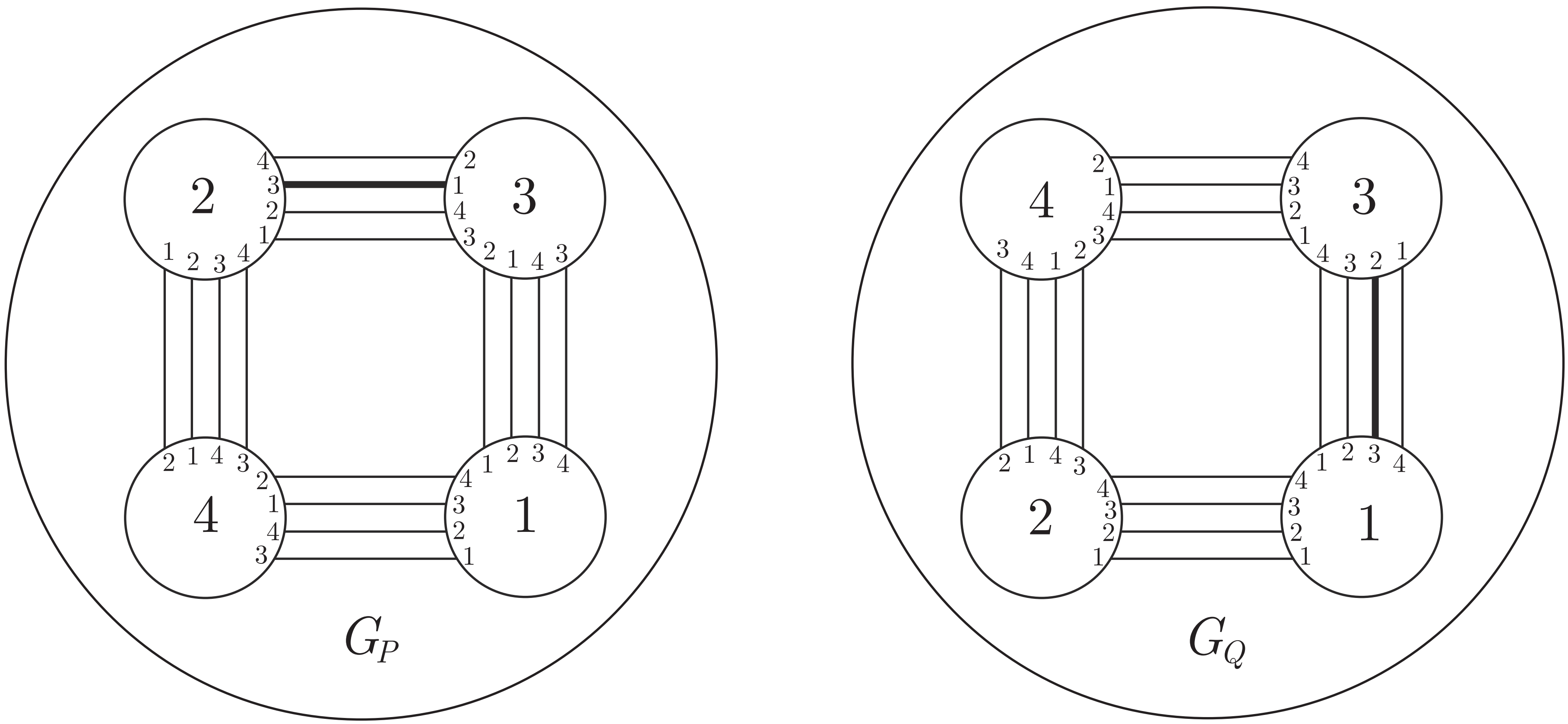
We assume that $\widehat{P}, \widehat{Q}$ had been chosen so that p, q are minimal.

Let $P = \widehat{P} \cap M$ and $Q = \widehat{Q} \cap M$.
Then P and Q are incompressible and ∂-incompressible.

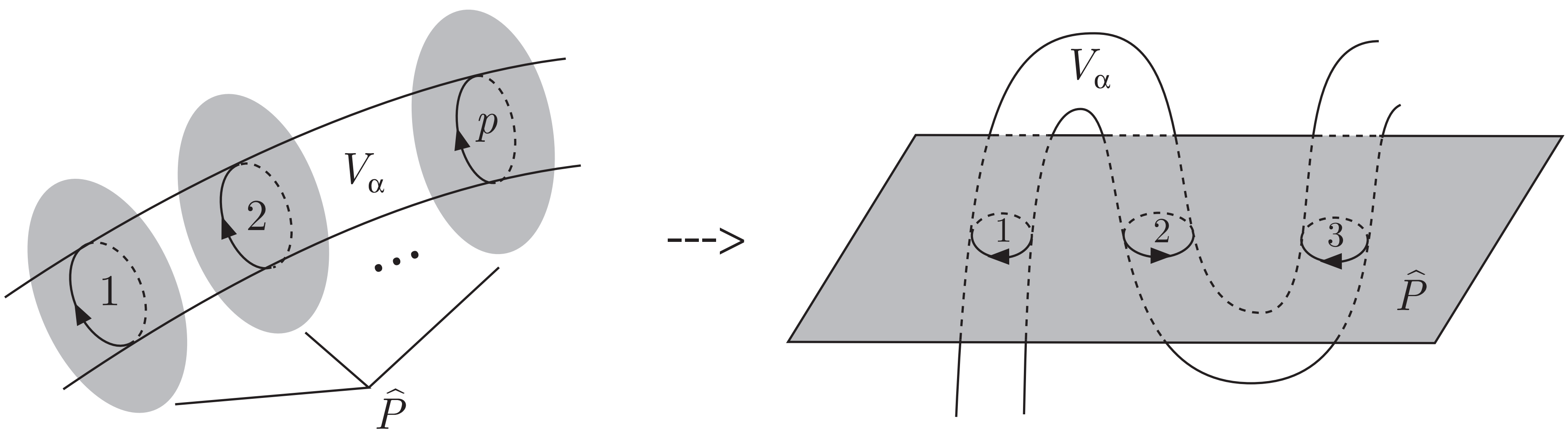
Isotope P or Q in M so that $P \pitchfork Q$.

The arc components of $P \cap Q$ define two labelled graphs G_P and G_Q.
No trivial edge by ∂-incompressibility.

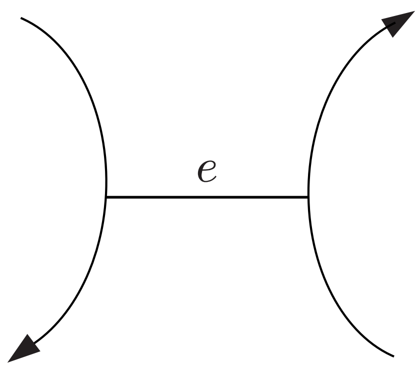
e.g. $\Delta=2,\ p=4,\ q=4$



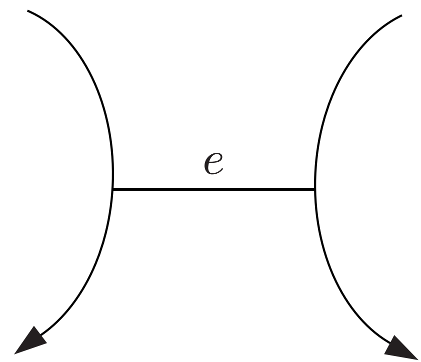
Orient ∂P so that all components of ∂P are homologous in $\partial V_\alpha = T \subset \partial M$.



Give a sign to each edge of G_P.



positive

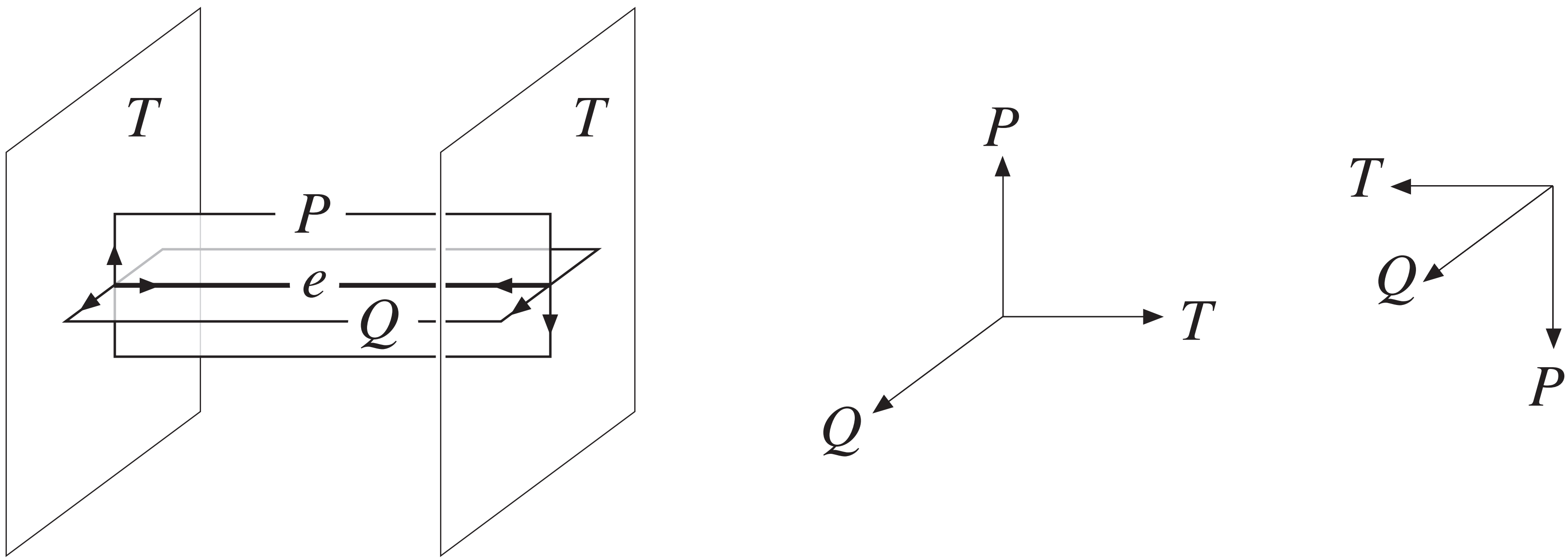


negative

Similarly for G_Q.

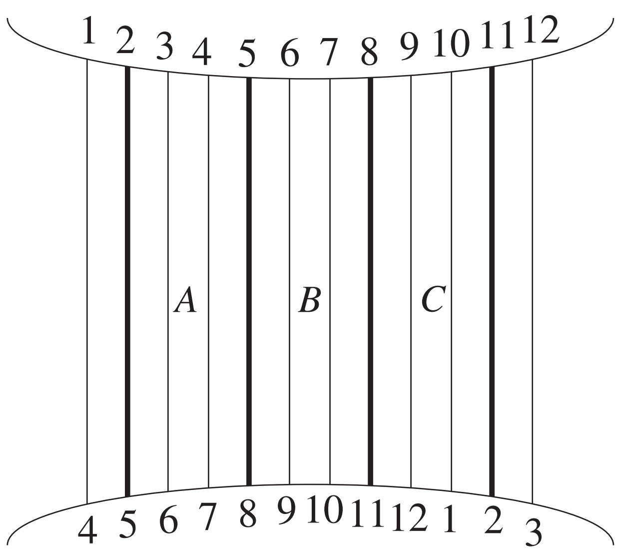
Parity Rule

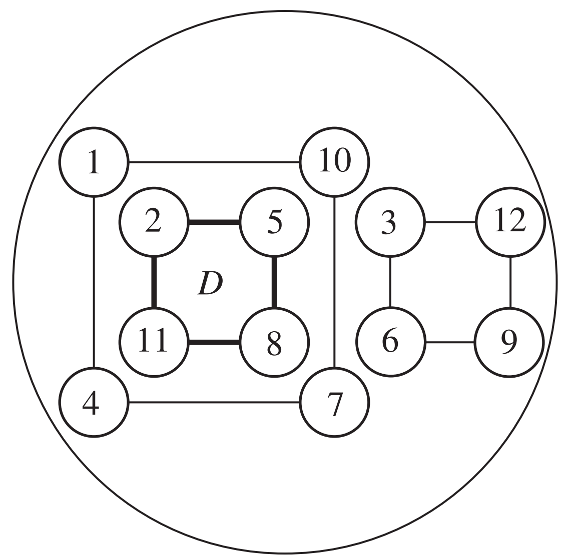
An edge is positive in one graph if and only if it is negative in the other.

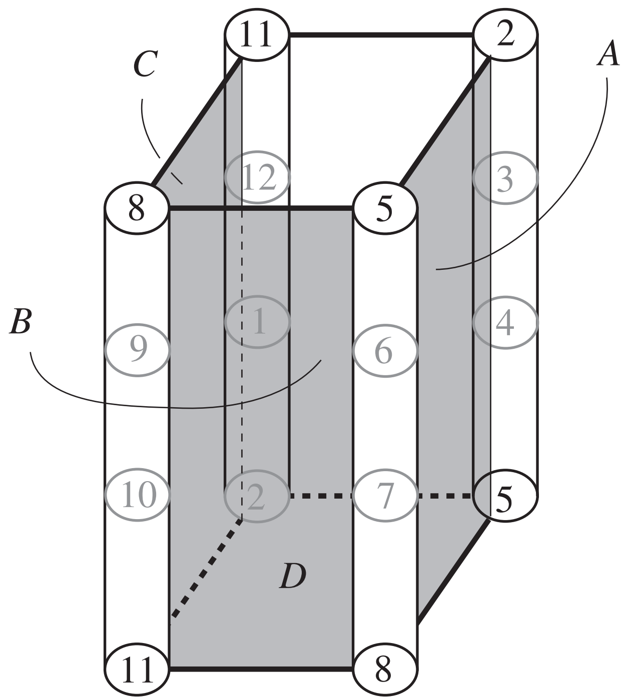


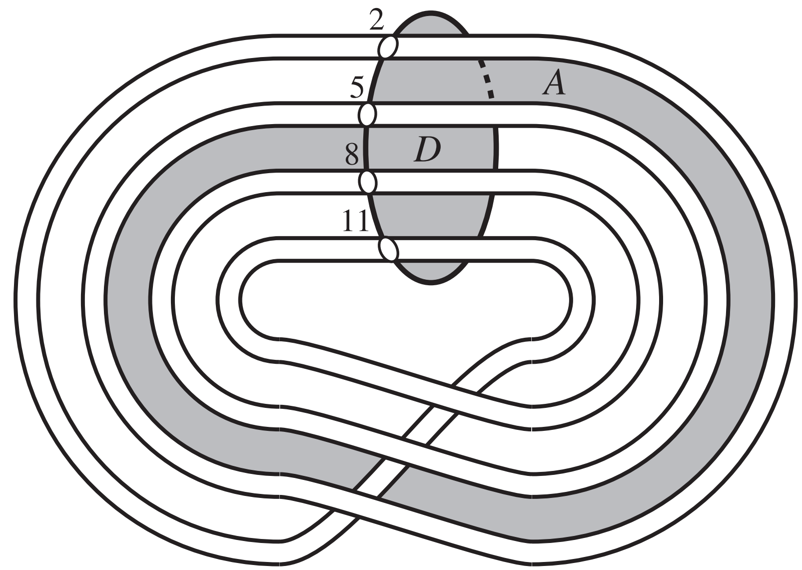
Lemma. *Any family of parallel negative edges in* G_P *contains at most* $q-1$ *edges.*

Proof. Assume G_P contains q parallel negative edges (assume $q = 12$).



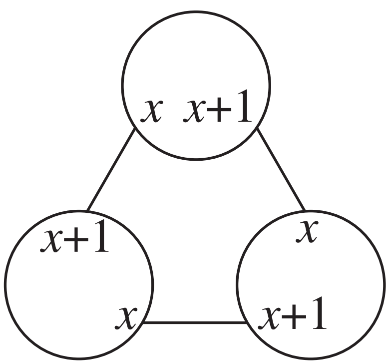




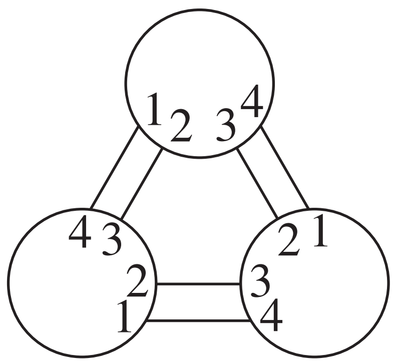


A neighborhood of $A \cup B \cup C \cup D \cup T$ in M is a cable space. This is impossible, since M is hyperbolic. $\square$

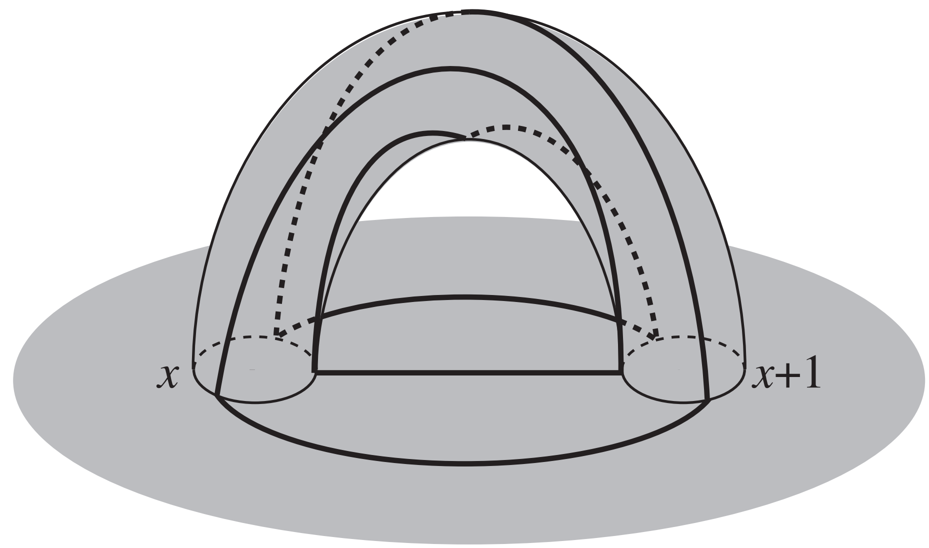
Scharlemann cycles and extended Scharlemann cycles



Scharlemann cycle



Extended
Scharlemann cycle



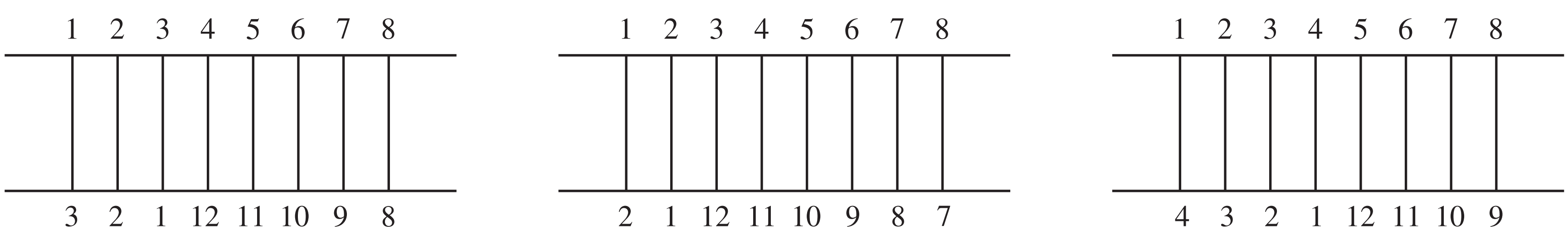
Punctured lens space

Lemma. *Any two Scharlemann cycles in G_P (resp. G_Q) have the same label pair.*

Lemma. *No extended Scharlemann cycle.*

Lemma. *Any family of parallel positive edges in G_P contains at most $q/2+1$ edges. If q is odd, then it contains at most $(q+1)/2$ edges.*

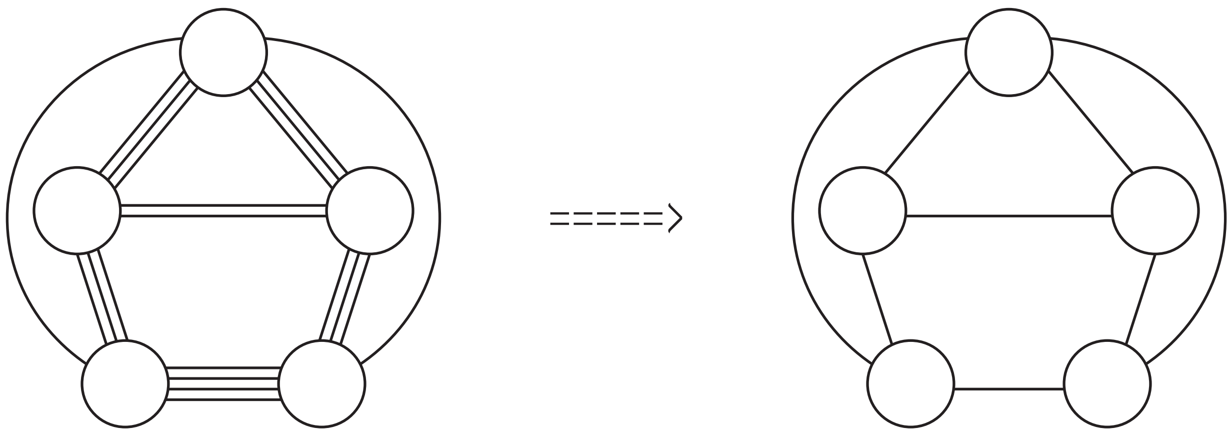
Proof. Assume G_P contains $q/2+2$ parallel positive edges (assume $q=12$).



□

Reduced graph

Let $\overline{G}_P$ denote the *reduced graph* of G_P, i.e., $\overline{G}_P$ is obtained from G_P by amalgamating each family of parallel edges into a single edge.



Lemma. *Let u_x be a vertex of G_P such that x is not a label of a Scharlemann cycle in G_Q. Then G_P contains at most $3q-6$ negative edges incident to u_x.*

Proof. Assume for contradiction that G_P contains more than $3q-6$ negative edges incident to u_x. Let $G_Q^+(x)$ be the subgraph of G_Q consisting of all positive x-edges. Let V, E, F be the number of vertices, edges, and disk faces of $G_Q^+(x)$, respectively. Then $V = q, E > 3q-6$, and

$$V - E + F \geq V - E + \sum_{f\text{:faces of } G_Q^+(x)} \chi(f) = \chi(\widehat{Q}) = 2.$$

Since G_Q contains no extended Scharlemann cycles, every disk face of $G_Q^+(x)$ has at least 3 sides. So, $2E \geq 3F \geq 3(E-V+2)$, which yields $E \leq 3V-6 = 3q-6$. This contradicts our assumption $E > 3q-6$. □

Lemma. *Any vertex of* $\overline{G}_P$ *has valence at least* 5.

Proof. Note that $q - 1 \geq q/2 + 1$ if $q \geq 4$ and that $q - 1 \geq (q+1)/2$ if $q = 3$. Hence any family of parallel edges in G_P contains at most $q - 1$. Therefore if some vertex of $\overline{G}_P$ has valence at most 4, then it has valence at most $4(q-1) = 4q - 4 (< \Delta \cdot q)$ in G_P. This is impossible. $\square$

Lemma. $\overline{G}_P$ *has at least* 3 *vertices of valence* 5.

Proof. Let V, E, F be the number of vertices, edges, and disk faces of $\overline{G}_P$, respectively. Then $V = q \geq 3, 2E \geq 3F$, and $F \geq E - V + 2$. Combining the last two inequalities, we obtain

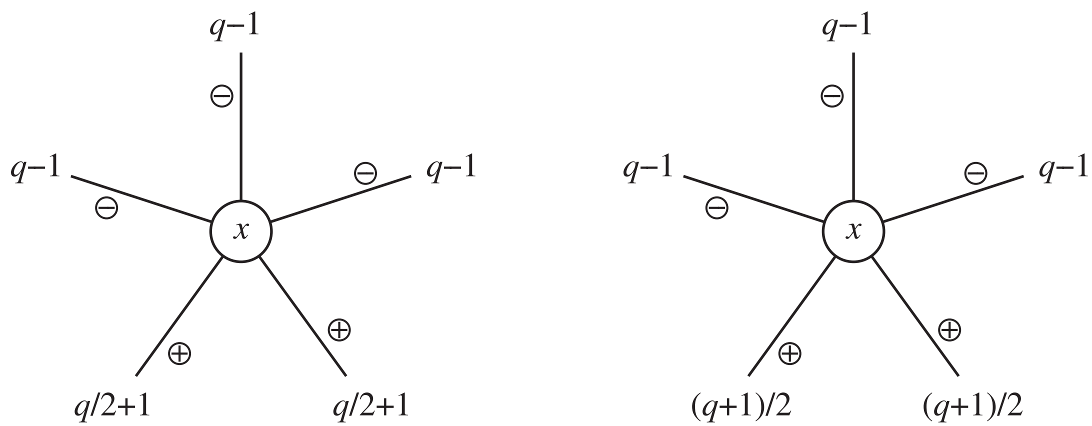
$$3V - 6 \geq E.$$

Suppose that all but two vertices of $\overline{G}_P$ has valence at least 6. Then $2E \geq 6(V-2) + 5 \times 2$ or

$$E \geq 3V - 1.$$

Two inequalities above conflict. □

Choose a vertex u_x of valence 5 in $\overline{G}_P$ such that x is not a label of a Scharlemann cycle in G_Q. Since G_P contains at most $3q-6$ negative edges incident to u_x, $\overline{G}_P$ contains at least 2 positive edges incident to u_x. Let N be the number of edge endpoints of G_P at u_x. Then



$3(q-1)+2(q/2+1) = 4q-1 \geq N \geq \Delta \cdot q,$ or
$3(q-1)+2((q+1)/2) = 4q-2 \geq N \geq \Delta \cdot q.$
Both are impossible, completing the proof of our theorem.

Conjecture. Let M be a hyperbolic 3-manifold with a torus boundary component T. Suppose that there are two distinct slopes α, β on T such that both $M(\alpha)$ and $M(\beta)$ are reducible. Then one of $M(\alpha)$ and $M(\beta)$ contains a reducing sphere which hits the core of the attached solid torus 4 times.

Large Manifolds

A 3-manifold M with a torus $T \subset \partial M$ is *large* if $H_2(M, \partial M - T) \neq 0$.

In particular, M is large if ∂M is not one or two tori.

Define
$\Delta^*(\mathcal{X}_1, \mathcal{X}_2) = \max\{\Delta(\alpha_1, \alpha_2) |$ there is a large hyperbolic 3-manifold M and slopes α_1, α_2 on some torus component of ∂M, such that $M(\alpha_i)$ is of type $\mathcal{X}_i, i = 1, 2\}$.

Δ	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Δ^*	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	0	0	1	1
$\mathcal{D}$		1	2	1
$\mathcal{A}$			4	4
$\mathcal{T}$				4

Manifold with boundary a union of tori

Define
$\Delta^k(\mathcal{X}_1, \mathcal{X}_2) = \max\{\Delta(\alpha_1, \alpha_2)|$ there is a hyperbolic 3-manifold M such that ∂M is a disjoint union of k tori, and slopes α_1, α_2 on some torus component of ∂M, such that $M(\alpha_i)$ is of type $\mathcal{X}_i, i = 1, 2\}$.

Δ	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Δ^2	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	2
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				5

Δ^3	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	0	0	1	1
$\mathcal{D}$		0	1	1
$\mathcal{A}$			3	3
$\mathcal{T}$				3

$\Delta^k (k \geq 4)$	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	0	0	1	1
$\mathcal{D}$		1	1	1
$\mathcal{A}$			2	2
$\mathcal{T}$				2

Research Aim

M : hyperbolic $\longrightarrow M(\alpha)$: not hyperbolic for finitely many slopes

Such slopes are called *exceptional slopes*.

Project. How many exceptional slopes?

Example. The figure-8 knot exterior has 10 exceptional slopes.

Conjecture (Gordon). There are at most 10 exceptional slopes for any hyperbolic 3-manifold.

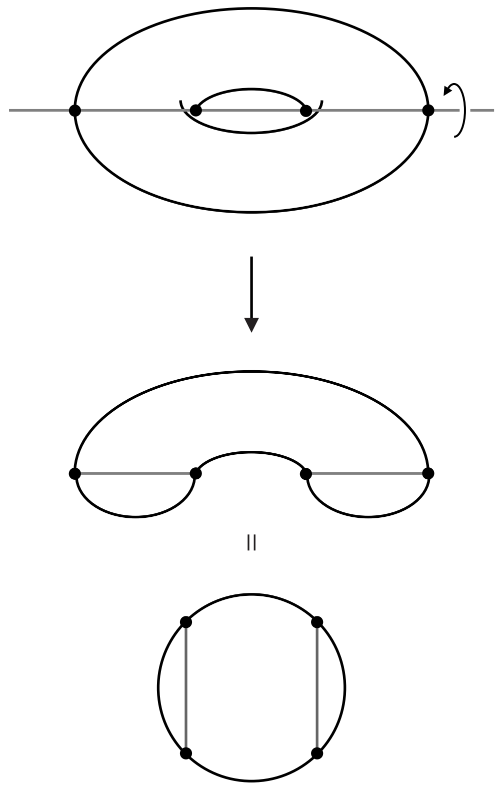
Let M be a hyperbolic 3-manifold with a torus boundary component T. Define

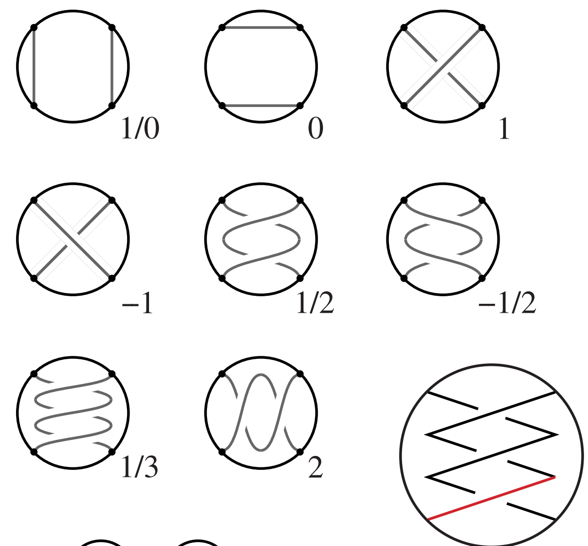
$$\mathcal{E}(M;T) = \mathcal{E}(M) = \{\alpha \subset T \mid M(\alpha) \text{ is not hyperbolic } \}$$

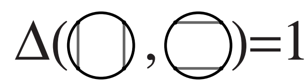
Then Gordon's conjecture is reformulated as follows.

Conjecture. $|\mathcal{E}(M)| \leq 10$. Moreover, $|\mathcal{E}(M)| \leq 8$ if M is not the figure-8 knot exterior.

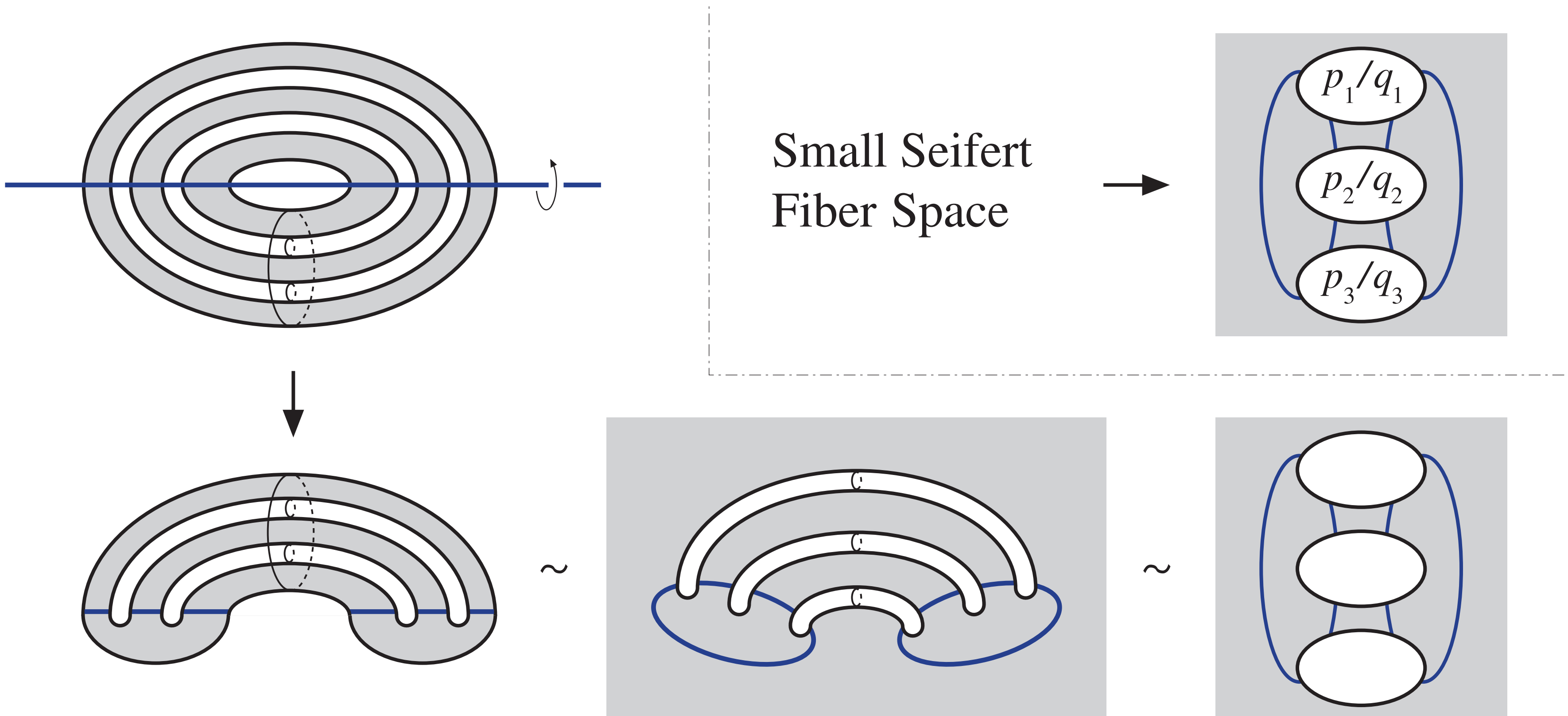
Double branched covering and Rational tangles





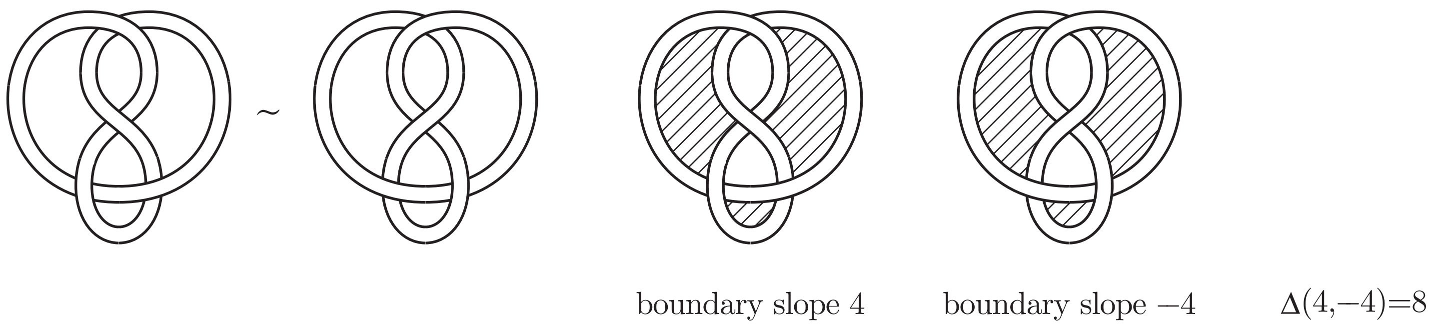


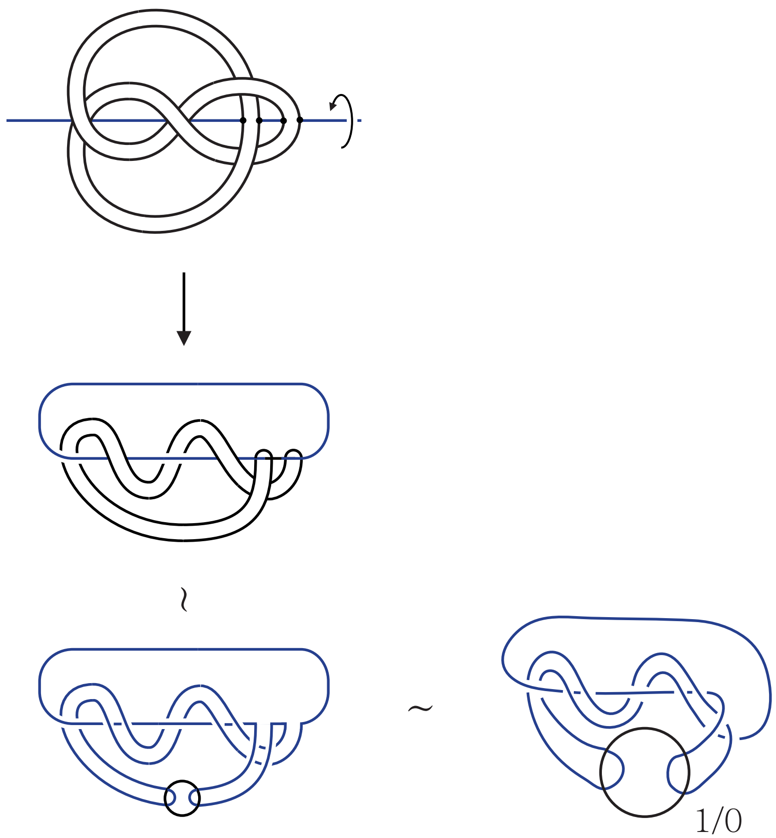
$$\Delta(p/q, r/s) = ps - qr$$

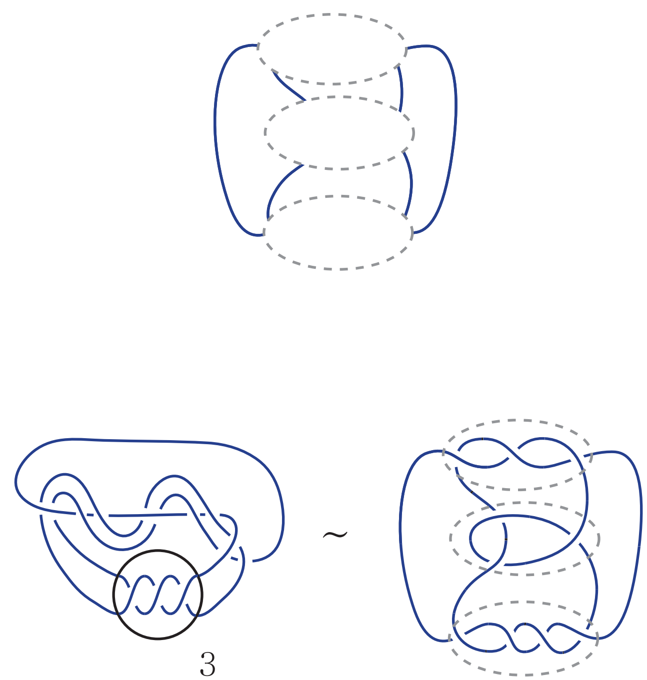

Small Seifert
Fiber Space
p_1/q_1
p_2/q_2
p_3/q_3
~
~

The figure-8 knot exterior and exceptional slopes

Let M be the exterior of the figure-8 knot.
Then $\mathcal{E}(M) = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 1/0\}$
Since the figure-8 knot is amphicheiral, $M(r) \cong M(-r)$.




1/0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$\Delta \leq ?$	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Upper bounds for Δ

Conjecture. $|\mathcal{E}(M)| \leq 10$. Moreover, $|\mathcal{E}(M)| \leq 8$ if M is not the figure-8 knot exterior.

$\Delta \leq ?$	0	1	2	3	4	5	6	7	8
$\sharp\{\text{slopes}\} \leq ?$	1	3	4	6	6	8	8	10	12

Theorem (Agol, Lackenby, 2000). *Let M be a hyperbolic 3-manifold with ∂M a single torus. Then $|\mathcal{E}(M)| \leq 12$.*

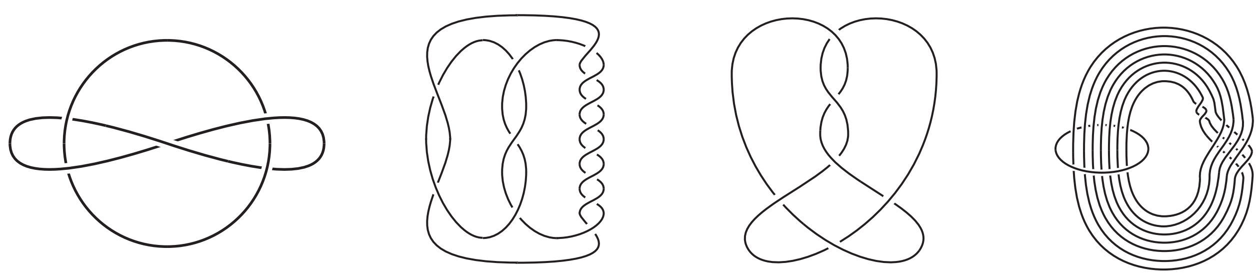
What if ∂M is not a single torus?

Suppose that M has a torus boundary component T and at least one other boundary component.

Examples

For hyperbolic 3-manifolds M with at least two boundary components, the maximal observed value for $|\mathcal{E}(M)|$ is 6.

The following links are the Whitehead link, the Whitehead sister link, the 2-bridge link associated to 3/10 in Conway's notation, and the Berge link.



Theorem (Martelli-Petronio). *Their exteriors have exactly* $\mathbf{6}$ *exceptional slopes.*

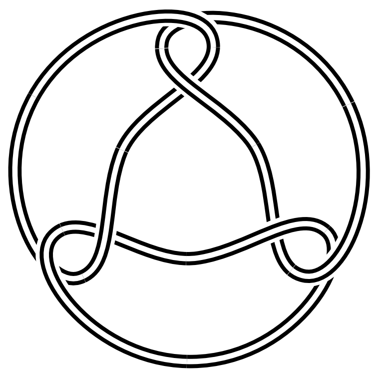
Theorem (Lee, 2007). *Let* M *be a hyperbolic* 3*-manifold with one torus boundary component and at least one other boundary component. Then*

$$|\mathcal{E}(M)| \leq 6.$$

Moreover, any two exceptional slopes have mutual distance no larger than 4 *unless* M *is the Whitehead sister link exterior.*

Magic manifold

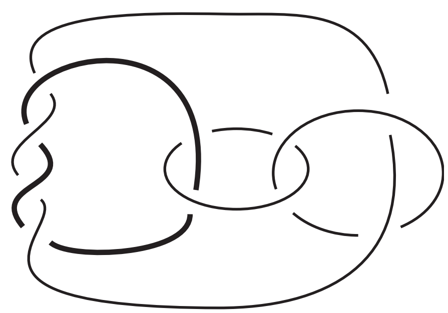
The exterior of the following link is called the *magic manifold*.



Exceptional slopes $= \{-3, -2, -1, 0, 1/0\}$.

Theorem (Lee and Teragaito). *Let M be a hyperbolic 3-manifold with ∂M a union of at least 4 tori. Then*

$$|\mathcal{E}(M)| \leq 4.$$



Dehn surgeries on knots in S^3

Conjecture. Let K be a hyperbolic knot in S^3. Then any exceptional Dehn surgery slope r is either (a) integral, or (b) half-integral and $K(r)$ is toroidal.

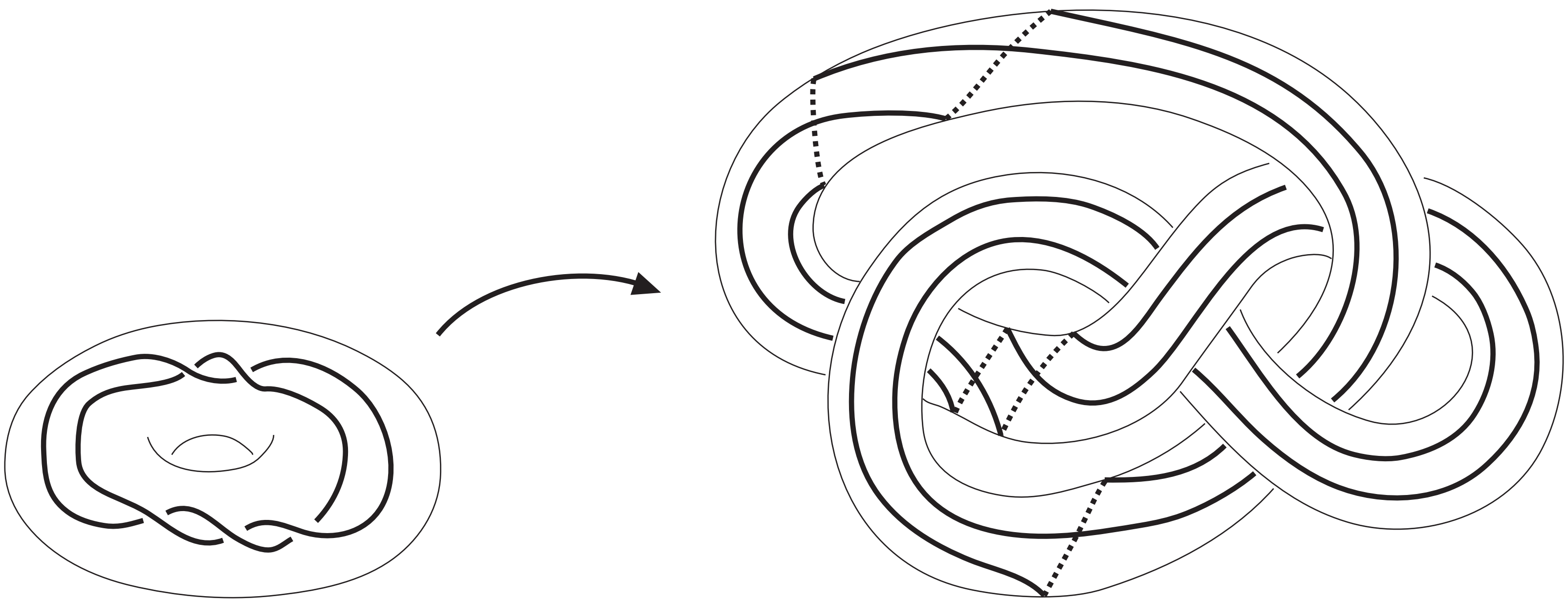
$\mathcal{L}(K) = \{r \in \mathcal{E}(K) | K(r)$ is a lens space$\}$
$\mathcal{S}(K) = \{r \in \mathcal{E}(K) | K(r)$ is a small Seifert fiber space$\}$
$\mathcal{T}(K) = \{r \in \mathcal{E}(K) | K(r)$ is toroidal$\}$

It is conjectured that

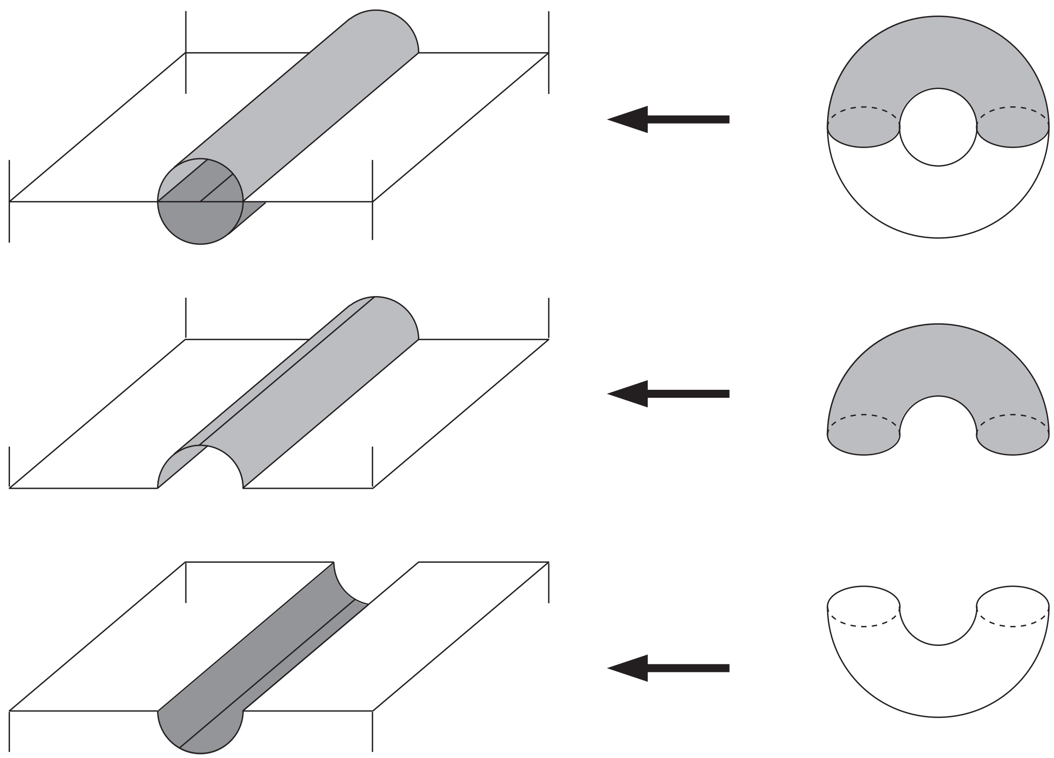
$$\mathcal{E}(K) = \mathcal{L}(K) \cup \mathcal{S}(K) \cup \mathcal{T}(K).$$

Cable knots

A cable knot is a satellite knot obtained by starting the satellite construction with a torus knot



Every cable knot admits a reducing Dehn surgery.

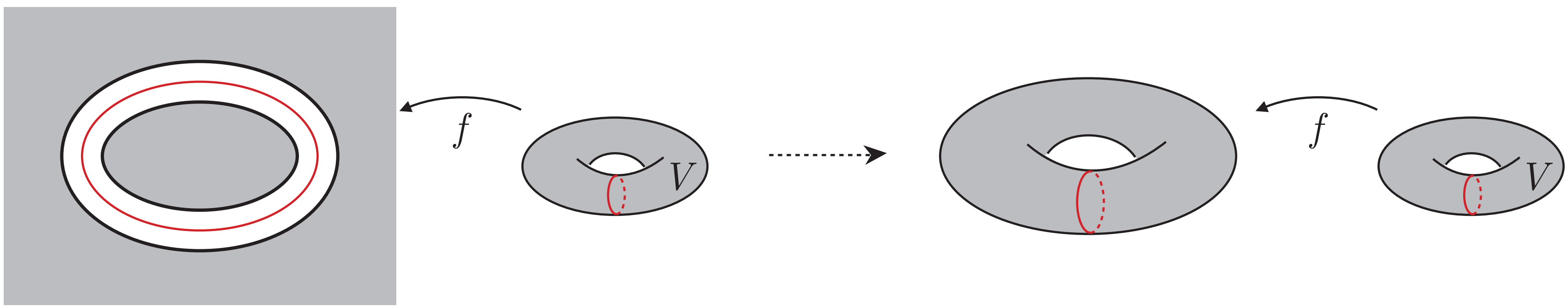


Cabling conjecture. If a manifold obtained by Dehn surgery on a knot $K \subset S^3$ is reducible, then K is a cable knot.

Known for :
- Satellite knots (Scharlemann)
- Alternating knots (Menasco-Thistlethwaite)
- Knots with at most 4 bridges (Hoffman)
- Symmetric knots (Eudave-Muñoz, Luft and Zhang,..., Hayashi and Shimokawa)
- Knots with at most 10 crossings (Brittenham)

Weak cabling conjecture. If a manifold obtained by Dehn surgery on a knot $K \subset S^3$ is reducible, then it is a composite manifold with only two summands.

Property R Conjecture

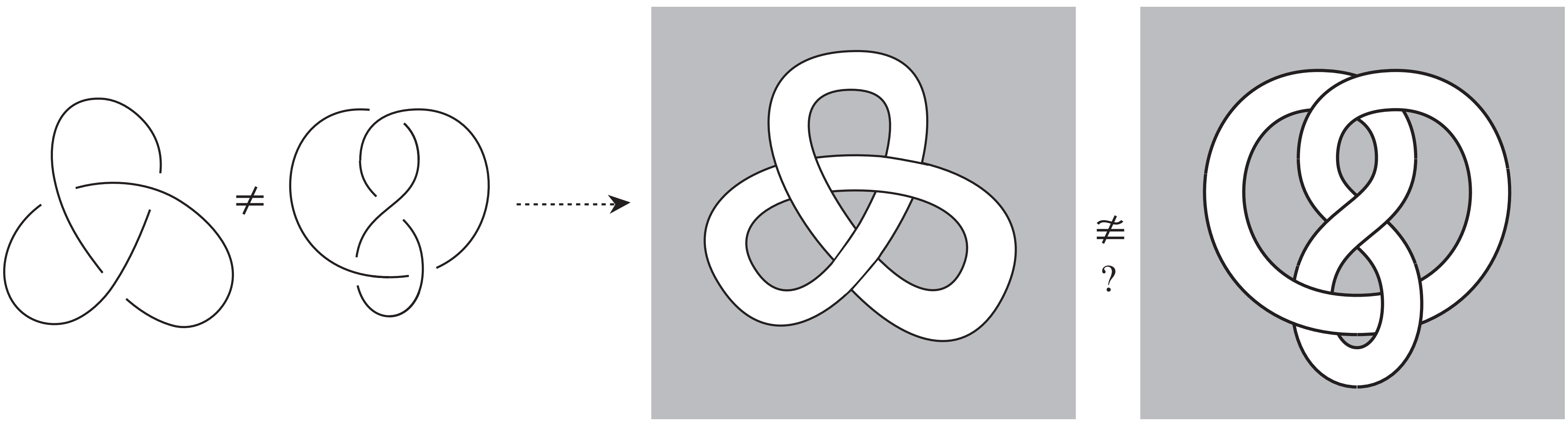


Conjecture. If $K \neq O$, then $K(r) \neq S^1 \times S^2$ for any slope r.

Theorem (Gabai, 1987). *The conjecture is true.*

He solved this problem by using the sutured manifold theory.

Knot Complement Problem



Problem. Are knots in S^3 are determined by their complements?

Theorem (Gordon and Luecke, 1989). *Yes.*

In fact, they showed the following, using a combinatorial technique.

Theorem. *If $K \neq O$, then $K(r) \neq S^3$ for any slope $r \neq 1/0$.*

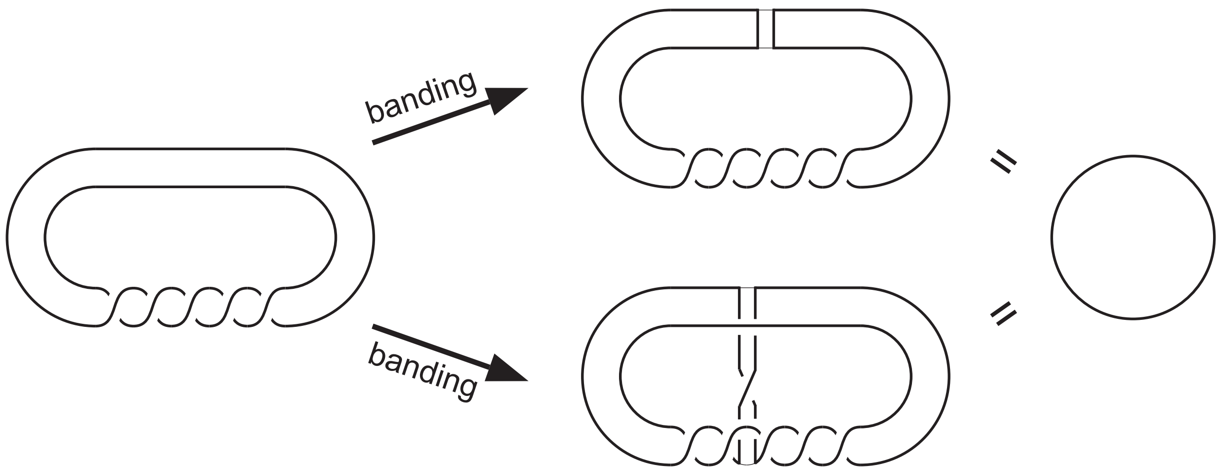
Property P Conjecture. $\pi_1(K(r)) \neq 1$ if $K \neq O$ and $r \neq 1/0$.

Theorem (Kronheimer and Mrowka, Ozsvath and Szabo, 2004). *Property P Conjecture is true.*

They used Heegaard Floer Homology Theory to prove the following.

Theorem. *If $K \neq O$, then $K(r) \neq L(2,1), L(3,1), L(4,1)$ for any slope r.*

Remark. A Lens space of order 5 is obtained by a Dehn surgery on a nontrivial knot.



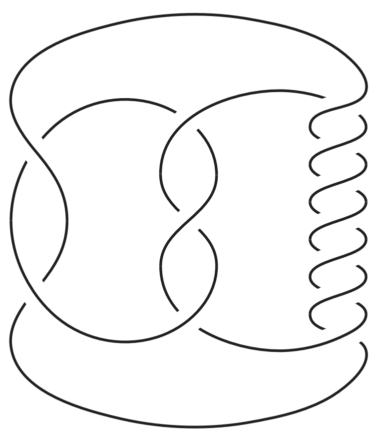
Theorem (Hirasawa and Shimokawa). *Let K be a nontrivial strongly invertible knot. Then no Dehn surgery on K can yield $L(2p, 1)$ for any integer p.*

Problem (Teragaito). $K(r) \neq L(4n, 2n \pm 1)$ if K is a hyperbolic knot? (Known for any integer $n \neq 4$: Tange)

$(-2, 3, 7)$-pretzel knot and exceptional surgery slopes

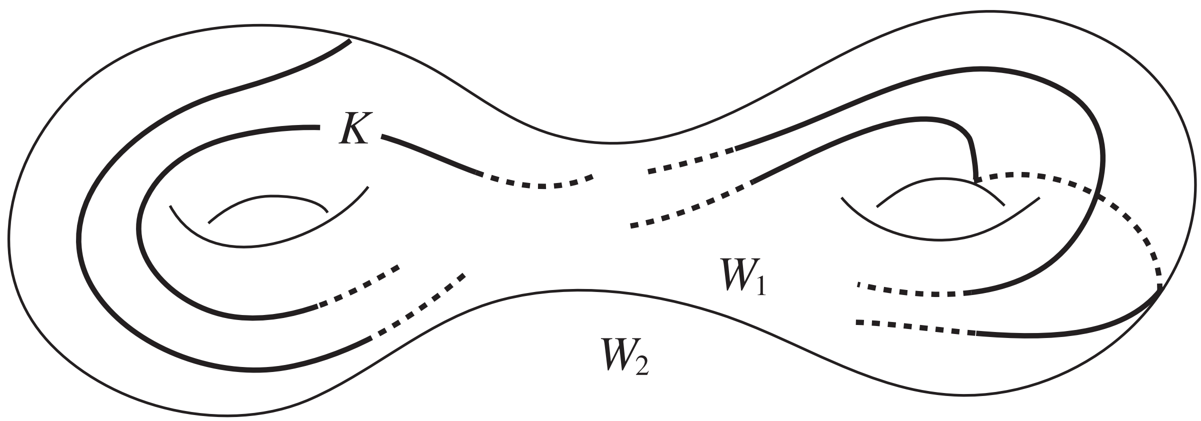
Exceptional slopes : $16, 17, 18, 37/2, 19, 20, 1/0$

- S^3 : $1/0$
- Lens space : $18, 19$
- Small Seifert fiber space : 17
- Toroidal manifold : $16, 37/2, 20$



Berge's construction

Let $W_1 \cup W_2$ be a genus 2 Heegaard splitting of S^3.
Let $K \subset \partial W_1 = \partial W_2$ be a knot such that $W_i \cup H(K)$ is a solid torus.
Then $K(r)$ is a lens space for some integral slope r.



Conjecture. If a knot K admits a lens space surgery, then K is a Berge's knot.

Lens space surgeries and genera of knots

Theorem (Culler, Gordon, Luecke, and Shalen). *Let K be a knot in S^3 which is not a torus knot. If $\pi_1(K(r))$ is cyclic, then r is an integral slope.*

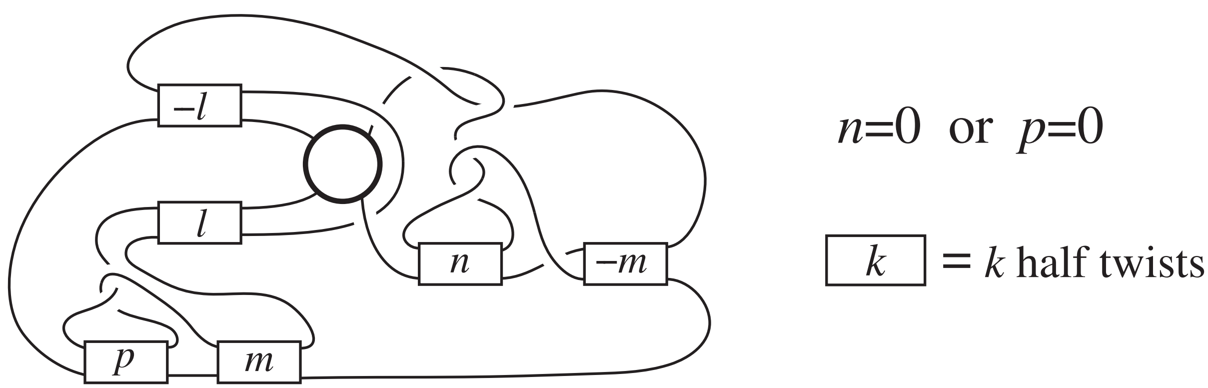
Conjecture (Goda and Teragaito). Let K be a hyperbolic knot in S^3. If $K(r)$ is a lens space, then K is fibered and $2g(K)+8 \leq |r| \leq 4g(K)-1$.

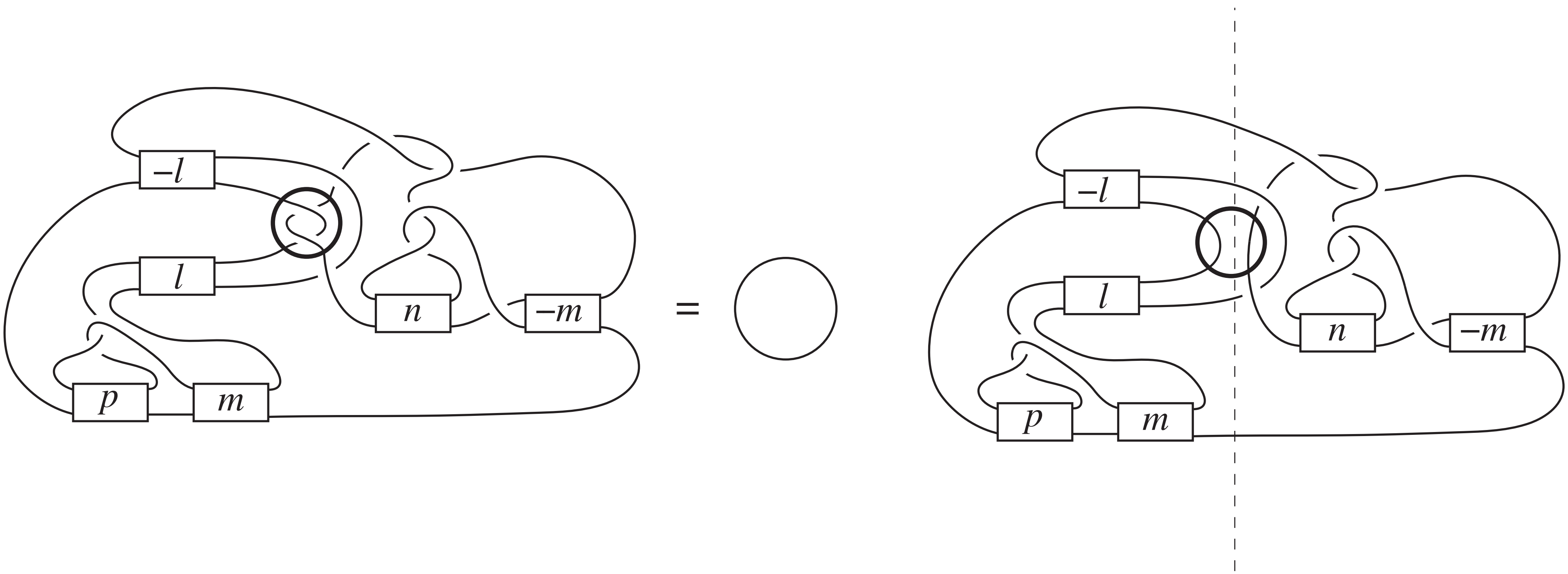
Theorem (Rasmussen). *Suppose that K is a nontrivial knot which admits a lens space surgery of slope r. Then $|r| \leq 4g(K)+3$.*

Toroidal Dehn surgeries

Theorem (Gordon and Luecke). *If a hyperbolic knot K admits a toroidal surgery of slope r, then r is either integral or half-integral. ($r = n$ or $n/2$ for some integer n.)*

Eudave-Muñoz gave infinitely many hyperbolic knots $k(l, m, n, p)$ which admits a half-integral toroidal surgery.





−l
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Other known results

Theorem (Boyer and Zhang).
- If $K(r)$ is toroidal Seifert fibered, then r is integral.
- If a 2-bridge knot K admits a toroidal surgery slope r, then $r \in 4\mathbb{Z}$.

Theorem (Boyer and Zhang, Patton). *If an alternating knot K admits a toroidal surgery slope r, then $r \in 4\mathbb{Z}$.*

Theorem (Brittenham and Wu). *Classification of Dehn surgeries on 2-bridge knots. (toroidal surgery $\Rightarrow$ genus one or Klein bottle surgery)*

Eudave-Muñoz knots

- strongly invertible
- tunnel number one
- The core of the attached solid torus hits the essential torus twice (minimally).
- $K(r)$ contains a unique essential torus (up to isotopy).

Theorem (Gordon and Luecke). *If $K(n/2)$ is toroidal, then K is a Eudave-Muñoz knot.*

Integral toroidal surgeries

K : a hyperbolic knot such that $K(r)$ is toroidal ($r \in \mathbb{Z}$)

• strongly invertible?
No. (Genus one knots which are not strongly invertible)

• tunnel number one?
Arbitrary high! (Eudave-Muñoz and Luecke)

• How many times (minimally) does the core of the attached solid torus hit an essential torus?
Arbitrary many! (Osoinach)

• How many non-isotopic essential tori in $K(r)$?
Unsolved.

Conjecture (Eudave-Muñoz). Any hyperbolic knot has at most 3 toroidal surgery slopes.

examples

The figure-8 knot : $-4, 0, 4$

The $(-2, 3, 7)$-pretzel knot : $16, 37/2, 20$

Toroidal surgeries and genera of knots

Conjecture (Teragaito). If a hyperbolic knot K admits a toroidal surgery of slope r, then $|r| \leq 4g(K)$.

Known for :
-genus one knots (Teragaito)
-alternating knots (Teragaito)
-genus two knots (Lee)

Theorem (Ichihara). $|r| < 3 \cdot 2^{7/4} g(K)$.

Known : $|r| \leq 6g(K) - 3$.

Seifert fibered surgeries

Not so much is known.

Conjecture (Eudave-Muñoz). Any Seifert fibered Dehn surgery on a hyperbolic knot is integral.

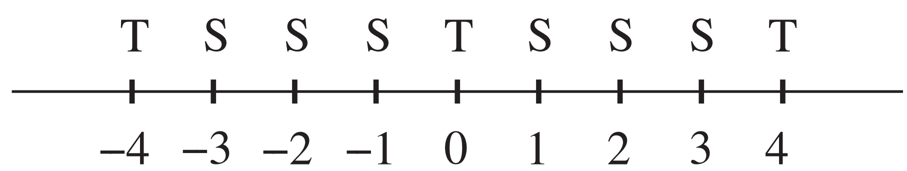
Conjecture (Motegi). If $K(r)$ is a Seifert fiber space, then there exists a knot c in S^3 disjoint from K such that c is unknotted and becomes a Seifert fiber in $K(r)$. (The knot c is called a *seiferter*.)

Known : If an r-surgery on K yielding a Seifert fiber space for some rational number r has a seiferter, then r is integral, except when K is a torus knot or a cable of a torus knot.

Examples

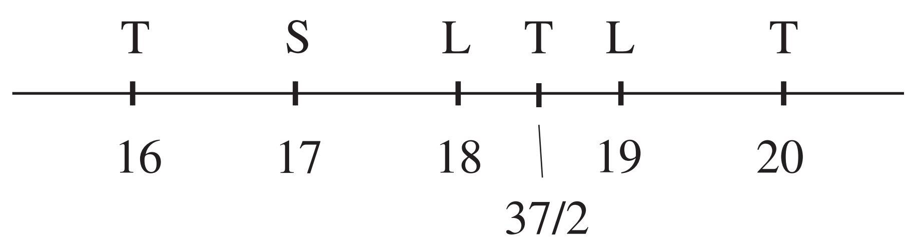
(1) figure-eight knot

$$\mathcal{E}(K) = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \infty\}$$



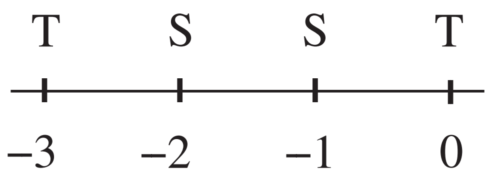
(2) $(-2, 3, 7)$-pretzel knot

$$\mathcal{E}(K) = \{16, 17, 18, 37/2, 19, 20, \infty\}$$

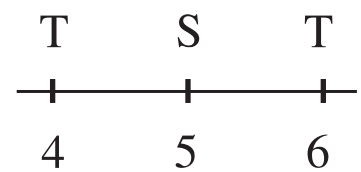


More examples (Wu)

(−1/2,1/3,2/11)-Montesinos knot



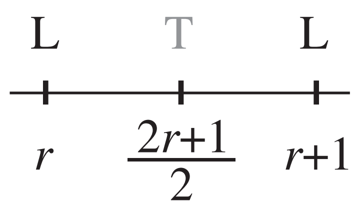
(−1/3,−2/5,2/3)-Montesinos knot



Conjecture (Teragaito). Integral exceptional slopes are consecutive. Moreover, integral toroidal slopes appear at the border, except figure-eight knot.

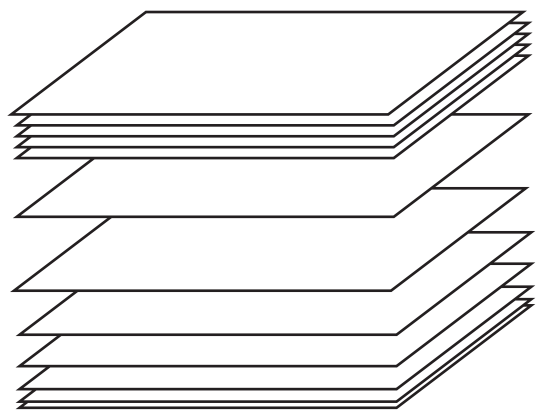
Theorem. *(Cyclic Surgery Theorem) If a hyperbolic knot has two lens spaces surgery slopes, then they are consecutive.*

Conjecture. If a hyperbolic knot has two lens spaces surgery slopes r and $r+1$, then $\frac{2r+1}{2}$ is a toroidal slope.



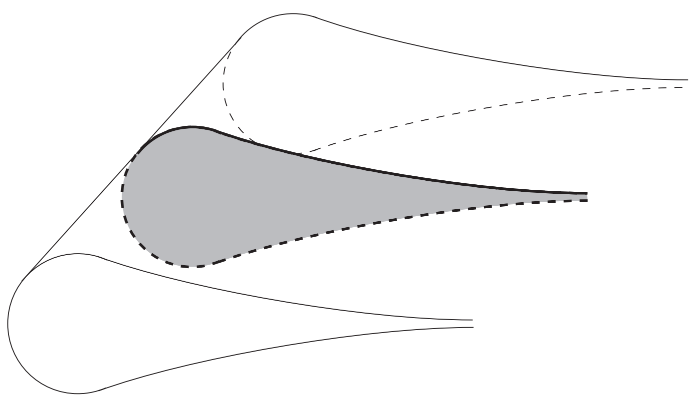
Laminations

A *lamination* λ on a 3-manifold M is a decomposition of a closed subset of M into surfaces called *leaves* so that M is covered by charts of the form $I^2 \times I$ where the leaves pass through a chart in slice of the form $I^2 \times \{\text{pt}\}$.

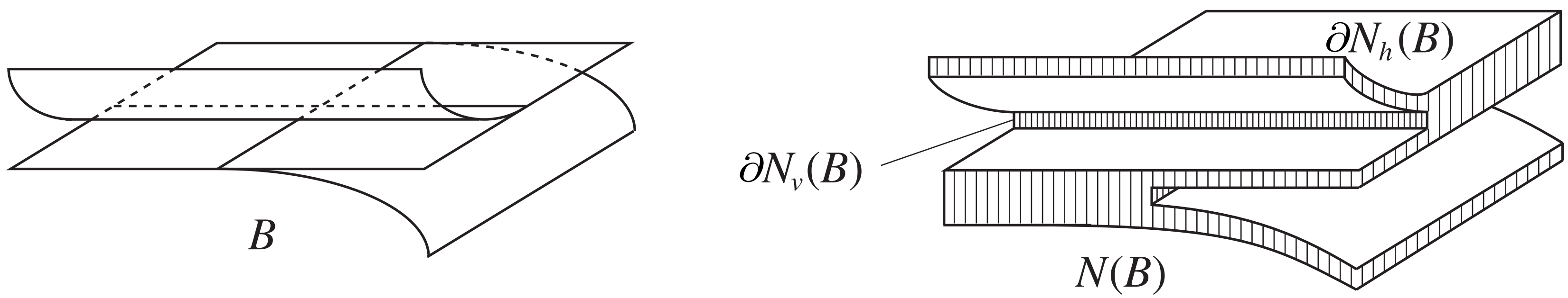


Essential laminations

The lamination λ is *essential* if no leaf is a sphere or a torus bounding a solid torus, M_λ is irreducible and ∂M_λ is both incompressible and end-incompressible in M_λ.



Branched surfaces

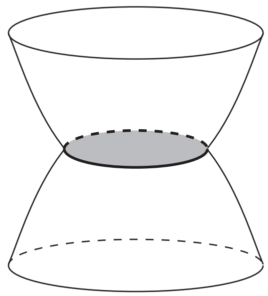


A lamination λ is *carried by* B if it can be isotoped into $N(B)$ everywhere transverse to an I-foliation $\mathcal{V}$ of $N(B)$. It is *fully carried* if it intersects every fiber of $\mathcal{V}$.

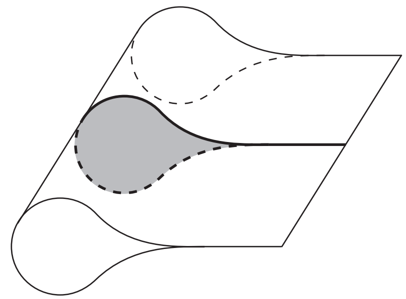
Essential branched surfaces

A closed branched surface B in a ∂-irreducible 3-manifold M is *essential* if it satisfies the following conditions.

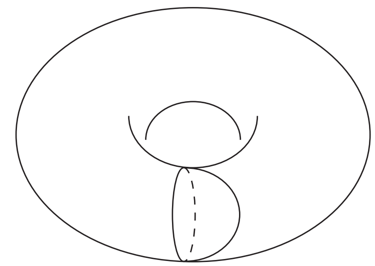
1. B has no disks of contact.
2. $\partial_h N(B)$ is incompressible in $E(B) = M - \text{Int}N(B)$.
3. There are no monogons in $E(B)$.
4. No component of $\partial_h N(B)$ is a sphere.
5. $E(B)$ is irreducible.
6. B contains no Reeb branched surface.
7. B fully carries a lamination.



disk of contact



monogon



Reeb branched surface

Theorem (Gabai and Oertel). *λ is an essential lamination if and only if it is fully carried by an essential branched surface.*

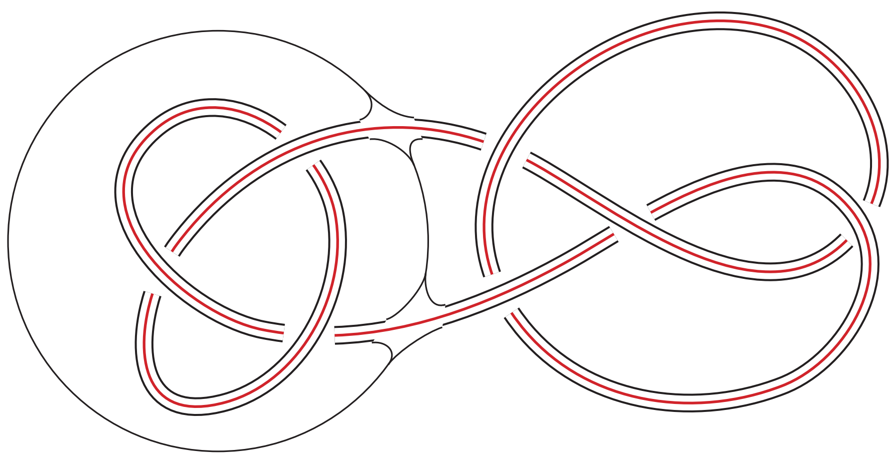
Theorem (Gabai and Oertel). *If a compact orientable 3-manifold contains an essential lamination, then its universal cover is homeomorphic to $\mathbb{R}^3$.*

Persistently laminar knots

A knot is *persistently laminar* if its complement contains an essential lamination and the lamination remains essential under all nontrivial Dehn surgeries.

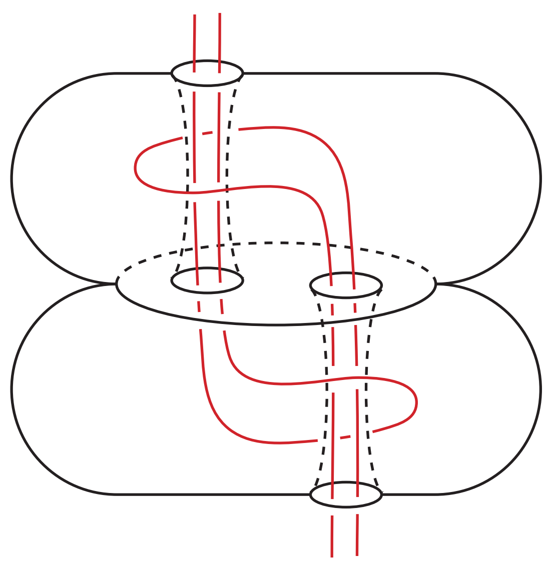
Persistently laminar knots have the strong Property P and satisfies the Cabling Conjecture.

All composite knots are persistently laminar.



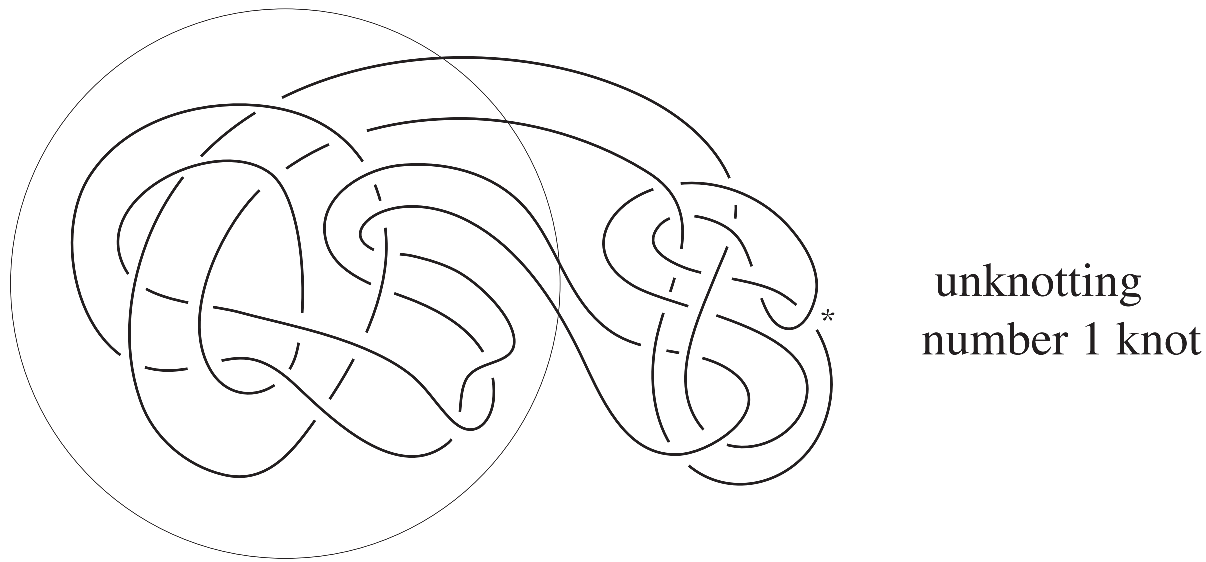
Theorem (Delman). *All non-torus 2-bridge knots are persistently laminar.*

Brittenham showed that any knot having the following tangle as its part is persistently laminar.



Applications of Dehn surgery theory

For a knot K in S^3,
$u(K) =$ min. # of self intersections needed to change K to O



Theorem (Gordon and Luecke). *The knots K with $cr(K) \leq 10$ and $u(K) = 1$ are completely determined.*

$K = O$

D : a disk in S^3 such that $\partial D \cap K = \emptyset$ and (min.$|\text{Int}D \cap K|$) ≥ 2

K_n : a knot obtained from K by performing $1/n$-surgery on ∂D.

(It is known that if (min.$|\text{Int}D \cap K|$) $= 2$, then K_n is prime.)

Theorem (Hayashi and Motegi). *If K_n is composite, then $n = \pm 1$.*