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# DILOGARITHM IDENTITIES

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ABSTRACT

We study the dilogarithm identities from algebraic, analytic, asymptotic, $K$-theoretic, combinatorial and representation-theoretic points of view. We prove that a lot of dilogarithm identities (hypothetically all!) can be obtained by using the five-term relation only. Among those the Coxeter, Lewin, Loxton and Browkin ones are contained. Accessibility of Lewin's one variable and Ray's multivariable (here for $n \le 2$ only) functional equations is given. For odd levels the $\widehat{sl_2}$ case of Kuniba-Nakanishi's dilogarithm conjecture is proven and additional results about remainder term are obtained. The connections between dilogarithm identities and Rogers-Ramanujan-Andrews-Gordon type partition identities via their asymptotic behavior are discussed. Some new results about the string functions for level $k$ vacuum representation of the affine Lie algebra $\widehat{sl_n}$ are obtained. Connection between dilogarithm identities and algebraic $K$-theory (torsion in $K_3(\mathbf{R})$) is discussed. Relations between crystal basis, branching functions $b_\lambda^{k\Lambda_0}(q)$ and Kostka-Foulkes polynomials (Lusztig's $q$-analog of weight multiplicity) are considered. The Melzer and Milne conjectures are proven. In some special cases we are proving that the branching functions $b_\lambda^{k\Lambda_0}(q)$ are equal to an appropriate limit of Kostka polynomials (the so-called Thermodynamic Bethe Ansatz limit). Connection between "finite-dimensional part of crystal base" and Robinson-Schensted-Knuth correspondence is considered.

## 0. Introduction.

In this paper we consider mainly the properties of dilogarithm related to the so-called accessibility problem, representation theory of Virasoro and Kac-Moody algebras, and Conformal Field Theory (CFT). The remarkable and mysterious dilogarithm [Za2], [Za3] will be the main hero of our paper, but we start with definition and list some applications of more general and may be even more amazing functions:

polylogarithm and hyperlogarithm.

**Definition 1** (G. Leibnitz, 1696; L. Euler, 1768). *The polylogarithm function $Li_k(x)$ is defined for $0 < x < 1$ by*

$$Li_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad (0.1)$$

*and admits analytic continuation to the complex plane as a multivalued analytical function on $x$.*

**Definition 2** (E. Kummer, 1840; I. Lappo-Danilevsky, 1910; K.-T. Chen, 1977). *For given vector $n \in \mathbf{Z}_+^l$, the hyperlogarithm of type $n$ is defined for a point $x$ lying in the unit cube $I_l = \{x = (x_1, \ldots, x_l) \mid |x_i| < 1,\ 1 \le i \le l\}$ by*

$$\Phi_n(x_1, \ldots, x_l) = \sum_{0<k_1<k_2<\cdots<k_l} \frac{x_1^{k_1} x_2^{k_2} \ldots x_l^{k_l}}{k_1^{n_1} k_2^{n_2} \ldots k_l^{n_l}}. \tag{0.2}$$

The function $\Phi_n(x)$ admits an analytic continuation to the complex space $\mathbf{C}^l$.

### 0.1. Ubiquity of polylogarithms.

The polylogarithm and hyperlogarithm have many intriguing properties and have appeared in various branches of mathematics and physics. We mention the following ones:

— volumes of polytopes in hyperbolic and spheric geometry (N. Lobachevsky (1836), L. Schläfli (1852), H. Coxeter (1935), G. Kneser (1936), ...)

— volumes of 3-dim hyperbolic manifolds (S. Kojima, J. Milnor, W. Neumann, T. Yoshida, D. Zagier, ...)

— combinatorial formula for characteristic classes (I.M. Gel'fand, R. MacPherson, A. Gabrielov, M. Losik, ...)

— cohomology of $GL_n(\mathbf{C})$ (A. Borel, J. Dupont, D. Quillen, A. Suslin, ...)

— special values of zeta functions for number fields/elliptic curves (S. Bloch, A. Beilinson, P. Deligne, A. Goncharov, D. Zagier, ...)

— algebraic $K$-theory (A. Borel, S. Bloch, A. Goncharov, A. Suslin, D. Zagier, ...)

— number theory : Rogers-Ramanujan's type identities, asymptotic behavior of partitions (S. Ramanujan, G. Hardy, D. Littlewood, B. Richmond, G. Meinardus, ...)

— representation theory of infinite dimensional algebras (A. Feingold, J. Lepowsky, V. Kac, D. Peterson, M. Wakimoto, B. Feigin, D. Fuchs, E. Frenkel, ...)

— Exactly Solvable Models (R. Baxter, V. Bazhanov, A.N. Kirillov, N.Yu. Reshetikhin, W. Nahm, A. Varchenko, ...)

— Conformal Field Theory (Al. Zamolodchikov, T. Klassen, E. Melzer, F. Ravanini, A.N. Kirillov, A. Kuniba, ...)

— Low dimensional topology; Vassiliev-Kontsevich's knot invariants; Drinfeld's associator, ...(V. Drinfeld, M. Kontsevich, D. Bar-Natan, T. Kohno, X.-S. Lin, T.Q.T. Le, J. Murakami, ...)

— Quantum dilogarithm (L.D. Faddeev, R. Kashaev, A.N. Kirillov, ...)

See also [Ca], [BGSV], [DS], [Go], [JKS], [Le5], [Mi2], [Mi3], [O], [V], [Za1]-[Za4].

### 0.2. Dilogarithm.

The main subject of this paper is the dilogarithm identities. More exactly, we are study the relations of the following type

$$\sum_{i=1}^{N} L(x_i) = c\frac{\pi^2}{6}, \tag{0.3}$$

where $L(x)$ is the Rogers dilogarithm, all $x_i$ are assumed to be the algebraic and $c$ is a rational number. The main problems we are interested in are the following:

*i*) how to find/classify the relations of type (0.3),

*ii*) how one can prove a dilogarithm identity.

Both problems have the deep connections with algebraic $K$-theory, representation theory of infinite dimensional algebras, combinatorics and mathematical physics.

The main purpose of the paper presented is to study the dilogarithm identities and related topics from the different points of view, namely, the algebraic, analytic, asymptotic, group-theoretic and combinatorial ones.

The first non-trivial examples of dilogarithm relations were obtained by Euler and Landen in the second half of 18-th century. And it happened only after 150 years that Coxeter and Watson had found the new examples of the type (0.3) dilogarithm relations (see (1.11) and Section 2.1.2). The most complicated among the Coxeter relations is the following one

$$L(\rho^{20}) - 2L(\rho^{10}) - 15L(\rho^4) + 10L(\rho^2) = \frac{\pi^2}{5}. \tag{0.4}$$

The methods used by Coxeter and Watson were entirely different. Coxeter's relations were developed from the properties of a certain infinite series, whereas Watson deduced his relations by "simple elimination" using Euler's and Abel's one-variable functional equations (i.e. (1.3) and (1.7)). Following L. Lewin, let us give

**Definition 3.** *A dilogarithm relation (0.3) is called to be accessible if it can be obtained by applying the five-term relation (1.4) a number of times.*

Although Watson's dilogarithm relations are accessible from the five term relations only, they are not obviously so. As Watson indicated, "he had long suspected the existence of certain results, and although his eventual proof is easy enough to follow, it is clear that it was not all that easy to come by" (see [Le5], p.5). This comment is applicable to the Coxeter relations as well. In fact, the last Coxeter relation had long been believed to be inaccessible (on the basis of many failed tries). However, it has been shown by Dupont in 1990 (see [Le5], p.52) that the relation (0.4) can be really deduced from the five-term relation (1.4) and the so-called factorization (or multiplication) formula

$$L(y^n) = n\sum_{l=1}^{n} L(\exp\left(\frac{2\pi i l}{n}\right) y).$$

There exist also the proofs of Coxeter's relations based on Lewin's single-variable ([Le4] and Theorem 1) and Ray's multivariable ([Ray] and Proposition E) functional equations. In fact, Ray multivariable functional equation is the very powerful means for proving the dilogarithm identities (see [Ray]). However, the known proof of Ray's functional equation [Ray] uses the analytical methods (e.g. the integral representation for $\mathrm{Li}_2$). Hence, the Ray functional equation can not be considered *ad hoc* as accessible.

Now we came to one of the main problems considered in the Section 1 of our paper, namely, to that of the dilogarithm identities accessibility. According to the strong version of Goncharov's conjecture (see (2.3.2)), any dilogarithm relation (0.3) with real algebraic $x_i$, $1 \le i \le N$, can be obtained as a linear combination with rational coefficients of the five-term relations:

$$\sum_{i=1}^{N} L(x_i) - cL(1)$$
$$= \sum_j n_j \left\{ L(a_j) + L(b_j) - L(a_j b_j) - L\left(\frac{a_j(1-b_j)}{1-a_j b_j}\right) - L\left(\frac{b_j(1-a_j)}{1-a_j b_j}\right) \right\},$$

where all $a_j, b_j$ lie in $\mathbf{R} \cap \overline{\mathbf{Q}}$ and all $n_j \in \mathbf{Q}$. Here $L(x)$ is the so-called single-valued dilogarithm (see (1.13)). It was indicated by Lichtenbaum ([Li], see also Section 1.1.3) that one should be careful working with the single-valued dilogarithm, because it does not satisfy the five-term relation in general (but only modulo $\pi^2$).

The main purpose of the Sections 1.1-1.3 is to prove accessibility of some well-known dilogarithm identities such as the Coxeter, Lewin, Loxton, Browkin ones. Let us note, that Browkin's identities have been found in [Bro2] and [Le3] "numerically", i.e. without, up to-date, any analytic proof. We give also the proofs of accessibility of the Lewin single-valued functional equations as well as the Ray multivalued functional equation (for simplicity, we consider in the paper presented only the case $n \le 2$, see Appendix).

A remarkable family of dilogarithm identities (see [Kir4], [BR], [Ku]) comes from a consideration of some Mathematical Physics problems. Namely, in physics, the dilogarithm appeared at first from a calculation of magnetic susceptibility in the $XXZ$-model at small magnetic field [KR], [Kir4], [BR]. More recently [Z], the dilogarithm identities (through Thermodynamic Bethe Ansatz (TBA)) appeared in the context of investigation of the ultra violet (UV) limit or the critical behavior of the integrable 2-dimensional quantum field theories and lattice models [Z], [DR], ... . The meaning of dilogarithm identities in question is to identify some theories via the central/effective charges and conformal dimension of primary fields computation. Even more, it was shown (e.g. [NRT], [DKKMM], [KMM], [KKMM1], [Tr], ...) using the Richmond-Szekeres method [RS] and the Kac-Wakimoto theorem, that the dilogarithm identities can be derived from an investigation of the asymptotic behavior of some characters of 2-dimensional CFT. To be more precise, it is necessary to have the Rogers-Ramanujan-left-hand-side type formulas for the characters (and branching functions) of the Virasoro and affine Lie algebras. This problem is very interesting one and is considered in many papers [LP], [FeSt]... .

In the Section 1.3 we give a new proof of the $A_n$-type dilogarithm identity [Kir7], mainly based on the multiple uses of the five-term relation.

In the Section 1.4 we discuss and partially prove the $\widehat{sl_2}$-case of Kuniba-Nakanishi's conjecture [KN]. This conjecture predicts the integrality and divisibility by 24 for the remainder term in some special linear combination of the principal-branch-valued Euler dilogarithm and logarithms and has many interesting connections with CFT, TBA and representation theory [KN], [KNS], ... . In principle, using the $A_n$-dilogarithm identity (1.28), one can reduce a proof of Kuniba-Nakanishi's identity for $\widehat{sl_n}$ to a proving that between the values of logarithms only. We intend to consider this interesting subject, including a generalization of Kuniba-Nakanishi's identity to a "fractional level", in a separate publication.

Summarizing, in the Section 1 we present mainly the algebraic method for proving the single-valued-dilogarithm identities by means of the multiple uses of the five-term relation (1.4). Presumably, any identity of type (0.3) can be obtained by this way (strong form of Goncharov's conjecture). But even if it is so, to have the more direct ways to produce the dilogarithm identities are still desirable. One of such ways, the so-called asymptotic method (S. Ramanujan, G. Hardy, G. Meinardus, B. Richmond, S. Szekeres, W. Nahm, B. MacCoy, M. Terhoeven, ...) and closely related with Rogers-Ramanujan type identities (L. Rogers, S. Ramanujan, I. Schur, G. Watson, L. Slater, B. Gordon, G. Andrews, D. Bressoud, B. Feigin, E. Melzer, ...) will be discussed in the Section 2.1. Another one, related with the algebraic $K$-theory (torsion in $K_3(\mathbf{R})$, S. Bloch, A. Suslin, D. Zagier, E. Frenkel, A. Szenes, ...) will be considered in Section 2.3. Finally, we discuss the so-called character-asymptotic method due to V. Kac and M. Wakimoto (see also E. Frenkel, A. Szenes, ...). This method is closely related with the following problems:

*i*) combinatorial (i.e. Rogers-Ramanujan's and Schur's type) formulae for the branching and string functions in the representation theory of integrable highest weight modules over affine Kac-Moody algebras [Kac];

*ii*) finite-dimensional version of crystal basis (see e.g. [FrSz2]) and its connection with Robinson-Schensted type correspondence;

*iii*) a natural "finitization" (or polynomial version) of the Weyl-Kac [Kac] and Feigin-Fuchs-Rocha-Caridi character formulae;

*iv*) branching functions as the Thermodynamic-Bethe-Ansatz-like limit of the Kostka-Foulkes polynomials (or Lusztig's $q$-analog of weight multiplicity).

We are going to discuss these problems in the Sections 2.2 and 2.4.

Finally, in the Section 2.5 we consider some properties of the quantum dilogarithm (cf. [FK], [FV]) and quantum polylogarithm.

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**1.1. Dilogarithm** (G. Leibnitz (1696), L. Euler (1768), L. Rogers (1906)).

**1.1.1.** Euler's and Rogers' dilogarithm.

The Euler dilogarithm function $Li_2(x)$ defined for $0 \le x \le 1$ by

$$Li_2(x) := \sum_{n \ge 1} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} dt \tag{1.1}$$

is one of the lesser transcendental function. Nonetheless, it has many intriguing properties and has appeared in various branches of mathematics and physics.

It is also convenient to introduce the Rogers dilogarithm function $L(x)$ defined for $0 < x < 1$ by

$$L(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \log x \cdot \log(1-x) = -\frac{1}{2} \int_0^x \left( \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right) dy. \tag{1.2}$$

The function $L(x)$ has an analytic continuation to the complex plane cut analog the real axis from $-\infty$ to 0 and 1 to $+\infty$.

**1.1.2.** Five-term relation and characterization of Rogers' dilogarithm.

**Theorem A** (W. Spence (1809), N. Abel (1830), C. Hill (1830), E. Kummer (1840)). *The function $L(x)$ belongs to the class $C^\infty((0,1))$ and satisfies the following functional equations*

$$1.\ \ L(x) + L(1-x) = \frac{\pi^2}{6}, \quad 0 < x < 1, \tag{1.3}$$

$$2.\ \ L(x) + L(y) = L(xy) + L\left( \frac{x(1-y)}{1-xy} \right) + L\left( \frac{y(1-x)}{1-xy} \right), \tag{1.4}$$

*where* $0 < x, \quad y < 1$.

**Theorem B** (Cf. [Ro], Section 4) *Let $f(x)$ be a function of class $C^3((0,1))$ which satisfies the relations (1.3) and (1.4). Then we have*

$$f(x) = L(x).$$

**1.1.3.** Single–valued dilogarithm.

We continue the function $L(x)$ to all real axis $\mathbf{R} = \mathbf{R}^1 \cup \{\pm\infty\}$ by the following rules

$$\begin{aligned} &L(x) = \frac{\pi^2}{3} - L(x^{-1}), \quad \text{if} \quad x > 1, \\ &L(x) = L\left(\frac{1}{1-x}\right) - \frac{\pi^2}{6}, \quad \text{if} \quad x < 0, \\ &L(0) = 0, \quad L(1) = \frac{\pi^2}{6}, \quad L(+\infty) = \frac{\pi^2}{3}, \quad L(-\infty) = -\frac{\pi^2}{6}. \end{aligned} \tag{1.5}$$

**Important remark.** The continuation of Rogers' dilogarithm function constructed in this way satisfies the functional equation (1.3), but does not satisfy in general the five-term relation (1.4). Namely, one can check LHS(1.4) − RHS(1.4) = 0, except the case $x < 0$, $y < 0$ and $xy > 1$. In the latter case LHS(1.4) − RHS(1.4) $= -\pi^2$. So, proving accessibility of a dilogarithm identity we must remember about an existence of the exceptional cases. We leave a checking of exceptional cases to a reader.

In the last part of this section we are going to show that (1.3) is a corollary of (1.4). Namely, let us take $x = \dfrac{a}{1-b}$ and $y = \dfrac{b}{1-a}$ in (1.4). It is clear that $a = \dfrac{x(1-y)}{1-xy}$, $b = \dfrac{y(1-x)}{1-xy}$ and the five-term relation (1.4) is reduced to the so–called Abel equation (see e.g. [Le])

$$L\left(\frac{a}{1-b}\right) + L\left(\frac{b}{1-a}\right) = L\left(\frac{a}{1-b}\cdot\frac{b}{1-a}\right) + L(a) + L(b). \tag{1.6}$$

From (1.6), with $1-a$ for $b$, we obtain

$$L(a) + L(1-a) = L(1),$$

as it was claimed. If in (1.4) we take $x = y$, we get the Abel duplication formula

$$L(x^2) = 2L(x) - 2L\left(\frac{x}{1+x}\right). \tag{1.7}$$

From (1.6), with $-a$ for $b$, and using (1.7), we obtain

$$L(a^2) + L\left(\frac{-a^2}{1-a^2}\right) = 0$$

and consequently,

$$L(a) + L(-a) = \frac{1}{2}L(a^2). \tag{1.8}$$

Finally, if in (1.6) we take $\dfrac{a-1}{a}$ for $b$, we get

$$\begin{aligned} &L(a^2) + L(a^{-2}) = 2L(1); \\ &L(-a^2) + L\left(\frac{-1}{a^2}\right) = -L(1). \end{aligned} \tag{1.9}$$

**1.1.4.** Dilogarithm of complex argument. Wigner-Bloch's function.

By using the integral representation

$$\mathrm{L}i_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt,$$

Euler's dilogarithm $\mathrm{L}i_2(x)$ can be analytically continued as a multivalued complex function to the complex plane cut along the real axis from 1 to $+\infty$.

**Proposition A** (E. Kummer (1840)). *Assume that $r, \theta$ are the real numbers and $|r| < 1$, then*

$$\begin{aligned}\mathrm{L}i_2(re^{i\theta}) = & -\frac{1}{2}\int_0^r \frac{\log(1-2x\cos\theta + x^2)}{x} dx \\ & + i\left\{w\log(r) + \frac{1}{2}Cl_2(2w) + \frac{1}{2}Cl_2(2\theta) - \frac{1}{2}Cl_2(2\theta+2w)\right\},\end{aligned}$$

*where*

$$w = \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right), \quad \text{and} \quad Cl_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2} = -\int_0^\theta \log|2\sin(t/2)| dt$$

*is Clausen's function.*

**Proposition B** (F. Newman (1847)).

$$\mathrm{L}i_2(e^{i\theta}) = \pi^2 \overline{B}_2\left(\frac{\theta}{2\pi}\right) + iCl_2(\theta),$$

*where* $\overline{B}_2(x) := -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}$ *is the second modified Bernoulli polynomial.*

In recent studies in $K$-theory the Wigner-Bloch dilogarithm function $D(z)$ has played an important role [Bl]. It can be related to Clausen's function via the formula

$$\begin{aligned}D(z) &:= \arg(1-z)\log|z| - \mathrm{Im}\int_0^z \frac{\log(1-t)}{t} dt \\ &= \frac{1}{2}\left\{Cl_2(2\theta) + Cl_2(2w) - Cl_2(2\theta+2w)\right\},\end{aligned}$$

where $\theta = \arg(z)$, $w = \arg(1-\overline{z})$ and $D(z) = 0$ for $z \in \mathbf{R} \cup \{\infty\}$.

The dilogarithm $D$: $\mathbf{C} \cup \{\infty\} \to \mathbf{R}$ is a real-analytic function on $\mathbf{C} \setminus [0,1]$, continuous on $\mathbf{C} \cup \{\infty\}$. We summarize the basic properties of Wigner-Bloch's dilogarithm in the next Proposition (see e.g. [Bl], [Bro2]).

**Proposition C.** *For* $z \in \mathbf{C} \cup \{\infty\}$, *we have*
*i)* $D(z) + D(1-z) = 0$, $D(z) + D(1/z) = 0$, $D(z) = D(\overline{z})$,
*ii)* $D(z^n) = n \sum_{k=0}^{n-1} D\left(\exp\left(\frac{2\pi k i}{n}\right) z\right)$,
*iii)* (Five-term relation) *If* $z, w \in \mathbf{C} \setminus [0,1]$, *then*

$$D(z) + D(w) = D(zw) + D\left(\frac{z(1-w)}{1-zw}\right) + D\left(\frac{w(1-z)}{1-zw}\right).$$

There is a geometric interpretation of the number $D(z)$ and consequently of $Cl_2(t) = D(e^{it})$. Namely, for a complex number $z$ , the volume of the asymptotic simplex with vertices $0, 1, z$ and $\infty$ in 3-dimensional hyperbolic space is equal to $|D(z)|$. Some formulae for the Wigner-Bloch dilogarithm can be interpreted as relations between the volumes of corresponding simplexes. For more details, see [Mi2], [Mi3], [Za1], [Za4].

**1.1.5.** Multivariable functional equations (L. Rogers (1906), G. Ray (1990)).

Using the analytic methods, Rogers [Ro] had proved the following dilogarithm identity.

**Proposition D** (L. Rogers). *Define the polynomial*

$$r(x,t) := 1 - t - \prod_{j=1}^{n} (1 - \alpha_j x)$$

*of degree* $n$ *in* $x$ *for nonzero complex numbers* $\alpha_1, \ldots, \alpha_n$. *Let* $\{x_l := x_l(t)\}$, $l = 1, \ldots, n$ *be the algebraic functions satisfying* $r(x_l, t) = 0$. *Then we have*

$$\sum_{j=1}^{n} \sum_{l=1}^{n} [L(\alpha_j/\alpha_l) - L(\alpha_j x_l)] = L(1-t).$$

The special case $n = 2$ leads to the form of Hill's equation (see Remark in Section 1.3) with two independent variables and five dilogarithmic terms.

**Example.** Let us assume $\alpha_1 = \ldots = \alpha_n = 1$ and $1 - t = y^n$. Then $x_l = \zeta_l y$, where $\zeta_l = \exp\left(\frac{2\pi i l}{n}\right)$, and Rogers' identity is reduced to the so-called factorization formula

$$L(y^n) = n \sum_{l=1}^{n} L(\zeta_l y).$$

A remarkable generalization of Rogers' multivariable identity for dilogarithm was discovered by G. Ray [Ray]. Since the statement of the most general dilogarithm identity in the paper [Ray] is rather involved, we present here only an important particular case of the general Ray's result.

**Proposition E** (G. Ray). *Consider the polynomial*

$$r(x,t) := (1-t)A\prod_{j=1}^{n}(1-\beta_j x) - \prod_{j=1}^{n}(1-\alpha_j x)$$

*of degree $n$ in $x$ for nonzero complex numbers $\{\alpha_j, A\}$ and some complex numbers $\{\beta_j\}$. Let $\{x_l := x_l(t)\}$, $1 \le l \le n$, be the algebraic functions satisfying $r(x_l,t)=0$. Then we have*

$$\sum_{j=1}^{n}\sum_{l=1}^{n}[L(\beta_j x_l) - L(\beta_j/\alpha_l) - L(\alpha_j x_l) + L(\alpha_j/\alpha_l)]$$
$$= L(A(1-t)) - L\left(\frac{A\beta_1\dots\beta_n}{\alpha_1\dots\alpha_n}(1-t)\right).$$

In Appendix we are going to prove the accessibility of this identity for $n=1,2$. General case will be considered elsewhere.

### 1.2. Accessible dilogarithm relations.

**1.2.1.** Accessibility of Coxeter's and Lewin's identities.

It is easy to see from (1.3) and (1.7) that

$$L(1)=\frac{\pi^2}{6},\ L(-1)=-\frac{\pi^2}{12},\ L(\frac{1}{2})=\frac{\pi^2}{12},\quad \text{(L.Euler, 1768)}; \tag{1.10}$$
$$L(\frac{1}{2}(\sqrt{5}-1))=\frac{\pi^2}{10},\quad L(\frac{1}{2}(3-\sqrt{5}))=\frac{\pi^2}{15},\quad \text{(J.Landen, 1780)}.$$

Proof. Let us put $\rho := \frac{1}{2}(\sqrt{5}-1)$. It is clear that $\rho^2+\rho=1$. So we have $L(\rho^2) = L(1-\rho) = L(1) - L(\rho)$. Now we use the Abel formula (1.7)

$$L(\rho^2) = 2L(\rho) - 2L\left(\frac{\rho}{1+\rho}\right) = 2L(\rho) - 2L(\rho^2).$$

Consequently, $3L(\rho^2) = 2L(\rho)$. But as we already saw,

$$L(1) = L(\rho) + L(\rho^2) = \frac{5}{3}L(\rho).$$

This proves the part of (1.10) due to J. Landen. ∎

The results of Euler and Landen in equation (1.10) were the only such expressions until Coxeter [Co] in 1935 paper proved the following formulae

$$L(\rho^6) = 4L(\rho^3) + 3L(\rho^2) - 6L(\rho) + \frac{7\pi^2}{30};$$
$$L(\rho^{12}) = 2L(\rho^6) + 3L(\rho^4) + 4L(\rho^3) - 6L(\rho^2) + \frac{\pi^2}{10}; \tag{1.11}$$
$$L(\rho^{20}) = 2L(\rho^{10}) + 15L(\rho^4) - 10L(\rho^2) + \frac{\pi^2}{5}.$$

Here we rewrote Coxeter's results using Rogers' dilogarithm. H. Coxeter obtained his results using the properties of a certain infinite series. Our nearest aim is to show that the Coxeter results (1.11) are accessible from the functional equation (1.4) only. We prove also the accessibility of the relation discovered by L. Lewin [Le3]

$$L(\rho^{24}) = 6L(\rho^8) + 8L(\rho^6) - 6L(\rho^4) + \frac{\pi^2}{30}. \tag{1.12}$$

In fact we will prove that the Lewin single-variable functional equations (see [Le4], or [Le5], Chapter 6, or below) are also accessible from the five-term relation (1.4) only.

Let us remark, that one can obtain both Coxeter's relations (1.11) and Lewin's one (1.12) by using the Ray multi-variable functional equation (Proposition E, see also Exercises to Section 1). It is necessary to underline, that the accessibility of the last Coxeter's relation among (1.11) was proven at first by J. Dupont in 1989 (see [Le5], p.52). A proof of accessibility of Lewin's relation (1.12) and Lewin's single-variable functional equations (see Theorem 1) seems to be new.

**Theorem 1.** *The following single–variable functional equations (1.13), (1.14) and (1.15) are accessible from the functional equation (1.4)*

$$\begin{aligned} L\left(-z^7\frac{1-z}{1+z}\right) &= 2L(z^2(1-z)) + L\left(\frac{-z^3}{1-z^2}\right) + 2L\left(\frac{z^3}{1+z}\right) \\ &\quad + L(-z(1-z^2)) + \frac{7}{4}L(z^4) - \frac{9}{4}L(z^2) + \frac{1}{2}L\left(z\frac{1-z}{1+z}\right) - \frac{1}{2}L\left(-z\frac{1+z}{1-z}\right); \end{aligned} \tag{1.13}$$

$$\begin{aligned} L(-z^6(1-z)^3) &= 2L(-z(1-z)(1-z^2)) + L\left(-\frac{z^4}{(1-z)(1+z)^2}\right) \\ &\quad + 3L(-z^2(1-z)) + 4L(z(1-z)) + L\left(\frac{-z^2}{1-z}\right) + 4L(z^2) - 2L(z); \end{aligned} \tag{1.14}$$

$$\begin{aligned} L\left(-\frac{z^9}{(1+z)^3}\right) &= L(-z(1-z)(1-z^2)) + 2L\left(-\frac{z^4}{(1-z)(1+z)^2}\right) \\ &\quad + 3L\left(\frac{-z^3}{1+z}\right) + 4L\left(\frac{-z^2}{1+z}\right) + L(-z(1+z)) + 3L(z^2) + 2L(z). \end{aligned} \tag{1.15}$$

Proof. We have the following series of relations coming from functional equation (1.4):

- $L(1-z^2+z^3) + L(1+z-z^3) - L((1-z^2+z^3)(1+z-z^3)) - L\left(\frac{1+z-z^2+z^4}{1+z}\right) - L\left(\frac{-z(1+z-z^3)}{1-z^2+z^4}\right) = 0;$
- $L((1-z^2+z^3)(1+z-z^3)) + L\left(\frac{1+z^2}{1+z}\right) - L\left(\frac{1+z+z^7-z^8}{1+z}\right) - L\left(\frac{1+z^6}{z^6}\right) - L\left(1-\frac{(1+z)(1-z^2+z^4)}{z^6}\right) = 0;$

- $L\left(-\frac{1+z}{z^5}\right)+L\left(\frac{(1+z)^2}{z(1+z+z^2)}\right)+L\left(\frac{1+z-z^3}{1+z}\right)-L\left(\frac{-z}{1-z^2+z^4}\right)$
  $-L\left(\frac{(1+z)(1-z^2+z^4)}{z^6}\right)=0;$
- $L\left(\frac{1+z^6}{z^6}\right)+L\left(\frac{-z(1+z-z^3)}{1-z^2+z^4}\right)-L\left(-\frac{1+z+z^2-z^5}{z^5}\right)-L\left(\frac{(1+z)(1+z^2)}{z(1+z+z^2)}\right)$
  $-L\left(\frac{1+z-z^3}{(1+z+z^2)(1-z^2+z^4)}\right)=0;$
- $L\left(\frac{-z^5}{1+z}\right)+L\left(\frac{-z^3}{1-z^3}\right)+L\left(\frac{z^2}{1+z+z^2}\right)-L(z^2(1-z))-L\left(\frac{-z^3}{1-z^2}\right)=0;$
- $L\left(\frac{1+z-z^3}{1+z}\right)+L\left(\frac{-(1+z)z^2}{1+z-z^3}\right)-L(-z^2)-L\left(\frac{1+z+z^2}{(1+z)(1+z^2)}\right)$
  $-L\left(\frac{-z^5}{1+z+z^2-z^5}\right)=0;$
- $L\left(\frac{1+z}{1+z+z^2}\right)+L\left(\frac{1+z}{z}\right)-L\left(\frac{(1+z)^2}{z(1+z+z^2)}\right)-L\left(\frac{-(1+z)z^2}{1+z-z^3}\right)$
  $-L\left(\frac{1+z}{1+z-z^3}\right)=0;$
- $L\left(\frac{z}{1+z}\right)+L\left(\frac{z^2}{1+z^2}\right)-L\left(\frac{z^3}{(1+z)(1+z^2)}\right)-L\left(\frac{z}{1+z+z^2}\right)$
  $-L\left(\frac{z^2}{1+z+z^2}\right)=0;$
- $L\left(\frac{1}{1+z+z^2}\right)+L\left(\frac{-1}{z}\right)-L\left(\frac{-1}{z(1+z+z^2)}\right)-L\left(\frac{1}{1+z^2}\right)$
  $-L\left(\frac{-z}{1+z^2}\right)=0;$
- $L\left(z^3\right)+L\left(\frac{z+z^2}{1+z+z^2}\right)+L\left(\frac{z^2}{1+z+z^2}\right)-L(z)-L(z^2)=0;$
- $\frac{1}{2}L\left(\frac{z(1-z)}{1+z}\right)+\frac{1}{2}L\left(\frac{-z(1+z)}{1-z}\right)-\frac{1}{2}L(-z^2)-\frac{1}{2}L\left(\frac{z}{1+z}\right)-\frac{1}{2}L\left(\frac{-z}{1-z}\right)=0.$

Now let us take a sum of all these equations. Using the identities (1.3), (1.8) and (1.9) the last sum can be simplified to yield

$$\mathrm{LHS}(1.13)-\mathrm{RHS}(1.13)=\frac{3}{2}L(-z^2)+L(z)+L(z^2)-\frac{1}{2}L\left(\frac{z}{1+z}\right)+$$
$$+\frac{1}{2}L\left(\frac{-z}{1-z}\right)-2L\left(\frac{z^2}{1+z^2}\right)-\frac{7}{4}L(z^4)+\frac{9}{4}L(z^2)=0.$$

The last equality follows from Abel's duplication formula (1.7) and (1.8). Note that the following identities had played the key role in validity of the first six equations

$$
\begin{aligned}
&1+z+z^7-z^8=(1+z^2)(1+z-z^3)(1-z^2+z^3),\\
&1+z+z^5=(1+z+z^2)(1-z^2+z^3),\\
&1+z+z^2-z^5=(1+z^2)(1+z-z^3).
\end{aligned}
$$

Now let us prove the identity (1.14). We use the following equalities coming from (1.4)

- $L(1-z+z^2)+L(1+z^2-z^3)-L((1-z+z^2)(1+z^2-z^3))$
  $-L\left(1+\frac{z}{(1-z)(1+z^2)}\right)-L\left(1-\frac{1}{(1-z)(1+z^2)}\right)=0;$
- $L(1+z(1-z)(1-z^2))+L((1-z+z^2)((1+z^2-z^3))-L(1+z^6(1-z)^3)$
  $-L\left(1-\frac{1+z}{z^5(1-z)}\right)-L\left(1+\frac{1+z^2}{z^5(1-z)}\right)=0;$
- $L\left(\frac{-1}{z(1-z)(1-z^2)}\right)+L\left(\frac{1+z^2}{1+z}\right)+L\left(\frac{1}{z^4}\right)-L\left(\frac{1+z}{z^5(1-z)}\right)$
  $-L\left(\frac{-z^4}{(1-z)(1+z)^2}\right)=0;$
- $L\left(\frac{z}{1+z^2}\right)+L\left(\frac{1}{1-z}\right)+L(-z^2)-L(z(1-z))-L\left(\frac{1}{(1-z)(1+z^2)}\right)$
  $-L(-z^2(1-z))=0;$
- $L\left(\frac{z^3}{1+z^2}\right)+L\left(\frac{-z}{1-z}\right)+L(-z^2)-L\left(\frac{-z}{(1-z)(1+z^2)}\right)-L\left(\frac{z^3}{1+z^2}\right)=0;$
- $L\left(\frac{-1}{z^2(1-z)}\right)+L\left(\frac{-1}{z^3}\right)+L\left(\frac{1+z^3}{1+z^2}\right)-L\left(-\frac{1+z^2}{z^5(1-z)}\right)=0;$
- $L(1-z+z^2)+L\left(\frac{1-z+z^2}{1+z^2}\right)+L\left(\frac{z}{1+z}\right)-L\left(\frac{1+z^2}{1+z}\right)-L\left(\frac{1+z^3}{1+z^2}\right)=0;$
- $L(-z^3)+L\left(\frac{-z^2}{1-z^2}\right)+L\left(\frac{z}{1+z}\right)-L\left(\frac{-z^2}{1-z}\right)-L(z(1-z))=0.$

Now let us take a sum of all these equations. Using the classical identities (1.3), (1.8) and (1.9) this sum can be simplified to yield

$$
LHS(1.14)-RHS(1.14)=2L(-z^2)+2L\left(\frac{z}{1+z}\right)+L\left(\frac{-z^2}{1-z^2}\right)+4L(z^2)-2L(z)-L(z^4).
$$

Using the duplication formula (1.8) to remove the terms with negative arguments, we obtain the value zero for the $RHS$ of the last equation, as it was stated.

Finally, let us prove the identity (1.15). We use the following series of equalities which are a direct consequence of the five-term relation (1.4)

- $-L\left(-\frac{(1+z)^3}{z^9}\right)-L\left(\frac{-z^4}{(1-z)(1+z)^2}\right)+L\left(\frac{1+z}{z^5(1-z)}\right)+L\left(\frac{1+z}{z^4(1+z^2)}\right)$

$$+ L\left(\frac{(1+z+z^2)(1+z+z^3)}{(1+z)^2(1+z^2)}\right) = 0;$$

- $-L(-z(1-z)(1-z^2)) - L\left(\frac{-z^4}{(1-z)(1+z)^2}\right) + L\left(\frac{z^5(1-z)}{1+z}\right) + L\left(\frac{-z^4}{1-z^4}\right)$
$$+ L\left(-\frac{z(1-z)}{1+z^2}\right) = 0;$$
- $-L\left(\frac{1+z}{z^4(1+z^2)}\right) - L\left(-\frac{z^3}{1+z}\right) + L\left(\frac{-1}{z(1+z^2)}\right) + L\left(\frac{1-z^3}{1+z}\right) + L\left(\frac{1}{z^3}\right) = 0;$
- $-L\left(\frac{z(1-z)}{1+z}\right) - L\left(\frac{-z^4}{1-z^4}\right) + L\left(\frac{-z^5}{(1+z)^2(1+z^2)}\right)$
$$+ L\left(\frac{z}{(1+z+z^2)(1+z+z^3)}\right) + L\left(1 - \frac{1+z}{(1-z^3)(1+z+z^3)}\right) = 0;$$
- $L\left(\frac{1+z}{(1-z^3)(1+z+z^3)}\right) + L\left(\frac{1+z+z^3}{1+z}\right) - L\left(\frac{1}{1-z^3}\right) - L\left(\frac{1}{1+z+z^3}\right)$
$$- L\left(\frac{z(1+z^2)}{1+z}\right) = 0;$$
- $L\left(\frac{z}{1+z+z^2}\right) + L\left(\frac{1}{1+z+z^3}\right) - L\left(\frac{z}{(1+z+z^2)(1+z+z^3)}\right)$
$$- L\left(\frac{1}{(1+z)(1+z^2)}\right) - L\left(\frac{z^2}{(1+z)(1+z^2)}\right) = 0;$$
- $L\left(\frac{z(1+z^2)}{1+z}\right) + L\left(\frac{1}{(1+z)(1+z^2)}\right) - L\left(\frac{z}{(1+z)^2}\right) - \left(\frac{1-z}{1+z^2}\right) - L(z^2) = 0;$
- $-L(1-z) - L\left(\frac{z}{1+z}\right) + L\left(\frac{z(1-z)}{1+z}\right) + L\left(\frac{z^2}{1+z^2}\right) + L\left(\frac{-1}{z(1+z^2)}\right) = 0;$
- $L\left(\frac{z^2}{(1+z)(1+z^2)}\right) + L\left(\frac{1+z^2}{z^2}\right) - L\left(\frac{1}{1+z}\right) - L\left(\frac{1+z+z^3}{z^3}\right)$
$$- L\left(\frac{1-z}{1+z^2}\right) = 0;$$
- $L\left(-\frac{1+z}{z^2}\right) + L\left(\frac{z}{(1+z)^2}\right) - L\left(\frac{-1}{z(1+z)}\right) - L\left(\frac{1}{1+z}\right) - L\left(\frac{-1}{z}\right) = 0;$
- $L\left(\frac{1+z+z^2}{1+z}\right) + L\left(\frac{1+z}{1-z^3}\right) - L\left(\frac{1}{1-z}\right) - L\left(\frac{z}{1+z+z^2}\right) - L\left(\frac{1+z^2}{1+z}\right) = 0;$
- $2L(z^3) + 2L\left(\frac{-z}{1-z}\right) + 2L\left(\frac{-z^2}{1-z^2}\right) - 2L(-z(1+z)) - 2L\left(\frac{-z^2}{1+z}\right) = 0.$

Taking the sum of these, simplifying and using the identities (1.3), (1.8) and (1.9), we obtain

$$LHS(1.15) - RHS(1.15) = -L\left(\frac{1}{1-z^3}\right) - L(z^2) + L(z) - L\left(\frac{1}{1+z}\right) - L\left(\frac{-1}{z}\right)$$

$$-L\left(\frac{1}{1-z}\right)+L(z^3)+2L\left(\frac{-z}{1-z}\right)+2L\left(\frac{-z^2}{1-z^2}\right)+3L(z^2)+2L(z)+2L(1).$$

Using the duplication formula (1.8), it is easy to see that the $RHS$ of the last equality is equal to zero, as we set out to prove.

Note that the following identities are crucial in the proof of (1.14) and (1.15)

$$\begin{aligned}
&(1+z)^3+z^9=(1+z+z^2)(1+z+z^3)(1+z(1-z)(1-z^2)),\\
&1+z-z^5+z^6=(1+z^2)(1+z(1-z)(1-z^2)),\\
&1+z-z^4-z^6=(1-z^3)(1+z+z^3).
\end{aligned}$$

■

Now let us return back to the Coxeter and Lewin identities (1.11) and (1.12) respectively. We start with a proving of accessibility of the first two identities among (1.11). The last Coxeter and Lewin identities are a direct consequence of the functional equations (1.13)-(1.15) when $z:=\rho$ and duplication formula (1.8). Using the relations

$$1+\rho+\rho^2=2,\quad 1+\rho^3=2\rho,\quad 2+2\rho^2-\rho^4=\rho^{-2},$$

and functional equation (1.4), it can be shown that

$$\begin{aligned}
&L(\rho)+L(\rho^2)=L(\rho^3)+L\left(\frac{\rho^2}{2}\right)+L\left(\frac{1}{2}\right);\\
&L\left(\frac{\rho^3}{1+\rho^3}\right)=L\left(\frac{\rho^2}{2}\right);\quad L\left(\frac{\rho^6}{1+\rho^6}\right)=L\left(\frac{\rho^4}{2(1+\rho^2)}\right);\\
&L\left(\frac{\rho^2}{2}\right)+L\left(\frac{\rho^2}{1+\rho^2}\right)=L\left(\frac{\rho^4}{2(1+\rho^2)}\right)+L(\rho^3)+L(\rho^4).
\end{aligned}$$

Hence,

$$\begin{aligned}
&i)\ L(\rho^6)-4L(\rho^3)-3L(\rho^2)+6L(\rho)=-2(L(\rho^3)+L\left(\frac{\rho^2}{2}\right))-3L(\rho^2)+6L(\rho)\\
&=4L(\rho)-5L(\rho^2)+L(1)=\frac{7}{5}L(1);\\
&ii)\ L(\rho^{12})-2L(\rho^6)-3L(\rho^4)-4L(\rho^3)+6L(\rho^2)\\
&=-2L\left(\frac{\rho^4}{2(1+\rho^2)}\right)-3L(\rho^4)-4L(\rho^3)+6L(\rho^2)\\
&=-2(L\left(\frac{\rho^2}{2}\right)+L(\rho^3))+4L(\rho^4)-2L(\rho^2)+2L(\rho^4)-3L(\rho^4)+6L(\rho^2)\\
&=2L(\rho^2)-2L(\rho)+L(1)=\frac{3}{5}L(1).
\end{aligned}$$

Thus the accessibility of Coxeter's and Lewin's dilogarithm identities are proved.

**1.2.2.** Accessibility of Watson's, Loxton's and Lewin's identities.

Let $\alpha$, $-\beta$ and $-\dfrac{1}{\gamma}$ be the three roots of the cubic $x^3+2x^2-x-1=0$, so that $\alpha=\dfrac{1}{2}\sec\dfrac{2\pi}{7}$, $\beta=\dfrac{1}{2}\sec\dfrac{\pi}{7}$ and $\gamma=2\cos\dfrac{3\pi}{7}$ all lie between 0 and 1. Then Watson's relations [W2] are

$$L(\alpha)-L(\alpha^2)=\frac{1}{7}L(1),\quad 2L(\beta)+L(\beta^2)=\frac{10}{7}L(1),\quad 2L(\gamma)+L(\gamma^2)=\frac{8}{7}L(1).$$

To prove these relations, first note that

$$\beta=\frac{1}{2}\sec\frac{\pi}{7}=\frac{1}{1+\alpha},\quad \gamma=2\cos\frac{3\pi}{7}=\frac{\alpha}{1+\alpha}.$$

Now we use the fact that $\alpha$ satisfies the equation $\alpha^3+2\alpha^2-\alpha-1=0$. Consequently,

$$\begin{aligned}&\frac{1}{1+\beta}=\frac{1+\alpha}{2+\alpha}=\alpha^2,\quad \beta+\gamma=1,\\ &\frac{\gamma}{1+\gamma}=\frac{\alpha}{1+2\alpha}=\frac{1}{(1+\alpha)^2}=\beta^2,\quad \alpha+\gamma^2=1.\end{aligned}$$

From the duplication formula (1.7),

$$\begin{aligned}&L(\alpha^2)=2L(\alpha)-2L(\gamma),\quad L(\beta^2)=2L(\beta)-2L(1-\alpha^2),\\ &L(1-\alpha)=L(\gamma^2)=2L(\gamma)-2L(\beta^2);\end{aligned}$$

and Watson's relations follow by using (1.3) and some easy elimination. It is interesting to compare the Watson results with the following one (see [Kir7])

$$\sum_{n=1}^{\frac{r-1}{2}}L\left(\left(\frac{\sin\dfrac{(j+1)\pi}{r+2}}{\sin\dfrac{(n+1)(j+1)\pi}{r+2}}\right)^2\right)=\frac{(3j+1)r-3j^2-1}{r+2}\,L(1),\tag{1.16}$$

$$r\equiv 1 \pmod 2,\quad 0\le j\le\frac{r-1}{2},\quad \text{g.c.d. }(r+2,j+1)=1.$$

If $r=3$, (1.16) is reduced to

$$L\left(\left(\frac{1}{2}\sec\frac{(j+1)\pi}{5}\right)^2\right)=\frac{9j-3j^2+2}{5}\,L(1),\quad j=0,1.\tag{1.17}$$

But $\rho=\dfrac{1}{2}\sec\dfrac{\pi}{5}$, $\rho^{-1}=\dfrac{1}{2}\sec\dfrac{2\pi}{5}$ and (1.17) is equivalent to the Landen result (1.10).

If $r = 5$, (1.16) is reduced to ($j = 0, 1, 2$)

$$L\left(\left(\frac{1}{2}\sec\frac{(j+1)\pi}{7}\right)^2\right) + L\left(\left(\frac{\frac{1}{4}\sec^2\frac{(j+1)\pi}{7}}{1-\frac{1}{4}\sec^2\frac{(j+1)\pi}{7}}\right)^2\right) = \frac{15j-3j^2+4}{7}\,L(1). \quad (1.18)$$

Now if $j = 0$ then (1.18) is reduced to ($\beta = \dfrac{1}{2}\sec\dfrac{\pi}{7}$)

$$L(\beta^2) + L\left(\left(\frac{\beta^2}{1-\beta^2}\right)^2\right) = \frac{4}{7}L(1). \quad (1.19)$$

But $\dfrac{\beta^2}{1-\beta^2} = 1-\beta$ and $\dfrac{1-\beta}{2-\beta} = \beta^2$. Consequently,

$$\text{LHS (1.19)} = L(\beta^2) + 2L(1-\beta) - 2L\left(\frac{1-\beta}{2-\beta}\right) = 2L(1) - 2L(\beta) - L(\beta^2)$$

and (1.19) is equivalent to the second Watson relation. If $j = 1$ then (1.18) is equivalent to

$$L(\alpha^2) + L\left(\left(\frac{\alpha^2}{1-\alpha^2}\right)^2\right) = \frac{4}{7}L(1). \quad (1.20)$$

But $\dfrac{\alpha^2}{1-\alpha^2} = 1+\alpha$ and $\dfrac{\alpha+1}{\alpha+2} = \alpha^2$. Consequently,

$$\begin{aligned}\text{LHS (1.20)} &= L(\alpha^2) + 2L(1+\alpha) - 2L\left(\frac{\alpha+1}{\alpha+2}\right) = 2L(1+\alpha) - L(\alpha^2)\\ &= 2L(\alpha) - L(\alpha^2) + 2L(1) - L(\alpha^2) = 2L(1) - 2\left(L(\alpha) + L(\alpha^2)\right)\end{aligned}$$

and (1.20) is reduced to the first Watson's identity.

Now we are going to prove the accessibility of the following identities due to Loxton [Lo1], [Lo2] and Lewin [Le2]

$$L(x^3) - 3L(x^2) - 3L(x) = -\frac{7}{3}L(1), \quad \text{(Loxton)}$$

$$L(y^6) - 2L(y^3) - 9L(y^2) + 6L(y) = -\frac{2}{3}L(1), \quad \text{(Lewin)}$$

$$L(z^6) - 2L(z^3) - 9L(z^2) + 6L(z) = \frac{2}{3}L(1). \quad \text{(Lewin)}$$

Here $x = \dfrac{1}{2}\sec\dfrac{\pi}{9}$, $y = \dfrac{1}{2}\sec\dfrac{2\pi}{9} = \dfrac{1}{1+x}$ and $z = 2\cos\dfrac{4\pi}{9} = \dfrac{x}{1+x}$ lie in $(0,1)$ and $x, -y, -\dfrac{1}{z}$ are the three roots of the cubic equation

$$x^3 + 3x^2 - 1 = 0.$$

The history of the discovery of the dilogarithm identities involving the three roots $x, -y, -\frac{1}{z}$ of the cubic $x^3+3x^2=1$ is amusing. The first was found by Loxton [Lo1], who applied the Richmond-Szekeres method ([RS] or Section 2.1.2, Lemma 4) to some special partition identity of the Rogers-Ramanujan type (see identity 92 from [Sl]). It was L. Lewin [Le2] who observed the parallel with Watson's three identities and conjectured the second and third relations of the triple. The second was proved by Loxton in the same paper [Lo1], but the third one was a numerical fact and for some time had no analytic derivation. Situation is changed only in 1990 when H. Gangl (non published) proved the last Loxton-Lewin identity, using methods based on Bloch groups. However, a direct proof of accessibility of the Loxton-Lewin three relations was open problem till now. The main purpose of this Section is to solve this problem and present an algebraic proof of Loxton-Lewin's dilogarithm identities, using the five-term relation (1.4) only. Our proof is based on the following identities which are a direct consequence of functional equation (1.4)

$$\begin{aligned}
&L(x^2)=2L(x)-2L(z);\\
&L(y^2)=2L(y)-2L(a);\\
&L(a^2)=2L(a)-2L(x^2);\\
&L(z^2)=2L(z)-2L\left(\frac{x}{1+2x}\right);\\
&L(x^2)+L(y^2)=L(z^2)+L(z)+L(b);\\
&L(y)+L(a)=L\left(\frac{x}{1+2x}\right)+L(x)+L(b);\\
&L(x)+L(x^2)=L(x^3)+L(a^2)+L(1)-L(c^2);\\
&L(c^2)=2L(c)-2L(y^2);\\
&L(c)=L(y)+\frac{1}{2}L(z^2);\\
&L\left(\frac{y}{3}\right)=L(a)+L(c^2)-L(y^2)-L(d);\\
&L(d)=2L(z^2)-2L(x)+L(1);\\
&L\left(\frac{z^2}{3}\right)=L(a^2)+L\left(\frac{x}{1+2x}\right)-L(z^2)-L(d);\\
&L(y^6)=2L(y^3)-2L\left(\frac{y}{3}\right);\\
&L(z^6)=2L(z^3)-2L\left(\frac{z^2}{3}\right).
\end{aligned}$$

Here the following notations and relations are used

$$a:=\frac{1}{x+2},\quad b:=\frac{x^2}{1+x},\quad c:=\frac{1}{x(2+x)},\quad d:=\frac{x}{(1+x)^2},$$

$$\frac{z^3}{1+z^3}=\frac{z^2}{3}=\frac{x}{(1+2x)(2+x)^2},\quad \frac{y^3}{1+y^3}=\frac{1}{x^2(2+x)^3}=\frac{y}{3}.$$

After elimination of $a, b, c$ and $d$ from the relations above, one can find

$$
\begin{aligned}
&L(x^2) = 2L(x) - 2L(z);\\
&L(a) = L(y) - \frac{1}{2}L(y^2);\\
&L(a^2) = 2L(y) - L(y^2) - 2L(x^2);\\
&L\left(\frac{x}{1+2x}\right) = L(z) - \frac{1}{2}L(z^2);\\
&L(b) = L(x^2) + L(y^2) - L(z^2) - L(z);\\
&L(z^2) = L(y^2) + \frac{2}{3}L(x^2) + \frac{2}{3}L(x) - \frac{4}{3}L(y);\\
&L(x^3) = 3L(x^2) + L(z^2) - L(y^2) + L(x) - L(1);\\
&L\left(\frac{y}{3}\right) = -\frac{7}{2}L(y^2) - L(z^2) + 2L(x) + 3L(y) - L(1);\\
&L\left(\frac{z^2}{3}\right) = -2L(x^2) - L(y^2) - \frac{7}{2}L(z^2) + 2L(x) + 2L(y) + L(z) - L(1).
\end{aligned}
$$

Consequently,

- $3L(x) + 3L(x^2) - L(x^3) = 3L(x) + 3L(x^2) - 3L(x^2) - L(z^2) + L(y^2) - L(x) + L(1)$
  $= 2L(x) - \frac{2}{3}L(x^2) - \frac{2}{3}L(x) + \frac{4}{3}L(y) + L(1) = \frac{4}{3}L(x) - \frac{2}{3}L(x^2) + \frac{4}{3}L(y) + L(1)$
  $= \frac{4}{3}L(1) + L(1) = \frac{7}{3}L(1);$
- $L(y^6) - 2L(y^3) - 9L(y^2) + 6L(y) = -2L\left(\frac{y}{3}\right) - 9L(y^2) + 6L(y)$
  $= 7L(y^2) + 2L(z^2) - 4L(x) - 9L(y^2) + 2L(1) = 2L(z^2) - 2L(y^2) - 4L(x) + 2L(1)$
  $= \frac{4}{3}L(x^2) + \frac{4}{3}L(x) - \frac{8}{3}L(y) - 4L(x) + 2L(1) = \frac{4}{3}\big(L(x^2) - 2L(x)\big) - \frac{8}{3}L(y) + 2L(1)$
  $= 2L(1) - \frac{8}{3}L(1) = -\frac{2}{3}L(1);$
- $L(z^6) - 2L(z^3) - 9L(z^2) + 6L(z) = -2L\left(\frac{z^2}{3}\right) - 9L(z^2) + 6L(z)$
  $= 4L(x^2) + 2L(y^2) + 7L(z^2) - 4L(x) = 4L(y) - 2L(z) - 9L(z^2) + 6L(z)$
  $= 4L(x^2) + 2L(y^2) - 2L(z^2) - 4L(x) - 4L(y) + 4L(z) + 2L(1)$
  $= 4L(x^2) - \frac{4}{3}L(x^2) - \frac{4}{3}L(x) + \frac{8}{3}L(y) - 4L(x) - 4L(y) + 4L(z) + 2L(1)$
  $= \frac{8}{3}\big(L(x^2) - 2L(x)\big) - \frac{16}{3}L(y) + 6L(1) = 6L(1) - \frac{16}{3}L(1) = \frac{2}{3}L(1).$

Finally, let us prove the accessibility of (1.16) for the case $r = 7$ and $j = 0$. Under this assumption, (1.16) is reduced to

$$L(x^2) + L(a^2) + L(z^2) = \frac{2}{3}L(1). \tag{1.21}$$

Using the relation $L(a^2) = 2L(y) - L(y^2) - 2L(x^2)$, one can obtain

$$\begin{aligned}\text{LHS } (1.21) &= L(x^2) + 2L(y) - L(y^2) - 2L(x^2) + L(z^2) \\ &= 2L(y) - L(x^2) - L(y^2) + L(y^2) + \frac{2}{3}L(x^2) + \frac{2}{3}L(x) - \frac{4}{3}L(y) \\ &= -\frac{1}{3}\big(L(x^2) - 2L(x)\big) + \frac{2}{3}L(y) = \frac{2}{3}\left(L(y) + L(z)\right) = \frac{2}{3}L(1).\end{aligned}$$

■

In fact, it can be shown that the Loxton-Lewin identities follow from (1.16) with $r = 7$, $j = 0, 1, 2$ and vice versa.

**1.2.3.** Lewin's dilogarithm identity related with $x = \frac{1}{2}(\sqrt{13} - 3)$.

In the paper [Le] based on the numerical calculations the following dilogarithm identity was suggested

$$4L(x) - L(x^2) - \frac{2}{3}L(x^3) + \frac{1}{6}L(x^6) = \frac{7}{6}L(1). \tag{1.22}$$

We are going to prove the accessibility of this identity. For this purpose let us set

$$a = \frac{x^3}{1+x^3}, \quad b = \frac{x}{1+x}, \quad c = \frac{-x}{1-x}, \quad d = \frac{-1}{1+x}.$$

Having in mind the fact that $x$ satisfies the quadratic equation $x^2 + 3x - 1 = 0$, one can check

$$\frac{x^3}{1+x^3} = \frac{x^2}{2(1+x)}, \quad 2 + 4x + x^2 = x^{-1}, \quad x(1+2x) = (1-x)^2, \quad (1+x)^2 = 2 - x.$$

Now, using the duplication formula (1.7) and functional equation (1.4), one can check the following relations

$$\begin{aligned}&L(x^2) = 2L(x) - 2L(a), \\ &L(x^2) = 2L(-x) - 2L(c), \\ &L(x^6) = 2L(x^3) - 2L(a), \\ &L\left(\frac{1}{2}\right) + L(b^2) = L(a) + L(x^3) + L((1-x)^2), \\ &L(b^2) = 2L(b) - 2L(c^2), \\ &L(c^2) = 2L(c) - 2L(d), \\ &L(d^2) = 2L(d) - 2L\left(\frac{-1}{x}\right), \\ &L((1-x)^2) = 2L(d^2) - 2L(x).\end{aligned}$$

After elimination $a, b, c, d$ from these relations, one can find

$$L(x^6) = 4L(x^3) - 2L(x^2) - 8L(x) + 8L(-x) - 8L\left(\frac{-1}{x}\right) - L(1).$$

Finally, if we substitute this expression for $L(x^6)$ to the LHS of (1.22) and use (1.9) to remove the terms with negative arguments, the result is

$$6\mathrm{LHS}(1.22) = -16L(-1) - L(1) = 7L(1),$$

as it was stated. ■

**1.2.4.** Browkin's dilogarithm identities.

Let $x = \frac{1}{6}(\sqrt{13} - 1)$ and $z = \frac{1}{6}(\sqrt{13} + 1)$ be the roots of the quadratic equations $3x^2 + x = 1$ and $3z^2 - z = 1$ correspondently. In the paper [Bro2], p.261 the following dilogarithm identities were suggested

$$L(x^6) - 6L(x^3) + L(x^2) + 18L(x) = 8L(1), \tag{1.23}$$

$$L(z^6) - 3L(z^3) - 6L(z^2) + 9L(z) = 2L(1). \tag{1.24}$$

The main aim of this section is to prove the accessibility of (1.23) and (1.24). Let us start with a proof of the Browkin first identity. It can be easily checked by using the functional equation (1.4) and relations

$$\frac{x^3}{1+x^3} = \frac{x}{4(1+x)}, \quad 1 - x + x^2 = 4x^2, \quad 3x + 4 = x^{-2}, \quad 1 - x^3 = 2x^2(x+2)$$

that the following identities are valid

$$L(x^6) = 2L(x^3) - 2L\left(\frac{x^3}{1+x^3}\right),$$

$$L(x) + L(x^2) = L(x^3) + L\left(\frac{x}{2(1+2x)}\right) + L\left(\frac{1+x}{2(1+2x)}\right),$$

$$L\left(\frac{1}{2}\right) + L\left(\frac{x}{1+2x}\right) = L\left(\frac{x}{2(1+2x)}\right) + L\left(\frac{x^2}{1+x}\right) + L(x),$$

$$L\left(\frac{1}{2}\right) + L\left(\frac{1+x}{1+2x}\right) = L\left(\frac{1+x}{2(1+2x)}\right) + L(x(1+x)) + L(x^2),$$

$$L\left(\frac{1}{4}\right) + L\left(\frac{x}{1+x}\right) = L\left(\frac{x}{4(1+x)}\right) + L(x^2) + L(x(1-x)),$$

$$L(x) + L\left(\frac{x}{1+x}\right) = L\left(\frac{x^2}{1+x}\right) + L\left(\frac{1-x}{2(1+x)}\right) + L\left(\frac{1}{2(1+x)}\right),$$

$$L\left(\frac{1}{2}\right)+L\left(\frac{1-x}{1+x}\right)=L\left(\frac{1-x}{2(1+x)}\right)+L(x(1-x))-L(x(1+x))+L(1),$$
$$L\left(\frac{1}{2}\right)+L\left(\frac{1}{1+x}\right)=L\left(\frac{1}{2(1+x)}\right)+L\left(\frac{1}{1+2x}\right)+L\left(\frac{x}{1+2x}\right),$$
$$L(x(1-x))+L\left(\left(\frac{1-x}{2x}\right)^2\right)+L\left(\frac{1}{4}\right)=L(1), \tag{1.25}$$
$$L\left(\frac{1}{1+x}\right)+L\left(\frac{1+x}{1+2x}\right)=L\left(\frac{1}{1+2x}\right)+L\left(\frac{1}{2(1+x)}\right)+L\left(\frac{1}{2}\right),$$
$$L\left(\left(\frac{1-x}{2x}\right)^2\right)=2L\left(\frac{1-x}{2x}\right)-2L\left(\frac{1-x}{1+x}\right). \tag{1.26}$$

Using these identities one can find

$$L(x^6)=2L(x^3)+2L(x^2)+2L(x(1-x))-2L\left(\frac{x}{1+x}\right)-2L\left(\frac{1}{4}\right),$$
$$L(x^3)=2L(x^2)+3L(x)+L\left(\frac{x}{1+x}\right)+L(x(1-x))-L\left(\frac{1-x}{1+x}\right)$$
$$-L\left(\frac{1}{2(1+x)}\right)-\frac{3}{2}L(1).$$

Consequently,

$$\mathrm{LHS}(1.23)=-2L(x(1-x))-2L\left(\frac{1}{4}\right)+4L\left(\frac{1-x}{1+x}\right)+4L\left(\frac{1}{2(1+x)}\right)+6L(1).$$

Using the relations (1.25) and (1.26) one can rewrite the last expression as

$$\mathrm{LHS}(1.23)=4L(1)+4\left\{L\left(\frac{1-x}{2x}\right)+L\left(\frac{1}{2(1+x)}\right)\right\}.$$

But it follows from the equation $3x^2+x=1$, that

$$\frac{1-x}{2x}+\frac{1}{2(1+x)}=1.$$

The proof of the Browkin first dilogarithm relation is over. ■

Now let us prove the second Browkin's identity. Having in mind an observation that $z=\dfrac{x}{1-x}$, one can rewrite the LHS of (1.24) in the following form (hint: $1+z+z^2=4x^2$)

$$-6L\left(\frac{x}{1-x}\right)+14L(x)+L\left(\frac{1}{4}\right)+L\left(\frac{1}{4x}\right)-2L\left(\frac{x^3}{1-3x+3x^2}\right). \tag{1.27}$$

In order to prove that the expression (1.27) is equal to $2L(1)$, we are going to use the relations

$$\frac{x^3}{1-3x+3x^2} = \frac{x(1+x)}{2}, \quad 3x-1 = \frac{x}{1+x}, \quad (2-3x)x = 3x-1,$$
$$1-2x = (3x-1)x, \quad (2+x)x^2 = (1-x)(1-x^2), \quad \frac{1}{1+2x} = \frac{x}{1-x^2},$$

which are a direct consequence of the quadratic equation for $x$, namely, $3x^2+x-1=0$, and, additionally, the following identities which can be obtained immediately from the functional equation (1.4)

$$2L\left(\frac{x(1+x)}{2}\right) + 2L\left(\frac{1+x}{2+x}\right) + 2L\left(\frac{x}{2+x}\right) - 2L(x) - 2L\left(\frac{1+x}{2}\right) = 0,$$
$$2L\left(\frac{1}{2}\right) + 2L(1-x) - 2L\left(\frac{1-x}{1+x}\right) - 2L\left(\frac{x}{1+x}\right) - 2L\left(\frac{1-x}{2}\right) = 0,$$
$$4L\left(\frac{x}{1-x}\right) - 4L(x) - 2L\left(\frac{1+x}{2+x}\right) = 0, \quad (\text{hint}: \ \frac{1+x}{2+x} = \left(\frac{x}{1-x}\right)^2),$$
$$L\left(\frac{4x-1}{4x}\right) - 2L\left(\frac{1-x}{2x}\right) + 2L\left(\frac{1-x}{1+x}\right) = 0, \quad (\text{hint}: \ \frac{4x-1}{4x} = \left(\frac{x-1}{2x}\right)^2),$$
$$2L\left(\frac{x}{1-x}\right) + 2L\left(\frac{1-x}{2x}\right) - 2L\left(\frac{1}{2}\right) - 2L\left(\frac{x}{1+x}\right) - 2L\left(\frac{3x-1}{1-x}\right) = 0,$$
$$2L(1-x^2) + 2L\left(\frac{x}{1-x^2}\right) - 2L(x) - 2L\left(\frac{x}{2+x}\right) - 2L\left(\frac{2}{3}\right) = 0.$$

Taking the sum of these, simplifying and using the relation $\dfrac{x}{1-x^2} = \dfrac{3x-1}{1-x}$, one can find

$$4L(x) - 4L\left(\frac{x}{1+x}\right) + 2L(1-x^2) + L\left(\frac{1}{4}\right) - 2L\left(\frac{2}{3}\right) + L(1).$$

The last expression can be simplified by using the duplication formula (1.7). The result is $2L\left(\frac{1}{2}\right) + L(1) = 2L(1)$, as it was claimed. ■

### 1.3. Dilogarithm identities and conformal weights.

Let us consider the following dilogarithm sum

$$\sum_{k=1}^{n-1}\sum_{m=1}^{r} L\left(\frac{\sin k\varphi \cdot \sin(n-k)\varphi}{\sin(m+k)\varphi \cdot \sin(m+n-k)\varphi}\right) := \frac{\pi^2}{6}s(j,n,r),$$

where $\varphi = \dfrac{(j+1)\pi}{n+r}$, $0 \le 2j \le n+r-2$ and g.c.d. $(j+1, \ n+r) = 1$.

**Theorem 2** (A.N. Kirillov [Kir7]). 1. *We have*

$$s(j,n,r) = 6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}\{\frac{1}{6} - \overline{B}_2\left((n-1-2k)\theta\right)\} - \frac{1}{4}\{2n^2+1+3(-1)^n\}, \tag{1.28}$$

*where* $\theta = \dfrac{j+1}{n+r}$, *and* $\overline{B}_2(x) = -\dfrac{1}{\pi^2}\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^2}$ *is the second modified Bernoulli polynomial.*

2. (Level - rank duality)

$$s(j,n,r) + s(j,r,n) = nr - 1$$

**Corollary 1.** *We have*

$$s(j,n,r) = c_r^{(n)} - 24h_j^{(r,n)} + 6\mathbf{Z}_+,$$

*where*

$$c_r^{(n)} = \frac{(n^2-1)r}{n+r}, \quad h_j^{(n,r)} = \frac{n(n^2-1)}{24}\cdot\frac{j(j+2)}{r+n}, \quad 0 \le j \le r+n-2$$

*are the central charge and conformal dimensions of the* $SU(n)$ *level* $r$ $WZNW$ *primary fields, respectively.*

Sketch of a proof of Theorem 2.

We are going to show that the dilogarithm identity (1.28) follows from the functional equation (1.4) and the well-known results (F. Newman, 1847)

$$\mathrm{ReL}(e^{i\theta}) = \pi^2\overline{B}_2\left(\frac{\theta}{2\pi}\right), \quad \mathrm{ReL}\left(\frac{e^{i\theta}}{2\cos\theta}\right) = -\pi^2\overline{B}_2\left(\frac{\theta}{\pi}+\frac{1}{2}\right). \tag{1.29}$$

(Hint: take $x := e^{2i\theta}$ in the Abel duplication formula (1.7) and use functional equation for the modified Bernoulli polynomial: $\overline{B}_2(nx) = n\displaystyle\sum_{k=0}^{n-1}\overline{B}_2\left(x+\frac{k}{n}\right)$.)

We start with a proving of the following accessible dilogarithm identity

$$\begin{aligned} \mathrm{L}\left(\frac{1-a^{-1}}{1-b^{-1}}\cdot\frac{1-ac}{1-bc}\right) &= \mathrm{L}\left(\frac{b-a}{1-a}\right) + \mathrm{L}\left(\frac{(b-a)c}{1-ac}\right) - \mathrm{L}\left(\frac{(b-1)ac}{1-ac}\right) \\ &- \mathrm{L}\left(\frac{(bc-1)a}{1-a}\right) + \mathrm{L}(abc) + \mathrm{L}(ba^{-1}) - \mathrm{L}\left(\frac{-b}{1-b}\right) - \mathrm{L}\left(\frac{-bc}{1-bc}\right), \end{aligned} \tag{1.30}$$

where $a,b,c \in \mathbf{C}$ are the complex numbers such that $a \neq 1, c^{-1}$, and $b \neq 1, c^{-1}$.

For this purpose let us apply three times the five-term relation (1.4):

$$\mathrm{L}\left(\frac{1-a^{-1}}{1-b^{-1}}\cdot\frac{1-ac}{1-bc}\right)+\mathrm{L}\left(\frac{1-bc}{1-ac}\right)-\mathrm{L}\left(\frac{1-a^{-1}}{1-b^{-1}}\right)-\mathrm{L}\left(\frac{(a-1)bc}{1-bc}\right)-\mathrm{L}\left(\frac{1-abc}{1-ac}\right)=0,$$

$$\mathrm{L}\left(\frac{(a-1)bc}{1-bc}\right)+\mathrm{L}\left(\frac{(bc-1)a}{1-a}\right)-\mathrm{L}(abc)-\mathrm{L}\left(\frac{-a}{1-a}\right)-\mathrm{L}\left(\frac{-bc}{1-bc}\right)=0,$$

$$\mathrm{L}\left(\frac{1-a^{-1}}{1-b^{-1}}\right)+\mathrm{L}\left(\frac{1-b}{1-a}\right)-\mathrm{L}(ba^{-1})-\mathrm{L}\left(\frac{-b}{1-b}\right)+\mathrm{L}\left(\frac{-a}{1-a}\right)-\mathrm{L}(1)=0.$$

Taking the sum of these, one can obtain the relation (1.30). The next step is to consider a specialization

$$a\longrightarrow \exp(2i\theta),\quad b\longrightarrow \exp(2i\varphi),\quad c\longrightarrow \exp(2i\psi)$$

of the dilogarithm identity (1.30). One can easily check that under this specialization the following rules are valid

$$\begin{aligned}
\frac{1-a^{-1}}{1-b^{-1}}\cdot\frac{1-ac}{1-bc} &\longrightarrow \frac{\sin\varphi}{\sin\theta}\cdot\frac{\sin(\varphi+\psi)}{\sin(\theta+\psi)},\\
\frac{b-a}{1-a} &\longrightarrow -\frac{\sin(\varphi-\theta)}{\sin\theta}\exp(i\varphi),\\
\frac{(b-a)c}{1-ac} &\longrightarrow -\frac{\sin(\varphi-\theta)}{\sin(\theta+\psi)}\exp(i(\varphi+\psi)),\\
\frac{(b-1)ac}{1-ac} &\longrightarrow -\frac{\sin\varphi}{\sin(\theta+\psi)}\exp(i(\varphi+\theta+\psi)),\\
\frac{(bc-1)a}{1-a} &\longrightarrow -\frac{\sin(\varphi+\psi)}{\sin\theta}\exp(i(\varphi+\theta+\psi)),\\
\frac{-b}{1-b} &\longrightarrow \frac{e^{i(\varphi+\frac{\pi}{2})}}{2\cos(\varphi+\frac{\pi}{2})}.
\end{aligned}$$

Finally, let us introduce a single-valued function (see e.g. [Le1], [KR1])

$$\mathrm{L}(x,\theta):=\mathrm{Re}\mathrm{L}(xe^{i\theta}).$$

Taking the real part of the both sides of specialized dilogarithm identity (1.30) and using the Newman formulae (1.29), one can obtain

$$\begin{aligned}
L\left(\frac{\sin\theta}{\sin\varphi}\cdot\frac{\sin(\theta+\psi)}{\sin(\varphi+\psi)}\right) &= \mathrm{L}\left(-\frac{\sin(\varphi-\theta)}{\sin\theta},\varphi\right)+\mathrm{L}\left(-\frac{\sin(\varphi-\theta)}{\sin(\theta+\psi)},\varphi+\psi\right) \quad (1.31)\\
&-\mathrm{L}\left(-\frac{\sin\varphi}{\sin(\theta+\psi)},\theta+\varphi+\psi\right)-\mathrm{L}\left(-\frac{\sin(\varphi+\psi)}{\sin\theta},\theta+\varphi+\psi\right)\\
&+\pi^2\left\{\overline{B}_2\left(\frac{\varphi+\theta+\psi}{\pi}\right)-\overline{B}_2\left(\frac{\varphi+\psi}{\pi}\right)+\overline{B}_2\left(\frac{\varphi-\theta}{\pi}\right)-\overline{B}_2\left(\frac{\varphi}{\pi}\right)\right\}.
\end{aligned}$$

As it was shown in [Kir7], the dilogarithm identity (1.28) follows from (1.31) and properties of the modified Bernoulli polynomial $\overline{B}_2(x)$.

**Remark.** One can show that the five-term relation (1.4) follows from the nine-term relation (1.30). Namely, let us take $x := a = b$ and $y := bc$ in (1.30) $(a, b, c \in \mathbf{R})$. Then (1.30) is reduced to the relation

$$L(xy) = L(x) + L(y) + L\left(-x\frac{1-y}{1-x}\right) + L\left(-y\frac{1-x}{1-y}\right),$$

which is a form of the five-term relation due to C. Hill (see e.g. [Le1]).

Hint: using the Landen result $\operatorname{Re}\left(L(x) + L\left(\frac{-x}{1-x}\right)\right) = 0$, replace the terms $L(a)$ and $L(b)$ in RHS of (1.6) on $-L\left(\frac{-a}{1-a}\right)$ and $-L\left(\frac{-b}{1-b}\right)$ correspondently. It is easy to check that $\frac{-a}{1-a} = -x\frac{1-y}{1-x}$ and $\frac{-b}{1-b} = -y\frac{1-x}{1-y}$.

### 1.4. Kuniba-Nakanishi's dilogarithm conjecture.

Recall that the dilogarithm function is defined by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad (|z| < 1).$$

By using the integral representation

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt, \tag{1.32}$$

$\mathrm{Li}_2(z)$ can be analytically continued as a multivalued complex function to the complex plane cut along the real axis from 1 to $+\infty$. In the sequel the principal branch of dilogarithm which can be defined by taking the principal branch of logarithm function

$$-\pi < \mathrm{Im}(\log(z)) = \arg(z) \leq \pi$$

in the previous integral is used. The integration contour in (1.32) does not pass through the branch cut of $\log z$. Let us consider also the principal branch of Rogers' dilogarithm using the integral representation

$$\mathrm{L}(z) = -\frac{1}{2}\int_0^z \left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right) dx, \tag{1.33}$$

where the principal branch of logarithm function is taken. The integration contour in (1.33) is assumed to be located in the branch of $\log x$.

Let us set

$$Q_m(\varphi) = \frac{\sin(m+1)\varphi}{\sin\varphi}, \quad f_m(\varphi) = 1 - \frac{Q_{m+1}(\varphi)Q_{m-1}(\varphi)}{Q_m^2(\varphi)} = \frac{1}{Q_m^2(\varphi)},$$

where $\varphi = \dfrac{\pi(j+1)}{r+2}$, $1 \le j \le r$ and g.c,d. $(j+1, r+2) = 1$.

Following [KN] and [KNS], let us define $c_0(\varphi)$, $d_m(\varphi)$, and $y(\varphi)$ as

$$\frac{\pi^2}{6} c_0(\varphi) = \sum_{m=1}^{r-1} \left( \mathrm{L}(f_m(\varphi)) - \frac{\pi i}{2} d_m(\varphi) \log(1 - f_m(\varphi)) \right),$$

$$\pi i d_m(\varphi) = \log f_m(\varphi) - 2 \sum_{k=1}^{r-1} A_{mk} \log(1 - f_k(\varphi)), \quad \text{where}$$

$$A_{mk} = \min(k, m) - \frac{km}{r}, \quad 1 \le m, k \le r-1,$$

$$y(\varphi) = \frac{1}{\pi i} \left\{ \sum_{k=1}^{r-1} k \log(1 - f_k(\varphi)) + r \log Q_{r-1}(\varphi) - (r-1) \log Q_r(\varphi) \right\}.$$

**1.4.1.** Kuniba-Nakanishi's conjecture for $sl(2)$.

$$N(j, r) := \frac{1}{24} \left\{ \frac{3r}{r+2} - 1 - \frac{6j(j+2)}{(r+2)} + \frac{6y^2}{r} - c_0(\varphi) \right\} \in \mathbf{Z}.$$

**Remarks.** 1) One can show that $y := y(\varphi)$ is an integer for any $\varphi$, whereas $d_m(\varphi)$ is an integer if $\varphi = \dfrac{\pi(j+1)}{r+2}$. Namely, it follows from definition that ($f_m := f_m(\varphi)$, $Q_m := Q_m(\varphi)$)

$$1 - f_m = \frac{Q_{m+1} Q_{m-1}}{Q_m^2}.$$

Consequently,

$$\prod_{k=1}^{r-1} (1 - f_k)^k Q_{r-1}^r Q_r^{-(r-1)} = 1,$$

that is equivalent to $\exp(2\pi i y(\varphi)) = 1$. Now, if $\varphi = \dfrac{\pi(j+1)}{r+2}$, then $Q_r = f_r = 1$ and

$$(1 - f_m)^2 = \frac{f_m^2}{f_{m-1} f_{m+1}}.$$

Using the fact that the matrix $A := (A_{mk})$ appears to be the inverse to the Cartan matrix $C_{r-1} = (2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1})$, $1 \le i, j \le r-1$, one can check

$$f_m = \prod_{k=1}^{r-1} (1 - f_k)^{2A_{mk}}.$$

Consequently, $\log(2\pi i d_m(\varphi)) = 1$, as it was claimed.

2) If $\varphi = \dfrac{\pi(j+1)}{r+2}$, $0 \le j \le r$, and g.c.d.$(j+1, r+2) = 1$, it can be shown that

$$y(\varphi) = r \min(j, r-j+(-1)^j).$$

3) It follows from the definitions that $(x \in \mathbf{R})$

$$\begin{aligned} \mathrm{L}(x) &= \frac{\pi^2}{3} - \mathrm{L}(x^{-1}) - \frac{\pi i}{2}\log x, \quad \text{if} \quad x \ge 1, \\ \mathrm{L}(x) &= -\frac{\pi^2}{6} + \mathrm{L}\left(\frac{1}{1-x}\right) + \frac{\pi i}{2}\log(1-x), \quad \text{if} \quad x \le 0. \end{aligned} \tag{1.34}$$

Thus (see (1.5)), the single-valued dilogarithm is the real part of $\mathrm{L}(x)$, if $x \in \mathbf{R}$.

Before stating our main results of this section, let us consider an explanatory example.

**Example.** Assume $r = 3$ and $j = 1$, then we have $\varphi = \dfrac{2\pi}{5}$ and

$$f_1 = f_2 = \left(2\sec\frac{2\pi}{5}\right)^2 = \rho^{-2} = \frac{3+\sqrt{5}}{2}; \quad 1 - f_1 = -\rho^{-1}.$$

First of all, $\pi i d_1 = \log f_1 - 2\log(1-f_1) = -2\pi i$. Thus, $d_1 = d_2 = -2$. Further, using the fact $Q_2 = 1 - Q_1^2 = \dfrac{1}{f_1 - 1}$, one can find

$$y = \frac{1}{\pi i}\left\{\log(1-f_1) + 2\log(1-f_2) + 3\log Q_2\right\} = \frac{3}{\pi i}(\log(1-f_1) - \log(f_1 - 1)) = 3.$$

Consequently,

$$\begin{aligned} \frac{\pi^2}{6}c_0(\varphi) &= 2(\mathrm{L}(f_1) + \pi i \log(1-f_1)) = 2(\frac{\pi^2}{3} - L(\rho^2) - \frac{\pi i}{2}\log f_1 + \pi i \log(f_1 - 1) - \pi^2) \\ &= 2\pi^2\left(\frac{1}{3} - \frac{1}{15} - 1\right) = -\frac{44}{5}L(1). \end{aligned}$$

Here we used (1.34) and Landen's result $L(\rho^2) = \dfrac{\pi^2}{15}$. Finally,

$$N(1,3) = \frac{1}{24}\left\{\frac{9}{5} - 1 - \frac{18}{5} + 18 + \frac{44}{5}\right\} = 1.$$

Even more, using the result (1.16) with $j = 0, 1, r-1, r$ and $r \equiv 1$ (mod2), it can be shown that $N(1,r) = \dfrac{r-1}{2}$, $N(r-1, r) = r$ and $N(0,r) = N(r,r) = 0$.

**Theorem 3.** *Let $r$ be an odd positive integer and $j$ be an integer such that $0 \le 2j + (-1)^{j+1} \le r$. Then*

$$N(r-j, r) = N(j, r) + j(-1)^{j-1}\frac{r + (-1)^{j+1}}{2}.$$

To obtain an additional information about the numbers $N(j, r)$ let us introduce a function

$$b_j(r) := \frac{j^2(r-1)}{2} - \frac{j^3 - j}{6}\left[\frac{r}{j+1}\right] - N(j, r),$$

where $[x]$ is the integer part of a real number $x$.

Note that according to Theorem 3, it is sufficient to consider a behavior of the function $b_j(r)$ only in the "stable region", namely when $r \ge 2j + (-1)^{j+1}$.

**Theorem 4.** *Let $r$ and $j$ be the positive integers such that $r \ge 2j + (-1)^{j+1}$ and $r \equiv 1 (\mathrm{mod}\ 2)$. Then $b_j(r)$ is an integer depending only on the residue of $r$ modulo $j+1$.*

If $r \equiv s (\mathrm{mod}\ j+1)$ and $0 \le s \le j$, let us denote by $\widetilde{b}_j(s)$ the "stable value" of the function $b_j(r)$. Thus for a given positive integer $j$ we reduced a problem of computing the infinite family of numbers $N(j, r)$, $r \ge 2j+1$, to that for finite collection of quantities $\widetilde{b}_j(s)$, $0 \le s \le j$, only.

**Theorem 5.** *i)* (Duality) If $0 \le s \le j-3$, then $\widetilde{b}_j(s) + \widetilde{b}_j(j-3-s) = \binom{j}{3} - \binom{j}{2}$,

*ii)* $\widetilde{b}_j(j) = \widetilde{b}_j(j-2) = \binom{j}{3}$.

In principle, using the identity (1.16) it is possible to compute the functions $\widetilde{b}_j(s)$ exactly (and consequently, to find the remainder term $N(j, r)$ in Kuniba-Nakanishi's dilogarithm sum $c_0(\varphi)$). We give here only partial results concerning the computation of the numbers $\widetilde{b}_j(s)$.

**Theorem 6.** *i)* $\widetilde{b}_j(0) = -\left[\left(\frac{j-1}{2}\right)^2\right]$,

*ii)* $\widetilde{b}_j(1) = \left[\frac{2j}{3}\right]\left[\frac{2j+2}{3}\right] - \frac{j(j-1)}{2}$,

*iii)* $\widetilde{b}_j(2) = 2\left[\frac{j}{4}\right]\left[\frac{3j+3}{4}\right] - \left[\left(\frac{j-1}{2}\right)^2\right]$.

*On the other hand ($r \equiv 1 (\mathrm{mod}\ 2)$),*

*iv)* $b_1(r) = 0$,

*v)* $b_2(r) = 0$, if $r \ge 3$,

*vi)* $b_3(r) = 1$, if $r \ge 7$.

**Corollary 2** (of Theorem 4). *If $r$ is an odd positive integer, then $N(j, r)$ is an integer for any admissible value of $j$ (i.e. g.c.d. $(j+1, r+2) = 1$).*

**Corollary 3** (of Theorem 6). *We have* ($r \equiv 1 (\mathrm{mod}\ 2)$)

*i)* $N(2,r) = 2(r-1) - \left[\dfrac{r}{3}\right]$, if $r \geq 3$,

*ii)* $N(3,r) = 9\dfrac{r-1}{2} - 4\left[\dfrac{r}{4}\right] - 1$, if $r \geq 7$.

**Conjecture 1.** If g.c.d. $(j+1, r+2) = 1$ and $r \not\equiv 0 (\mathrm{mod}\ j)$, then $N(j,r)$ is a positive integer.

### 1.5. General $A_1$-type dilogarithm identity [Kir7].

Let $p$ be a rational number and $k$ be a natural number. Let us consider a decomposition of $p/k$ into continued fraction

$$\frac{p}{k} = b_r + \cfrac{1}{b_{r-1} + \cfrac{1}{\cdots + \cfrac{1}{b_1 + \cfrac{1}{b_0}}}}, \tag{1.35}$$

where $b_i \in \mathbf{N}$, $0 \leq i \leq r-1$ and $b_r \in \mathbf{Z}$. Using the decomposition (1.35) we define:

$$\begin{aligned}
&y_{-1} = 0, \quad y_0 = 1, \quad y_1 = b_0, \ldots, y_{i+1} = y_{i-1} + b_i y_i,\\
&m_0 = 0, \quad m_1 = b_0, \quad m_{i+1} = |b_i| + m_i,\\
&r(j) := r_k(j) = i, \quad \text{if} \quad km_i \leq j < km_{i+1} + \delta_{i,r},\\
&n_j := n_k(j) = ky_{i-1} + (j - km_i)y_i, \quad \text{if} \quad km_i \leq j < km_{i+1} + \delta_{i,r},
\end{aligned}$$

where $0 \leq i \leq r$. Using the decomposition (1.35) let us consider the following dilogarithm sum

$$\sum_{j=1}^{km_{r+1}} (-1)^{r(j)} L_k\Big(y_{r(j)}\theta,\ (n_j + y_{r(j)})\theta\Big) = (-1)^r \frac{\pi^2}{6} s(l, k+1, p), \tag{1.36}$$

where $\theta = \dfrac{(l+1)\pi}{ky_{r+1} + (k+1)y_r}$, and

$$L_k(\theta, \varphi) := 2L\left(\frac{\sin\theta \cdot \sin k\theta}{\sin\varphi \cdot \sin(\varphi + (k-1)\theta)}\right) - \sum_{j=0}^{k-1} L\left(\left(\frac{\sin\theta}{\sin(\varphi + j\theta)}\right)^2\right). \tag{1.37}$$

**Theorem 7.** *We have*

$$(i) \quad s(0, k+1, p) := c_k = \frac{3(p+1-k)}{p+k+1}, \quad k \geq 1, \tag{1.38}$$

$$(ii) \quad s(l, k+1, p) = c_k - \frac{6k\ l(l+2)}{p+k+1} + 6\mathbf{Z}, \tag{1.39}$$

If $k = 1$ one can obtain an exact formula for the remainder term in (1.39) (see [Kir7] or Exercise 11).

**1.6. Exercises to Section 1.**

**1.** Using the integral representation (1.1) for the Euler dilogarithm $\mathrm{Li}_2(x)$, prove the simple single-variable functional equations

*i)* $\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2)$,

*ii)* $\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \mathrm{Li} - 2(1) - \log z \log(1-z)$, (L. Euler, 1768),

*iii)* $\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = 2\mathrm{Li}_2(-1) - \frac{1}{2}\log^2(z)$.

**2.** Prove the following five-term, two-variable functional equations

$$i)\ \mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] = \mathrm{Li}_2\left[\frac{x}{1-y}\right] + \mathrm{Li}_2\left[\frac{y}{1-x}\right] - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \log(1-x)\log(1-y),$$

(W. Spence, 1809; N. Abel, 1830)

$$ii)\ \mathrm{Li}_2(x) - \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{y}{x}\right) + \mathrm{Li}_2\left[\frac{1-x}{1-y}\right] - \mathrm{Li}_2\left[\frac{y(1-x)}{x(1-y)}\right] = \mathrm{Li}_2(1) - \log x \log\left[\frac{1-x}{1-y}\right],$$

(W. Schaeffer, 1846)

$$iii)\ \mathrm{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right] = \mathrm{Li}_2\left[-x\cdot\frac{1-y}{1-x}\right] + \mathrm{Li}_2\left[-\frac{1}{y}\cdot\frac{1-y}{1-x}\right] + \mathrm{Li}_2\left[\frac{x(1-y)}{y(1-x)}\right] + \mathrm{Li}_2\left[\frac{1-y}{1-x}\right] + \frac{1}{2}\log^2(y),$$

(E. Kummer, 1840)

$$iv)\ L(x) + L(y) - L(xy) = L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right).$$

(L. Rogers, 1906)

**3.** Prove the following nine-term, three-variable functional equations

$$i)\ \mathrm{Li}_2\left(\frac{vw}{xy}\right) = \mathrm{Li}_2\left(\frac{v}{x}\right) + \mathrm{Li}_2\left(\frac{w}{y}\right) + \mathrm{Li}_2\left(\frac{v}{y}\right) + \mathrm{Li}_2\left(\frac{w}{x}\right) + \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(v) - \mathrm{Li}_2(w) + \frac{1}{2}\log^2\left(\frac{-x}{y}\right),$$

subject to the constraint $(1-v)(1-w) = (1-x)(1-y)$ (W. Mantel, 1898),

$$ii)\ L\left(\frac{(1-x)(1-y)}{(1-v)(1-w)}\right) + L\left(\frac{1-x}{1-vy^{-1}}\right) + L\left(\frac{1-x}{1-wy^{-1}}\right) - L\left(\frac{1-v}{1-vy^{-1}}\right) - L\left(\frac{1-w}{1-wy^{-1}}\right)$$

$+L(x) + L(y) - L(v) - L(w) = L(1)$, subject to the constraint $xy = vw$, $x, y, v, w \in \mathbf{R}$ (compare with (1.30)).

Prove the accessibility of these identities.

**4.** Prove ($\mathrm{L}(r,\theta) := \mathrm{Re}\mathrm{L}(re^{i\theta})$, $r, \theta \in \mathbf{R}$)

$$\mathrm{L}\left(\frac{\sin\theta}{\sin(\varphi+\theta)}, \varphi\right) + \mathrm{L}\left(\frac{\sin\varphi}{\sin(\varphi+\theta)}, \theta\right) = \pi^2\left\{\overline{B}_2(\varphi+\theta) - \overline{B}_2(\theta) - \overline{B}_2(\varphi)\right\}.$$

**5.** Prove the accessibility of Rogers' functional equation (Proposition D) for $n = 3$.

**6.** Prove the accessibility of the following dilogarithm identities

*i)* $6L\left(\frac{1}{3}\right) - L\left(\frac{1}{9}\right) = \frac{\pi^2}{3}$,

*ii)* $\sum_{k=2}^{n} L\left(\frac{1}{k^2}\right) + 2L\left(\frac{1}{n+1}\right) = \frac{\pi^2}{6}$,

*iii)* If $\alpha = \sqrt{2} - 1$, then

$$4L(\alpha) - L(\alpha^2) = \frac{\pi^2}{4}, \qquad \text{(L. Lewin)}$$

$$4L(\alpha) + 4L(\alpha^2) - L(\alpha^4) = \frac{5\pi^2}{12}, \qquad \text{(L. Lewin)}$$

$$5L(\alpha^2) - L(\alpha^4) = \frac{\pi^2}{6}. \qquad \text{(L. Lewin)}$$

*iv)* Let us take $a = \frac{\sqrt{3} - 1}{2}$ and $c = \sqrt{3} - 1$, then

$$12L(a) + 3L(a^2) - 2L(a^3) = \frac{5\pi^2}{6}, \qquad \text{(J. Loxton)}$$

$$12L(c) - 9L(c^2) - 2L(c^3) + L(c^6) = \frac{\pi^2}{2}. \qquad \text{(J. Loxton)}$$

*v)* Let $u$ be the solution in $(0, 1)$ of the quintic $u^5 + u^4 - u^3 + u^2 = 1$. Then

$$L(u^6) + L(u^5) - 2L(u^3) + 8L(u^2) - 5L(u) = \frac{\pi^2}{6}.$$

**7.** Prove the accessibility of the so-called $\theta$-family of single-variable functional equations (see [Le4], Chapter 6, Section (6.4)).

**8.** Prove the Coxeter and Lewin relations (1.11) and (1.12) using the Ray multivariable functional equation (Proposition E).

Hint: take $r(x, t)$ in the following forms

$$(1 - \rho x)(1 + \rho^3 x) - (1 - \rho^3 x)(1 - t), \quad t := \rho^2;$$

$$(1 + \rho^6 x) - (1 + \rho^2 x)(1 - t), \quad t := \rho^3;$$

$$(1 - \rho^2 x)(1 + \rho^{10} x) - (1 - \rho^4 x)^2(1 - t), \quad t := \rho^4;$$

$$(1 + \rho^{12} x)(1 + \rho^3 x) - (1 + \rho^4 x)^2(1 - t), \quad t := \rho^6, \quad \text{or}$$

$$(1 + \rho^{12} x)(1 + \rho^6 x) - (1 - \rho^8 x)(1 + \rho^4 x)(1 - t), \quad t := \rho^6.$$

**9.** Prove the following properties of the Wigner-Bloch dilogarithm $D(z)$.
*i*) For $0 < \theta < \pi$ the function $D(z)$ is positive and attains its maximum value on the half line $L_\theta := \{z \mid \arg z = \theta\}$ at $z = e^{i\theta}$.
*ii*) The function $D(z)$ attains its maximum on the complex plane $\mathbf{C}$ at $z = e^{\frac{2\pi i}{3}}$.
*iii*) $\lim_{z\to\infty} D(z) = 0$.
*iv*) $D(z) = \frac{1}{2}\{\mathrm{Cl}_2(2\theta) + \mathrm{Cl}_2(2w) - \mathrm{Cl}_2(2\theta + 2w)\}$, where $\theta = \arg z$, $w = \arg(1 - \overline{z})$.
Hint: using the five-term relation for $D(z)$, to show

$$D\left(\frac{z}{\overline{z}}\right) + D\left(\frac{1-\overline{z}}{1-z}\right) = 2D(z) + D\left(\frac{z(1-\overline{z})}{\overline{z}(1-z)}\right).$$

**10.** Prove the accessibility of the following dilogarithm identities (the Lewin Conjectures, [Le3]).
*i*) Let $\alpha = \dfrac{5-\sqrt{21}}{2}$ be the root of the quadratic equation $x^2 - 5x + 1 = 0$. Then

$$42L(\alpha) - 3L(\alpha^2) - 6L(\alpha^3) + L(\alpha^6) = \frac{5\pi^2}{3};$$

*ii*) Let $\beta = 4 - \sqrt{15}$ be the root of the quadratic equation $x^2 - 8x + 1 = 0$. Then

$$30L(\beta) + 2L(\beta^2) - 2L(\beta^3) - L(\beta^4) = \frac{5\pi^2}{6};$$

*iii*) Let $\gamma = 5 - 2\sqrt{6}$ be the root of the quadratic equation $x^2 - 10x + 1 = 0$. Then

$$46L(\gamma) - 15L(\gamma^2) - 2L(\gamma^3) + L(\gamma^6) = \pi^2.$$

**11.** (We use the notation of Section 1.5). Let us define for a given positive rational number $p$ the set of integers $\{s_k\}$, $k =, 1, 2, \ldots$ such that $\left[\dfrac{j+1}{p+2}\right] = k$ iff $s_k \le j < s_{k+1}$, $s_0 := 0$. For dilogarithm sum (1.37) to show

$$s(j, 2, p) = \frac{3p}{p+2} - \frac{6j(j+2)}{p+2} + 6t(j, p),$$

where $t(j, p) = (2k+1)j + k - 2\sum_{a=0}^{k} s_a$ iff $s_k \le j < s_{k+1}$.

**12.** Let us define Rogers' trilogarithm as

$$L_3(x) = \mathrm{Li}_3(x) - \log|x| L(x) - \frac{1}{6}\log^2|x|\log(1-x).$$

Prove

*i*) (Landen's third-order functional equations)

$$L_3(x)+L_3(1-x)+L_3\left(\frac{-x}{1-x}\right)=\zeta(3),$$

$$L_3(x)+L_3(-x)=\frac{1}{4}L_3(x^2).$$

*ii*) $L_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3),\ \ L_3(\rho^2)=\frac{4}{5}\zeta(3).$ (J. Landen, 1780)

*iii*) (Nine-term relation)

$$2L_3(x)+2L_3(y)+2L_3\left(\frac{x(1-y)}{x-1}\right)+2L_3\left(\frac{y(1-x)}{y-1}\right)+2L_3\left(\frac{1-x}{1-y}\right)$$
$$+2L_3\left(\frac{x(1-y)}{y(1-x)}\right)-L_3(xy)-L_3\left(\frac{x}{y}\right)-L_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)=2\zeta(3).$$

(E. Kummer, 1840)

Does this functional equation define the trilogarithm uniquely?

*iv*) Let $\alpha=\frac{1}{2}(5-\sqrt{21})$. Then

$$\frac{1}{36}L_3(\alpha^6)-\frac{1}{4}L_3(\alpha^2)+7L_3(\alpha)=\frac{77}{18}\zeta(3).$$

### 2.1. Dilogarithm and partitions.

**2.1.1.** Rogers-Ramanujan and Gordon-Andrews identities

**Theorem C.** *Let $a$ be an integer, $1\le a\le k+1$. Then*

*i*) (Gordon-Andrews)

$$\sum_{n_1,\ldots,n_k}\frac{q^{N_1^2+\ldots+N_k^2+N_a+\ldots+N_k}}{(q)_{n_1}\cdots(q)_{n_k}}=\prod_{n\not\equiv 0,\ \pm a \bmod (2k+3)}(1-q^n)^{-1}= \tag{2.1.a}$$

$$=(q)_\infty^{-1}\sum_{m\in\mathbf{Z}}(-1)^m q^{\frac{1}{2}[m(m+1)(2k+3)-am]}, \tag{2.1.b}$$

*ii*) (Göllnitz-Gordon-Andrews)

$$\sum_{n_1,\ldots,n_k}\frac{(-q;q^2)_{N_1}q^{N_1^2+N_2^2+\cdots+N_k^2+N_a+\ldots+N_k}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_k}}$$
$$=\prod_{\substack{n\not\equiv 2(\text{mod } 4)\\ n\not\equiv 0,\ \pm(2a-1)\ (\text{mod } 4k+4)}}(1-q^n)^{-1}, \tag{2.2}$$

*where* $N_i = n_i + \ldots + n_k$, $(q)_n := (q;q)_n$ *and* $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$.

The Rogers-Ramanujan identities correspond to $k = 1$, $a = 1$, or 2 in (2.1$a$).

Note that the second equality (2.1.b) follows from the Jacobi triple-product formula

$$\sum_{m\in\mathbf{Z}} x^m q^{\frac{1}{2}m(m+1)} = \prod_{m\geq 1}(1-q^m)(1+xq^m)(1+x^{-1}q^{m-1}),$$

which follows from the Cauchy identity (2.6) after passing to the limit $N \to \infty$.

There exist essentially four methods of proving the partition identities of Gordon-Andrews' type, namely, analytical one, using the transformation properties of $q$-series (L. Rogers, G. Watson, I. Schur, W. Bailey, L. Slater, G. Andrews, D. Bressoud, J. Stembridge, ...), algebraic one, based on the recurrence relations technique (L. Rogers, S. Ramanujan, G. Andrews, R. Baxter, P. Forrester, ...), combinatorial one, based on an explicit construction of a bijection between some sets of partitions (F. Franklin, G. Andrews, W. Durfee, A. Garsia, S. Milne, D. Bressoud, W. Burge, ...) and group-theoretical one, based on an investigation of the (integrable) highest-weight modules over the Kac-Moody or Virasoro algebras and the Weyl-Kac character formula (I. Macdonald, A. Feingold, J. Lepowsky, R. Wilson, M. Primc, V. Kac, S. Milne, B. Feigin, D. Fuchs, E. Frenkel, ...). We give here an analytical proof of (2.1) due to D. Bressoud [Bre2].

**Proof of the Gordon-Andrews identity** (for $a = k+1$).

It is convenient to divide the proof into three steps.

$1^0$. Reduction to finite dimensional (polynomial) case. Note that it is sufficient to prove the following polynomial identity (finite dimensional analog of (2.1))

$$(q)_N \sum_{n_1,\ldots,n_k} q^{n_1^2+\cdots+n_k^2} \begin{bmatrix} 2N \\ N-n_1, n_1-n_2, \ldots, n_{k-1}-n_k, n_k \end{bmatrix}_q$$

$$= \sum_m (-1)^m q^{\frac{1}{2}((2k+3)m^2+m)} \begin{bmatrix} 2N \\ N+m \end{bmatrix}_q, \qquad (2.3)_N$$

where $\begin{bmatrix} M \\ m_1,\ldots,m_k \end{bmatrix}_q := \dfrac{(q)_M}{(q)_{m_1}\ldots,(q)_{m_k}(q)_{M-m_1-\ldots-m_k}}$ is the $q$-analog of multinomial coefficient.

Indeed, it is easy to check that

$$\lim_{N\to\infty}(2.3)_N = (2.1.b).$$

$2^0$. Generalization. It is clear that one can deduce $(2.3)_N$ from the more general identity

$$\sum_{n_1,\ldots,n_{k+1}} \frac{q^{n_1^2+\cdots+n_k^2}\prod_{m=1}^{n_{k+1}}(1+xq^m)(1+x^{-1}q^{m-1})}{(q)_{N-n_1}(q)_{n_1-n_2}\cdots(q)_{n_k-n_{k+1}}(q)_{2n_{k+1}}}$$

$$= \frac{1}{(q)_{2N}}\sum_m x^m q^{\frac{1}{2}((2k+3)m^2+m)} \begin{bmatrix} 2N \\ N+m \end{bmatrix}_q. \qquad (2.4)$$

Indeed, let us take $x=-1$ in (2.4). Then the non zero terms in the LHS(2.4) happen to appear only if $n_{k+1}=0$.

$3^0$. Induction. The proof of the identity (2.4) is based on the following two Lemmas.

**Lemma 1** (Classical identities).

*i)* ($q$-binomial theorem)

$$(qx;q)_n=\sum_{k=1}^{n}(-1)^k q^{k(k+1)}x^k\begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2.5}$$

*ii)* (Euler identity)

$$\frac{1}{(qx;q)_n}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\frac{x^k q^k}{(xq;q)_k}, \tag{2.6}$$

*iii)* (Cauchy identity)

$$\sum_{k=-n}^{n} x^k q^{\frac{1}{2}k(k+1)}\begin{bmatrix} 2n \\ n+k \end{bmatrix}_q=\prod_{k=1}^{n}(1+xq^k)(1+x^{-1}q^{k-1}). \tag{2.7}$$

**Lemma 2.** *Given $N\in\mathbf{N}$, $a\in\mathbf{C}$, then*

$$\sum_{m=-N}^{N}\frac{x^m q^{am^2}}{(q)_{N-m}(q)_{N+m}}=\sum_{l=0}^{N}\frac{q^{l^2}}{(q)_{N-l}}\sum_{n=0}^{l}\frac{x^n q^{(a-1)n^2}}{(q)_{l-n}(q)_{l+n}}. \tag{2.8}$$

The statements of Lemma 1 are well-known (see e.g. [An1] or [GR]). Let us prove the Lemma 2. For this purpose let us take $n:=N-m$, $x:=q^{2m}$ in (2.6) and multiply the both sides of Euler's identity (2.6) on $(q)_{2m}^{-1}$. We get

$$(q)_{n+m}^{-1}=\sum_{k}\begin{bmatrix} n-m \\ k \end{bmatrix}_q\frac{q^{k^2+2km}}{(q)_{k+2m}}. \tag{2.9}$$

Now let us substitute the expression for $(q)_{n+m}^{-1}$ from (2.9) to the LHS of (2.8):

$$\mathrm{LHS}(2.8)=\sum_{m}\frac{x^m q^{am^2}}{(q)_{N-m}}\sum_{k}\frac{q^{k^2+2mk}(q)_{N-m}}{(q)_{k+2m}(q)_k(q)_{N-m-k}}=\sum_{m,k}\frac{q^{(m+k)^2}}{(q)_{N-m-k}}\cdot\frac{x^m q^{(a-1)m^2}}{(q)_k(q)_{k+2m}}.$$

Finally, let us take $l:=m+k$ in the last expression. ■

Now let us return back to a proving of (2.4). For this purpose we apply Lemma 2 to the RHS of (2.4):

$$\mathrm{RHS}(2.4)=\sum_{m}\frac{(xq^{\frac{1}{2}})^m q^{\frac{1}{2}(2k+3)m^2}}{(q)_{N-m}(q)_{N+m}}=\sum_{n_1}\frac{q^{n_1^2}}{(q)_{N-n_1}}\sum_{m}\frac{(xq^{\frac{1}{2}})^m q^{\frac{1}{2}(2k+1)m^2}}{(q)_{n_1-m}(q)_{n_1+m}}$$

$$=\dots\ k\text{ times}\dots=\sum_{n_1,\dots,n_{k+1}}\frac{q^{n_1^2+\dots+n_{k+1}^2}}{(q)_{N-n_1}(q)_{n_1-n_2}\dots(q)_{n_k-n_{k+1}}}\sum_{m}\frac{(xq^{\frac{1}{2}})^m q^{\frac{1}{2}m^2}}{(q)_{n_{k+1}-m}(q)_{n_{k+1}+m}}.$$

Now let us apply the Cauchy identity (2.6) to simplify the sum over $m$. The result is the LHS(2.4), as we set out to prove. ■

**2.1.2.** Partitions.

Let us consider the following classes of partitions:

$$A_{k,n,a}=\left\{(m_1,\ldots,m_p)\in\mathbf{Z}_+^p\ \middle|\ \begin{matrix} m_1\geq\ldots\geq m_p\geq 1, \quad m_i-m_{i+k}\geq 2, \quad i+k\leq p, \\ \sum m_i=n, \quad \text{and at most } a-1 \text{ of } b_i \text{ equal } 1 \end{matrix}\right\},$$

$$B_{k,n,a}=\left\{(r_1,\ldots,r_s)\in\mathbf{Z}_+^s\ \middle|\ r_i=0,\ \text{if } i\equiv 0,\ \pm a(\text{mod } (2k+3)),\quad \sum ir_i=n\right\},$$

$$C_{n,k,a}=\left\{(f_1,\ldots,f_l)\in\mathbf{Z}_+^l\ |\ \sum jf_j=n,\ f_1\leq a-1,\ f_j+f_{j+1}\leq k,\ \forall j\right\},$$

$$D_{k,n,a}=\left\{(\lambda_1,\ldots,\lambda_r)\in\mathbf{Z}_+^r\ |\ \lambda\vdash n;\ \text{if } \lambda_j\geq j \text{ then } -a+2\leq\lambda_j-\lambda_j'\leq 2k-a+1\right\}.$$

**Lemma 3.**

$$\sum_n \#|A_{k,n,a}|q^n=\text{LHS of Gordon – Andrews}' \text{ identity},$$

$$\sum_n \#|B_{k,n,a}|q^n=\text{RHS of Gordon – Andrews}' \text{ identity}.$$

Proof. The second statement of Lemma 3 is clear. Let us prove the first one for $k=1$ and $a=2$. In fact, we are going to prove a slightly more general result $(k=1,a=2)$. Namely, let us consider the set

$$C_{1,N}^d=\left\{(m_1,\ldots,m_p)\in\mathbf{Z}_+^p\ \middle|\ \begin{matrix} m_1\geq m_2\geq\cdots\geq m_p\geq d, \\ m_i-m_{i+1}\geq d, \quad \text{if} \quad i+1\leq p, \quad \sum m_i=N \end{matrix}\right\}.$$

Then

$$\sum_{n\geq 0}\frac{q^{dn^2}}{(q)_n}=\sum_{(m)\in C_{1,N}^d} q^{\sum m_i}.$$

Indeed, one can check

$$\sum_{n\geq 0}\frac{q^{dn^2}}{(1-q)\ldots(1-q^n)}=\sum_{\{k_i\geq 0\}} q^{dn^2+k_1+2k_2+\cdots+nk_n}. \tag{2.10}$$

Now let us consider the collection of integer numbers $\{k_i\geq 0\}_{i=1}^n$ appearing in the RHS of (2.10) and define the partition $m=(m_1,m_2,\ldots,m_n)$ by the following rules

$$m_1=d(2n-1)+k_1+\ldots+k_n, \quad m_2=d(2n-3)+k_2+\ldots+k_n, \quad \ldots,$$
$$m_i=d(2n-2i+1)+k_i+\ldots,k_n, \quad \ldots, \quad m_n=d+k_n, \quad N=m_1+\cdots,m_n.$$

It is easy to check that the partition $m$ constructed by this way belongs to the set $C_{1,N}^d$ and vice versa. ■

**Theorem D.** (A. Garsia and S. Milne [GM], W. Burge [Bur]). *There exist the natural bijections*

$$A_{k,n,a} \longleftrightarrow B_{k,n,a} \longleftrightarrow C_{k,n,a} \longleftrightarrow D_{k,n,a}.$$

From Theorem D there follows a combinatorial proof of the Gordon-Andrews identity.

**Remark.** Bijection $A_{k,n,a} \leftrightarrow C_{k,n,a}$ is obvious; bijection $C_{k,n,a} \leftrightarrow D_{k,n,a}$ was constructed by W. Burge (see e.g. [Bur]). Further information can be found in Exercise 7.

**Example.**

$$\begin{aligned}
A_{2,6,3} &= \{\ (3,2,1),\ (4,1,1),\ (6),\ (5,1),\ (4,2),\ (3,3)\ \},\\
B_{2,6,3} &= \{\ (1^6),\ (2,1^4),\ (2^2,1^2),\ (2^3),\ (5,1),\ (6)\ \},\\
C_{2,6,3} &= \{\ (4,1^2),\ (3^2),\ (3,2,1),\ (4,2),\ (5,1),\ (6)\ \},\\
D_{2,6,3} &= \{\ (3,1,1,1),\ (4,1,1),\ (3,2,1),\ (2,2,2),\ (3,3),\ (4,2)\ \}.
\end{aligned}$$

The next step (S. Ramanujan, G. Hardy, G. Meinardus, G. Andrews, B. Richmond, G. Szekeres, TBA (Thermodynamic Bethe's Ansatz, Al. Zamolodchikov, ...) ...) is to examine the asymptotic behavior of the numbers $\#|C_{k,n,a}|$ and $\#|B_{k,n,a}|$ as $n \to \infty$. It is well known from the theory of partitions (see e.g. [An1]) that if we introduce the partition function $p(n)$ using the expansion of the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1},$$

then $\log p(n) = \pi\sqrt{\frac{2n}{3}} + o(\sqrt{n})$ (more exactly, $p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$).

Consequently,

$$\log(\#|B_{k,n,a}|) = \pi\sqrt{\frac{2k}{2k+3}} \cdot \sqrt{\frac{2n}{3}} + o(\sqrt{n}).$$

In order to find the asymptotic behavior of $\log(\#|C_{k,n,a}|)$ as $n \to +\infty$ one can use the saddle point method.

**Lemma 4.** *Assume that*

$$\sum_{n=(n_1,\ldots,n_k)\in \mathbf{Z}_+^k} \frac{q^{nBn^t}}{(q)_{n_1}\cdots(q)_{n_k}} = \sum_{N=0}^{\infty} a_N q^N, \tag{2.11}$$

*where $B$ is a symmetric and positively definite rational matrix. Then*

$$\log^2 a_N = 4N \sum_{i=1}^{k} L(z_i) + o(N),$$

*where* $z_i$, $1 \le i \le k$, *satisfy the following system of algebraic equations*

$$z_i = \prod_{j=1}^{k} (1 - z_j)^{2B_{ij}}.$$

Proof. Using Cauchy's theorem, $a_{N-1}$ can be expressed as the integral

$$a_{N-1} = \sum_{n=(n_1,\ldots,n_k)} \oint \frac{q^{nBn^t - N}}{(q)_{n_1} \cdots (q)_{n_k}}. \tag{2.12}$$

In the sequel we are follow to the paper [RS] (see also [NRT], [Tr2], [DKMM]). In order to analyze the behavior of the integral (1.12), we are going to use the saddle point approximation, i.e. a crude estimate of the integral can be obtained from the integrand evaluated at its saddle point. First of all, we rewrite each summand of (2.12) in the exponential form

$$\exp((nBn^t - N)\log q - \sum_{j=1}^{k} \sum_{i=1}^{n_k} \log(1 - q^i)). \tag{2.13}$$

Using the Euler-Maclaurin formula, we first approximate

$$\log(q)_{n_k} = \sum_{i=1}^{n_k} \log(1 - q^i) \simeq \int_0^{n_k} \log(1 - q^t)dt. \tag{2.14}$$

Hence, we replace (2.13) on

$$\exp((nBn^t - N)\log q - \sum_{j=1}^{k} \int_0^{n_k} \log(1 - q^t)dt). \tag{2.15}$$

Furthermore, we replace the summation in (2.12) by the integration over $dn$, treating the $n_i$'s as continuous variables, so that the RHS of (2.12) is approximately given by

$$\oint \frac{dq}{2\pi i} \int dn \exp(F(q,n)), \tag{2.16}$$

where $F(q,n) = (nBn^t - N)\log q - \sum_{j=1}^{k} \int_0^{n_j} \log(1-q^t)dt$. Now the saddle point conditions with respect to $n$, namely, $\partial_{n_i} F(q,n) = 0$, $1 \le i \le k$, give the set of constraints

$$2(nB)_i \log q - \log(1 - q^{n_i}) = 0. \tag{2.17}$$

If we put $x_i := q^{n_i}$, then we obtain a system of algebraic equations on $x_i$:

$$1 - x_i = \prod_{j=1}^{k} x_j^{2B_{ij}}. \tag{2.18}$$

Further, the value of the function $F(q,n)$ at the critical point $x=(x_1,\dots,x_k)$ can be found using the formulae:

$$\int_0^{n_i} dt\log(1-q^t) \overset{z=q^t}{=} \frac{1}{\log q}\int_1^{x_i}\log(1-z)\frac{dz}{z} = \frac{1}{\log q}\{Li_2(1-x_i)+\log x_i\cdot\log(1-x_i)\}$$

(hint : $\quad Li_2(x)+Li_2(1-x)=L(1)-\log x\log(1-x)$, L.Euler);

$$nBn^t\log q = \frac{1}{2\log q}\sum_i \log x_i\cdot\log(1-x_i).$$

The result is

$$F(q,n_{\text{crit}}) = -N\log q - \frac{1}{\log q}\sum_{j=1}^k L(1-x_j).$$

Now the equation $\partial_q F(q,n_{\text{crit}})=0$ fixes $q$ at the saddle point

$$(\log q)^2 = \frac{1}{N}\sum_{j=1}^k L(1-x_j),$$

so that finally, the asymptotic behavior of $a_N$ is given by

$$a_N \sim \exp\{2(N\sum_{j=1}^k L(1-x_j))^{\frac{1}{2}}\}.$$

Now if we put $z_j := 1-x_j$, then $z_j$ satisfy the following system of algebraic equations

$$z_i = \prod_{j=1}^n (1-z_j)^{2B_{ij}}.$$

■

**Corollary 4** ( L. Lewin, B. Richmond, G. Szekeres, A.N. Kirillov, N. Reshetikhin ).

$$\sum_{n=1}^k L\left(\left(\frac{\sin\dfrac{\pi}{k+2}}{\sin\dfrac{(n+1)\pi}{k+2}}\right)^2\right) = \frac{3k}{k+2}\cdot\frac{\pi^2}{6}.$$

The interesting applications of the Corollary 4 to the study of thermodynamic properties of the $XXX$ model one can find in [KR1] and [BR].

**Exercises to Section 2.1.**

**1.** Using the Jacobi triple product identity, prove
*i*) (Euler's pentagonal number theorem)

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{m\in\mathbf{Z}}(-1)^m q^{\frac{1}{2}m(3m-1)}.$$

*ii*) (Gauss' identities)

$$\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}=\prod_{m=1}^{\infty}\frac{1-q^m}{1+q^m},$$

$$\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\prod_{m=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m-1}}.$$

*iii*) (Gauss' identity)

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{m\geq 0}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}}.$$

More generally, [FeSt], (we use the notation of Section 2.2)

$$\sum_{n=(n_a)\in\mathbf{Z}_+^{2(k-1)}}\frac{q^{\frac{1}{2}n(A_2\otimes T_{k-1}^{-1})n^t}}{\prod_a (q)_{n_a}}=\frac{1}{(q)_\infty^3}\sum_{n\in\mathbf{Z}}((2k+2)n+1)q^{(k+1)n^2+n}. \qquad (1)$$

The Gauss identity *iii*) corresponds to $k=1$. If $k=2$, then (1) takes the form

$$\sum_{m\in\mathbf{Z}}q^{m^2}=\frac{1}{(q)_\infty^2}\left(\sum_{m\in\mathbf{Z}}(6m+1)q^{3m^2+m}\right).$$

Proofs and further details see in [FeSt].
*iv*) (Problem). Find a polynomial analog for the identity (1) (compare with Exercise 5).

**2.** Prove a $q$-analog of the binomial series

$$\sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(q;q)_n}=\frac{(az;q)_\infty}{(z;q)_\infty},\quad |z|<1,\ |q|<1. \qquad \text{(E. Heine, 1847)}$$

Hint: both sides satisfy the functional equation $(1-z)f(z)=(1-az)f(qz)$.

**3.** Using the result of Exercise 2, prove the Ramanujan ${}_1\psi_1$ summation formula

$$\sum_{n\in\mathbf{Z}} \frac{(a;q)_n z^n}{(b;q)_n} = \frac{(b/a;q)_\infty (az;q)_\infty (q/az;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/az;q)_\infty (b;q)_\infty (z;q)_\infty}. \qquad \text{(S. Ramanujan, 1915)}$$

**4.** Deduce from Ramanujan's ${}_1\psi_1$-summation formula the following identities

*i*) $$\sum_{n\in\mathbf{Z}} \frac{(-1)^n q^{\frac{n(n-1)}{2}} z^n}{(b;q)_n} = \frac{(z;q)_\infty (q/z;q)_\infty (q;q)_\infty}{(b/z;q)_\infty (b;q)_\infty}.$$

*ii*) (Gauss-Jacobi's identity)

$$\sum_{k\in\mathbf{Z}} (-1)^k z_1^{\frac{k(k+1)}{2}} z_2^{\frac{k(k-1)}{2}} = \prod_{n\ge 1} (1-z_1^n z_2^n)(1-z_1^n z_2^{n-1})(1-z_1^{n-1} z_2^n).$$

*iii*) (Watson's identity)

$$\sum_{k\in\mathbf{Z}} z_1^{\frac{3k^2+k}{2}} \left( z_1^{3k^2-2k} - z_2^{3k^2-4k+1} \right) =$$

$$= \prod_{n\ge 1} (1-z_1^n z_2^{2n})(1-z_1^n z_2^{2n-1})(1-z_1^{n-1} z_2^{2n-1})(1-z_1^{2n-1} z_2^{4n-4})(1-z_1^{2n-1} z_2^{4n}).$$

(G. Watson, 1928)

*iv*) (Kac-Van der Leur-Wakimoto's identity, [KW3])

$$\frac{1}{(q)_\infty^2} \left( \sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty} \right) x^n y^m q^{mn} = \frac{(xy)_\infty (qx^{-1}y^{-1})_\infty}{(x)_\infty (y)_\infty (qx^{-1})_\infty (qy^{-1})_\infty}.$$

Hint: consider the Laurent series $F_1(x,y) = (q)_\infty^2 \cdot$RHS and $F_2(x,y) = (q)_\infty^2 \cdot$LHS. Show that these series satisfy the same functional equations $(i=1,2)$

$$F_i(qx,y) = y^{-1} F_i(x,y), \quad F_i(x,qy) = x^{-1} F_i(x,y).$$

*v*) (Problem) Find the polynomial analog for Watson's and Kac-Van der Leur-Wakimoto's identities.

**5.** Let $[x]$ denote the largest integer $\le x$. Prove

*i*) (Polynomial analog of Euler's pentagonal number theorem)

$$\sum_{k\in\mathbf{Z}} (-1)^k q^{\frac{k(3k-1)}{2}} \begin{bmatrix} N \\ \left[ \frac{N+1-3k}{2} \right] \end{bmatrix}_q = 1, \quad N \in \mathbf{Z}_+. \qquad \text{(I. Schur, 1917)}$$

*ii*) (Polynomial analog of Rogers-Ramanujan's identities, $a=0$ or $1$).

$$\sum_{j\ge 0} q^{j^2+aj} \begin{bmatrix} N-a-j \\ j \end{bmatrix}_q = \sum_{k\in\mathbf{Z}} (-1)^k q^{\frac{1}{2}k(5k+1)-2ak} \begin{bmatrix} N \\ \frac{1}{2}(N-5k)+a \end{bmatrix}_q.$$

(I. Schur, 1917; G. Andrews, 1970)

Hint: check that each side satisfies the recurrence relation $f_N = f_{N-1} + q^{N-1} f_{N-2}$.

**6.** Deduce from Exercise 5 the Rogers-Ramanujan identities ($a = 0$ or $1$)

$$\sum_{j\geq 0} \frac{q^{j^2+aj}}{(q)_j} = \prod_{j\geq 0} \frac{1}{(1-q^{5j+1+a})(1-q^{5j+4-a})}. \tag{2}$$

• Prove, RHS(2)$= \lim_{N\to\infty} \sum_{\substack{\sigma_i \in \{0,1\},\ \sigma_i\sigma_{i+1}=0 \\ \sigma_1 = a,\ \sigma_{N+1}=0}} q^{\sum_{j=1}^{N-1} j\sigma_{j+1}}$

$$= \frac{1}{(q)_\infty} \sum_{k\in\mathbf{Z}} \left( q^{k(10k+1+2a)} - q^{(2k+1)(5k+2-a)} \right). \tag{3}$$

**7.** (Fusion rules and Rogers-Ramanujan's type identities).

**A.** Let $r \geq 3$ be an odd integer and $l := \dfrac{r-3}{2}$. Let us introduce
*i*) matrix $M := M(x) \in \mathrm{Mat}_{l+1\times l+1}(\mathbf{Z}[x])$, where $(1 \leq i, k \leq l+1)$

$$M_{ik} := \begin{cases} 0, & \text{if } i+k < l+2; \\ x^{l-k+1}, & \text{if } i+k \geq l+2; \end{cases}$$

*ii*) the highest weight vector $|j >= e_{l-j+1} + \ldots + e_{l+1}$, where $0 \leq j \leq l$ and $e_i$ is the basis vector in $(\mathbf{R}^l)^*$, namely, $e_i^t = (\delta_{ik})$, $1 \leq k \leq l+1$;
*iii*) vector $a_{j,N} := a_{j,N}(x;q) = (a_{j,N}^{(1)}, \ldots, a_{j,N}^{(l+1)})$ by the following rule

$$M(q^{N-1}x)M(q^{N-2}x)\cdots M(qx)M(x)|j >= a_{j,N}^t.$$

• Prove, $a_{j,N}^{(p)}(x,q) =$

$$\sum_{m=(m_1,\ldots,m_l)\in\mathbf{Z}_+^l} x^{N_1+\ldots+N_l} q^{N_1^2+N_2^2+\ldots+N_l^2-N_1-\ldots-N_j} \prod_{k=1}^{l} \begin{bmatrix} P_k(m;j,p)+m_k \\ m_k \end{bmatrix}_q,$$

where $N_i := m_i + m_{i+1} + \ldots + m_l$, $1 \leq p \leq l+1$, and

$$P_k(m;j,p) := kN + \min(k, l-j) - \max(k+1-p, 0) - 2\sum_{i=1}^{l} \min(i,k) m_i.$$

It is well-known (see e.g. [FrSz1]) that

$$\lim_{N\to\infty} a_{j,N}^{(p)}(q,q) = \chi_{1,1+j}^{(2,r)}(q).$$

Using the Feigin–Fuchs–Rocha-Caridi character formula for $\chi_{1,1+j}^{(2,r)}(q)$ (see e.g. [FF] or formula (5)), and the exact expression for $a_{j,N}^{(p)}(x,q)$, we obtain the group-theoretic proof/explanation of the Gordon-Andrews identity. Even more, one can consider the polynomial $(x;q)_N \cdot a_{j,N}^{(l+1)}(x,q)$ as a "natural finitization" of the Watson-Andrews identity (9). However, the exact computation of polynomials $(x;q)_N \cdot a_{j,N}^{(l+1)}(x,q)$ seems to be very difficult (even for $x = 1$). In this direction one can obtain the following partial result.

**Theorem 8** (A.N. Kirillov). *We have* $(x;q)_N a_{j,N}^{(l+1)}(x,q) \equiv$

$$\sum_{m\geq 0}(-1)^m x^{(l+1)m} q^{\frac{1}{2}(2l+3)m(m+1)-(l+2+j)m} \begin{bmatrix} (l+1)(N-2m)+l+2+j \\ m \end{bmatrix}_q$$

$$\cdot \left(1-(q^{2m}x)^{1+j}\right) \frac{(x;q)_m}{(q;q)_m} \pmod{\deg_q N}.$$

Taking the limit $N\to\infty$ one can obtain the Watson–Andrews identity (9).

**B.** (Fusion algebra). Let us define the level $r$ fusion algebra $\mathcal{F}_r$ as a finite dimensional algebra over rational numbers $\mathbf{Q}$ with generators $\{v_j \mid j=0,\frac{1}{2},1,\frac{3}{2},\ldots,\frac{r-2}{2}\}$ and the following multiplication rule (the level $r$ Clebsch-Gordan series):

$$v_{j_1}\widehat{\otimes} v_{j_2} = \sum_{j=|j_1-j_2|,\ j-j_1-j_2\in\mathbf{Z}}^{\min(j_1+j_2,\ r-2-j_1-j_2)} v_j. \tag{4}$$

It is well known (see e.g. [Kac]), that the fusion algebra $\mathcal{F}_r$ is a commutative and associative one. Note also, that the fusion rules (4) correspond to a decomposing the tensor product $V_{j_1}\widehat{\otimes} V_{j_2}$ of the restricted representations [Ros] $V_{j_1}$ and $V_{j_2}$ of the Hopf algebra $U_q(sl(2))$ when $q$ is the root of unity $q=\exp\left(\frac{2\pi i}{r}\right)$ into the irreducible parts (see e.g. [Lu3]).

Further, let us denote by $\mathrm{Mult}_{V_k}(V_{j_1}\widehat{\otimes}\cdots\widehat{\otimes} V_{j_N})$ the coefficients which appear in the decomposition of the product $v_{j_1}\widehat{\otimes}\cdots\widehat{\otimes} v_{j_N}$ in the fusion algebra $\mathcal{F}_r$:

$$v_{j_1}\widehat{\otimes}\cdots\widehat{\otimes} v_{j_N} = \sum_k \mathrm{Mult}_{V_k}(v_{j_1}\widehat{\otimes}\cdots\widehat{\otimes} v_{j_N})\cdot v_k.$$

*i*) Prove the following relation (see Exercise 7, part A, $0\leq j\leq l$)

$$a_{j,N}^{(l+1)}(1,1) = \mathrm{Mult}_{V_{r(j)}}(V_s^{\widehat{\otimes}(N+1)}), \quad s := \left[\frac{l+1}{2}\right],$$

where for given $j$, $r(j)$ is the unique integer such that

$$0\leq r(j)\leq \frac{r-3}{2} \quad \text{and} \quad j\frac{r-3}{2} \equiv \pm r(j) \pmod{(r-2)}.$$

*ii*) Prove the following multiplicity formula (cf. [Kir6]):

$$\mathrm{Mult}_{V_k}(V_{j_1}\widehat{\otimes}\cdots\widehat{\otimes} V_{j_N}) = \sum_{\{\nu\}}\prod_{n\geq 1}\binom{P_{n,r}(\nu_i j)+m_n(\nu)}{m_n(\nu)},$$

where the summation is taken over all partitions $\nu=(\nu_1\geq\nu_2\geq\cdots\geq 0)$ such that *a*) $|\nu| := \nu_1 + \nu_2 + \ldots = \sum_{s=1}^{N} j_s - k$,

*b*) (inequalities for vacancy numbers)

$$P_{n,r}(\nu; j) := \sum_{s=1}^{N} \min(n, 2j_s) - \max(n + 2k + 2 - r, 0) - 2Q_n(\nu) \geq 0.$$

Here $Q_n(\nu) := \sum_{j \geq 1} \min(n, \nu_j)$ and $m_n(\nu)$ is the number of parts of partition $\nu$ which are equal to $n$.

**C.** (Restricted Kostka-Foulkes polynomials).

For the given natural number $l$, partition $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$ and composition $\mu$, we define the level $l$ (or restricted) Kostka-Foulkes polynomial $K^{(l)}_{\lambda,\mu}(q)$ by the following way

$$K^{(l)}_{\lambda,\mu}(q) := \sum_{\nu,\ l(\nu') \leq l} q^{2n(\nu)} \prod_{n \geq 1} \begin{bmatrix} P_n(\nu;\mu) + m_n(\nu) \\ m_n(\nu) \end{bmatrix}_q,$$

where summation is taken over all partitions $\nu$ such that

*i*) $|\nu| = \nu_1 + \nu_2 + \cdots = \lambda_2$, $\nu_1 \leq l$,

*ii*) $P_n(\nu;\mu) := \sum_j \min(n, \mu_j) - 2Q_n(\nu) \geq 0$, $\forall n$.

It follows from the Exercise 7, A, that

$$a^{(l+1)}_{0,N}(x;q) = \sum_{0 \leq m \leq \frac{Nl}{2}} (qx)^m K^{(l)}_{(Nl-m,m),(l^N)}(q).$$

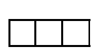
• Prove, that if $m \leq l$, then

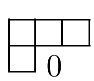
$$K^{(l)}_{(Nl-m,m),(l^N)}(q) = \overline{K}_{(Nl-m,m),(l^N)}(q) = \begin{bmatrix} m + l - 1 \\ l - 1 \end{bmatrix}_q.$$

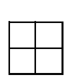
• Using the definition of restricted Kostka polynomials $K^{(l)}_{\lambda,\mu}(q)$, prove that the $q$-binomial coefficients $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are the unimodal and symmetric polynomials (cf. [Kir5]).

**Conjecture 2.** *Polynomials $K^{(l)}_{(Nl-m,m),(l^N)}(q)$ are unimodal.*

Let us consider a simple example with $l = 2$, $N = m = 4$. One can compute the Kostka polynomial $\overline{K}_{(4,4),(2^4)}(q)$ by using the Theorem 10. Namely, there exist three configurations

 $0, \qquad \overline{c} = 1 + 1 + 1 + 1 - 4 = 0,$

 $0, \qquad \overline{c} = 3 + 1 + 1 + 1 - 4 = 2,$

 $0, \qquad \overline{c} = 3 + 3 + 1 + 1 - 4 = 4.$

Consequently, $\overline{K}_{(4,4),(2^4)}(q) = 1 + q^2 + q^4$ and this is a non-unimodal polynomial, whereas the level 2 Kostka polynomial $K^{(2)}_{(4,4),(2^4)}(q) = q^4$ is the unimodal one.

**D.** (Polynomial analog of Gordon-Andrews' identities).

Let us remind the basic facts about the minimal series irreducible representations of the Virasoro algebra. Given two (coprime) positive integers $p' > p \geq 2$, then central charge and highest weights of the corresponding irreducible representations $\mathcal{V}^{(p,p')}_{r,s}$ of the Virasoro algebra are (see e.g. [FF])

$$c^{(p,p')} = 1 - \frac{6(p'-p)^2}{pp'} \quad \text{and}$$
$$\Delta^{(p,p')}_{r,s} = \frac{(rp'-sp)^2 - (p'-p)^2}{4pp'}, \quad 1 \leq r \leq p-1, \ 1 \leq s \leq p'-1.$$

The characters of these representations are ([R-C], [FF])

$$\chi^{(p,p')}_{r,s}(q) := q^{-\Delta^{(p,p')}_{r,s}} \mathrm{Tr}_{\mathcal{V}^{(p,p')}_{r,s}} q^{L_0} = \frac{1}{(q)_\infty} \sum_{k \in \mathbf{Z}} \left( q^{k(kpp'+rp'-sp)} - q^{(kp+r)(kp'+s)} \right). \tag{5}$$

Note the symmetry of the "conformal grid"

$$\{(r,s) \mid 1 \leq r \leq p-1, \ 1 \leq s \leq p'-1\}, \tag{6}$$
$$(r,s) \leftrightarrow (p-r, p'-s): \ \Delta^{(p,p')}_{r,s} = \Delta^{(p,p')}_{p-r,p'-s} \Rightarrow \chi^{(p,p')}_{r,s}(q) = \chi^{(p,p')}_{p-r,p'-s}(q).$$

Note also that the RHS(3) is precisely $\chi^{(2,5)}_{1,2-a}(q)$ as given by the RHS(5).

We are interested in finding a natural polynomial analog for the character $\chi^{(p,p')}_{r,s}(q)$. For this purpose let us introduce the polynomials (cf. [Me2])

$$B^{(N)}_{p,p';r,s}(q) := \sum_{k \in \mathbf{Z}} \left\{ q^{k(kpp'+rp'-sp)} \begin{bmatrix} N \\ \left[\frac{N+s-r-d(p,p')}{2}\right] - p'k \end{bmatrix}_q \right.$$
$$\left. - q^{(pk+r)(p'k+s)} \begin{bmatrix} N \\ \left[\frac{N-s-r-d(p,p')}{2}\right] - p'k \end{bmatrix}_q \right\}, \tag{7}$$

where $d(p,p') := \left[\frac{p'-p}{2}\right]$.

It is clear that in the "physical region" (6)

$$\lim_{N \to \infty} B^{(N)}_{p,p';r,s}(q) = \chi^{(p,p')}_{r,s}(q).$$

- Prove the following properties of the "bosonic" polynomials (7) *i*) (Symmetry) If $p \equiv p'$ (mod 2), $1 \le r \le p-1$ and $1 \le s \le p'-1$, then

$$B^{(N)}_{p,p';r,s}(q) = B^{(N)}_{p,p';p-r,p'-s}(q).$$

*ii*) (Positivity) All coefficients of the polynomials $B^{(N)}_{p,p':r,s}(q)$ in the "physical region" (6) are the non-negative integers.
*iii*) (Recurrence relations and initial conditions)

$$B^{(N)}_{p,p';r,s}(q) = B^{(N-1)}_{p,p';r.s}(q) + \begin{cases} q^{\frac{N+s-r-d(p,p')}{2}} B^{(N-1)}_{p,p';r-1,s}(q), & \text{if } N \equiv s-r-d(p,p') (\text{mod } 2); \\ q^{\frac{N-s+r+d(p,p')+1}{2}} B^{(N-1)}_{p,p';r+1,s}(q), & \text{if } N \not\equiv s-r-d(p,p') (\text{mod } 2); \end{cases}$$

$$B^{(0)}_{p,p';r,s}(q) = \delta_{s,r+d(p,p')} + \delta_{s,r+1+d(p,p')}.$$

*iv*) (Fusion multiplicities) Assume that $p' \equiv 1(2)$, $l = (p-3)/2$ and $\{V_0, V_1, \ldots, V_l\}$ are the odd dimensional restricted representations of the quantum group $U_q(sl(2))$ at the root of unity $q = \exp\left(\dfrac{2\pi i}{p'}\right)$ (see [Lu3] or Exercise 7, A and B). If $-d \le r \le p + \widetilde{d}$, $1 \le s \le p'-1$ then $B^{(N)}_{p,p';r,s}(1)$ is a linear combination with non-negative integer coefficients of the "fusion multiplicities" $\text{Mult}_{V_j}(V_l^{\widehat{\otimes} N})$, $0 \le j \le l$.

- Prove, for example, that $(1 \le s \le l+1)$

$$B^{(N)}_{2,2l+3;1,s}(1) = \sum_{j=1}^{s} \text{Mult}_{V_{r(j;l)}} \left(V_l^{\widehat{\otimes} N}\right),$$

where $r(j;l) := \left[\dfrac{l+1}{2}\right] + (-1)^{l+j} \left[\dfrac{j}{2}\right]$;

$$B^{(N)}_{p,p';r,1}(1) = \begin{cases} \text{Mult}_{V_{\left[\frac{r+d}{2}\right]}} \left(V_l^{\widehat{\otimes} N}\right), & N \equiv 0(2), \\ \text{Mult}_{V_{\left[\frac{p-r+\widetilde{d}}{2}\right]}} \left(V_l^{\widehat{\otimes} N}\right), & N \equiv 1(2), \end{cases}$$

where $\widetilde{d} := \widetilde{d}(p,q) = \left[\dfrac{p'-p-1}{2}\right]$, $-d \le r \le p + \widetilde{d}$.

- Prove that

$$B^{(N)}_{p,p';r,s}(1) = B^{(N)}_{p,p';r+1,s}(1), \quad \text{if} \quad s + r \equiv \frac{1+(-1)^N}{2} (\text{mod } 2).$$

- Let us define a matrix $\mathcal{B}^{(N)}_{p,p'}$ of the size $p' \times (p'-1)$ by the following manner

$$\left(\mathcal{B}^{(N)}_{p,p'}\right)_{i,j} = B^{(N)}_{p,p';i-1-d,j}(1), \quad 1 \le i \le p', \quad 1 \le j \le p'-1.$$

Prove that $(p' \equiv 1(2))$

$$\mathcal{B}_{2,p'}^{(N)} = \mathcal{B}_{3,p'}^{(N)} = \cdots = \mathcal{B}_{p'-1,p'}^{(N)}.$$

• Let us put $\mathcal{B}_{p'}^{(N)} := \mathcal{B}_{p,p'}^{(N)}$, where $p' \equiv 1 \pmod 2$ and $1 \le p \le p'-1$.

Prove $(1 \le i \le p',\ 1 \le j \le p'-1)$

$j$) If $i+j \equiv N+1 \pmod 2$, then

$$(\mathcal{B}_{p'}^{(N)})_{ij} = \begin{cases} \sum_{k=1}^{\min(i-1,p'-i+1,j,p'-j)} \mathrm{Mult}_{V_{k-1+\frac{1}{2}|p'-i-j+1|}}(V_l^{\widehat{\otimes}^N}), & \text{if } N \equiv 1 \pmod 2; \\ \sum_{k=1}^{\min(i-1,p'-i+1,j,p'-j)} \mathrm{Mult}_{V_{k-1+\frac{1}{2}|i-j-1|}}(V_l^{\widehat{\otimes}^N}), & \text{if } N \equiv 0 \pmod 2. \end{cases}$$

$jj$) If $i+j \equiv N \pmod 2$, then

$$(\mathcal{B}_{p'}^{(N)})_{ij} = \begin{cases} \sum_{k=1}^{\min(i,p'-i,j,p'-j)} \mathrm{Mult}_{V_{k-1+\frac{1}{2}|p'-i-j|}}(V_l^{\widehat{\otimes}^N}), & \text{if } N \equiv 1 \pmod 2; \\ \sum_{k=1}^{\min(i,p'-i,j,p'-j)} \mathrm{Mult}_{V_{k-1+\frac{1}{2}|i-j|}}(V_l^{\widehat{\otimes}^N}), & \text{if } N \equiv 0 \pmod 2. \end{cases}$$

Let us remark that

$$B_{p,p';r,s}^{(N)}(1) = (\mathcal{B}_{p'}^{(N)})_{r+d+1,s}.$$

It is an interesting problem to find a pure combinatorial interpretation in terms of some kind of partitions for the numbers $B_{p,p';r,s}^{(N)}(1)$ in the region $\{-d \le r \le p+\widetilde{d},\ 1 \le s \le p'-1\}$, as well as their natural $q$-analogs.

**A proof of part** $ii$**).** Following the paper [ABBBFV], we give a combinatorial interpretation of the "bosonic" polynomials $B_{p,p';r,s}^{(N)}(q)$.

**Definition 4.** *Let $\lambda$ be a partition/Young diagram. For any box $x \in \lambda$ lying in the $i$-th row and $j$-th column of the Young diagram $\lambda$, let us define the hook difference at the $x$ (notation $hd(x)$) as follows*

$$\mathrm{hd}(x) = \lambda_i - \lambda_j'.$$

**Definition 5.** *We say that a box $x := (i,j) \in \lambda$ lies on diagonal* $c$, if $i-j=c$.

**Definition 6.** *For given positive integers $A, B,\ p, p',\ r$ and $s$ let us define $\mathcal{R}_{p',s}(A,B;p,r;n)$ to be the set of partitions of $n$ into at most $B$ parts each $\le A$ such that the hook differences on diagonal $1-r$ are $\ge 1+r-s$ and on diagonal $p-r-1$ are $\le p'-s-p+r-1$.*

The related generating function is, of course, a polynomial

$$D_{p',s}(A,B;p,r;q)=\sum_{n\geq 0}\#|\mathcal{R}_{p',s}(A,B;p,r;n)|q^n.$$

**Theorem** ([ABBBFV]). *Assume that* $1\leq r<p$ *and* $1\leq 2s\leq p'$. *Then*

$$B^{(N)}_{p,p';r,s}(q)=D_{p',s}(N-B,B;p,r;q),$$

*where* $\quad B=\left[\dfrac{N+s-r-d(p,p')}{2}\right]$.

**Remark.** It seems plausible that for given integers $p'>p\geq 2$, $N\geq 1$ and $r$, the polynomials $B^{(N)}_{p,p';r,s}(q)$ have the nonnegative coefficients for all $s$, $1\leq s\leq p'-1$, if and only if $r$ satisfies the inequalities $0\leq r\leq p$. For example,

$$B^{(9)}_{4,9;0,1}(q)=q^4(1+q^4)(1+q^3+q^6)\frac{1-q^7}{1-q}.$$

However, one can prove that $\lim\limits_{N\to\infty}B^{(N)}_{p,p';r,s}(q)=0$, if $r=0$ or $p$ and $1\leq s\leq p'-1$.

• It is an interesting task to find a natural representation of the "restricted Hecke algebra" $\widetilde{H}_N(q)$, $q=\exp\left(\dfrac{2\pi i}{p'}\right)$, $p'\equiv 1(\mathrm{mod}\ 2)$ (see e.g. [GW]), in the linear space generated by the finite set $\coprod\limits_n\mathcal{R}_{p',s}(N-B,B;p,r;n)$.

• Another interesting problem is to find the "fermionic" representation (see e.g. [Me2]) for $B^{(N)}_{p,p';r,s}(q)$. Such representation is known in the following three cases:

*i)* $(p,p'):=(p,p+1)$, $d(p,p')=0$, E. Melzer [Me2] (conjecture), A. Bercovich [Ber] (proof the Melzer conjecture for all pairs $(r,s)$ with $s=1$).

*ii)* $(p,p'):=(2,2l+3)$, $d(p,p')=l$, Y.-H. Quano, A.N. Kirillov.

*iii)* $(p,p'):=(2l+2)$, $d(p,p')=l$, Y.H. Quano.

**Theorem 9*** (A.N. Kirillov). *We have the following equivalent expressions for the RHS(7) (we use the notation from Exercise 7, A,B,C,D. Note also, that in our case* $r=1$ *and we assume that* $0\leq s\leq l+1$*):* $B^{(N)}_{2,2l+3;1,s}(q)=$

$$=\sum_{(m_1,\ldots,m_l)\in\mathbf{Z}^l_+}q^{N_1^2+\cdots+N_l^2+N_s+\cdots+N_l}\prod_{k=1}^{l}\begin{bmatrix}P_k(m;s,N)+m_k\\ m_k\end{bmatrix}_q \tag{8}$$

$$=\sum_m q^{2m}\sum_{\nu\vdash m}q^{2n(\nu)-Q_{s-1}(\nu)}\prod_{k=1}^{l}\begin{bmatrix}N-\max(k+1-s,0)-2Q_k(\nu)+m_k(\nu)\\ m_k(\nu)\end{bmatrix}_q,$$

* After finishing this work, I was informed by A. Kato about the recent preprint of Y.-H. Quano [Q1] which contains the proof of Theorem 9. Also in [Q1] a polynomial analog of Bressoud's identity (10) is given.

*where ( the so-called vacancy numbers)*

$$P_k(m;s,N) := N - \max(k+1-s,0) - 2\sum_{j=1}^{l} \min(k,j)m_j \geq 0.$$

Our proof is based on the Theorem [ABBBFV], the combinatorial description of the RHS(8) in terms of the special type partitions (see e.g. [AB], [Bre4], [Bre5]) and that in terms of the rigged configurations (see e.g. [Kir3]).

**Problems.** *i*) To find the corner-transfer-matrix type representation for the "bosonic" sum (7) (see e.g. [ABF], [FB], [Me2]).

*ii*) To find a natural polynomial analog for the Göllnitz-Gordon-Andrews identity (2.2). It is well-known, that the RHS(2.2) is the character of a representation of the Neveu-Schwarz algebra.

**8.** Prove the following generalizations of the Gordon-Andrews identity (2.1):

$$i)\ \sum_{n_1,\ldots,n_k} \frac{z^{N_1+\cdots+N_k} q^{N_1^2+\cdots+N_k^2+N_j+\cdots+N_k}}{(q)_{n_1}\ldots(q)_{n_k}} \tag{9}$$

$$= \frac{1}{(qz)_\infty}\left\{\sum_{m\geq 0}(-1)^m z^{(k+1)m} q^{(2k+3)\frac{m(m+1)}{2}-jm}(1-(q^{2m+1}z)^j)\frac{(qz)_m}{(q)_m}\right\},$$

(L. Rogers and S. Ramanujan, 1920; G. Watson, 1928; G. Andrews, 1974)

$$ii)\ \sum_{n_1,\ldots,n_k} \frac{(zq^{-1})^{N_1+\cdots+N_k} q^{N_1^2+\cdots+N_k^2}(a;q^{-1})_{N_1}(b;q^{-1})_{N_1}}{(q)_{n_1}\ldots(q)_{n_k}}$$

$$= \sum_{m\geq 0}(-1)^m z^{(k+1)m} q^{(2k+3)\frac{m(m+1)}{2}} \frac{(a;q^{-1})_m(b;q^{-1})_m(azq^m)_\infty(bzq^m)_\infty(1-zq^{2m-1})}{(q)_m(zq^{m-1})_\infty(abz)_\infty}.$$

(G. Andrews, J. Stembridge)

**9.** Prove the following identity

$$\sum_{n_1,\ldots,n_l} \frac{q^{N_1^2+\ldots+N_l^2+N_s+\ldots+N_l}}{(q)_{n_1}\cdots(q)_{n_{l-1}}(q^2;q^2)_{n_l}} = \prod_{n\not\equiv 0,\pm s \pmod{2l+2}} (1-q^n)^{-1}. \tag{10}$$

(D. Bressoud, 1980)

- Prove the polynomial analog of Bressoud's identity (10), $1 \leq s \leq l+1$ (see [Q1]),

$$B^{(N)}_{2,2l+2;1,s}(q) =$$

$$\sum_{m=(m_1,\ldots,m_l)\in\mathbf{Z}_+^l} q^{N_1^2+\cdot+N_l^2+N_s+\cdot+N_l} \prod_{k=1}^{l-1}\begin{bmatrix} P_k(m;s,N)+m_k \\ m_k\end{bmatrix}_q \begin{bmatrix}\left[\frac{N+s-l-1}{2}\right]-N_{l-1} \\ m_l\end{bmatrix}_{q^2},$$

where $P_k(m;s,N) := N+1-\min(l-s+1,k)-\delta_{s,1}-\delta_{s,l+1}-2\sum_{j=1}^{l}\min(j,k)m_j \geq 0.$

**10.** Prove the following identity

$$\sum_{n_1,\ldots,n_l} \frac{q^{2(N_1^2+\cdots+N_l^2+N_s+\cdots+N_l)}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_l}(-q)_{2n_l}} = \frac{1}{(-q)_\infty} \prod_{n\not\equiv 0,\pm 2s \pmod{4k+3}} (1-q^n)^{-1}$$

(L. Rogers, 1894; A. Selberg, 1936; P. Paule, 1985)

**11.** Prove the following polynomial identities ($a=0,1$)

$$\sum_{k\in\mathbf{Z}}(-1)^k q^{4k^2-(2a+1)k}\begin{bmatrix}2N+a\\ N+k\end{bmatrix}_{q^2} = (q^{2N+2};q^2)_{N+a}\sum_{k=0}^{N}\frac{q^{2k^2+2ak}}{(-q;q^2)_{k+a}}\begin{bmatrix}N\\ k\end{bmatrix}_{q^2}. \tag{11}$$

(P. Paule, 1985)

It is easy to see that in the limit $N\to\infty$ these identities become

$$\sum_{k\geq 0}\frac{q^{2k^2+2ak}}{(-q;q^2)_{k+a}(q^2;q^2)_k} = \frac{1}{(q^2;q^2)_\infty}\prod_{n\equiv 0,\pm(3-2a) \pmod 8}(1-q^n).$$

(L. Slater, 1950; H. Göllnitz, 1967; B. Gordon, 1961)

**12.** (Bailey's transform and Rogers-Ramanujan's type identities).
1) (Bailey's transform). Let $a$ be indeterminate and $i,j\geq 0$ be integers. Let us consider the matrices $M$ and $M^*$, where

$$M_{ij} := (q)_{i-j}^{-1}(aq)_{i+j}^{-1}; \tag{12a}$$
$$M_{ij}^* := (-1)^{i-j}q^{\frac{(i-j)(i-j-1)}{2}}(1-aq^{2i})(aq)_{i+j-1}(q)_{i-j}^{-1}. \tag{12b}$$

Prove that $M$ and $M^*$ are inverse, infinite, lower-triangle matrices. That is

$$\sum_{j\leq k\leq i} M_{ik}M_{kj}^* = \delta_{ij}.$$

Hint: use the terminating very well-poised ${}_4\varphi_3$ summation theorem (see e.g. [GR]).
2) Let $\alpha=\{\alpha_n\}$ and $\beta=\{\beta_n\}$, $n\geq 0$ be sequences of functions in $q$. Let $M$ and $M^*$ be as in (12). We say (cf. [An3]) that $\alpha$ and $\beta$ form a Bailey pair relative to $a$ if

$$\beta_n = \sum_{k=0}^{n} M_{nk}\alpha_k, \quad \text{for all} \quad n\geq 0.$$

It is clear that (Bailey's pair inversion rule)

$$\alpha_n = \sum_{k=0}^{n} M^*_{nk} \beta_k.$$

**Theorem** (G. Andrews [An2]) (Bailey's lemma). *Let the sequences* $\alpha = \{\alpha_n\}$ *and* $\beta = \{\beta_n\}$ *form a Bailey pair. If* $\alpha' = \{\alpha'_n\}$ *and* $\beta' = \{\beta'_n\}$ *are defined by*

$$\alpha'_n := \frac{(\rho_1)_n(\rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n}(aq/\rho_1\rho_2)^n \alpha_n,$$
$$\beta'_n := \sum_{k=0}^{n} \frac{(\rho_1)_k(\rho_2)_k(aq/\rho_1\rho_2)_{n-k}}{(q)_{n-k}(aq/\rho_1)_k(aq/\rho_2)_k}(aq/\rho_1\rho_2)^k \beta_k,$$

*then* $\alpha'$ *and* $\beta'$ *also form a Bailey pair.*

A proof can be found in [An3].

- Prove the following corollary of Bailey's lemma:

$$\sum_{k\geq 0} a^k q^{k^2} \beta_k = \frac{1}{(aq)_\infty} \sum_{k=0}^{\infty} a^k q^{k^2} \alpha_k, \tag{13}$$

for any Bailey pair $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$.

Hint: take the limit $n, \rho_1, \rho_2 \to \infty$ in the Bailey pair defining relation

$$\beta'_n = \sum_{k=0}^{n} \frac{\alpha'_k}{(q)_{n-k}(aq)_{n+k}}.$$

More generally, prove that if $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ is a Bailey pair, then

$$\sum_{m_1,\ldots,m_k} \frac{a^{N_1+\cdots+N_k} q^{N_1^2+\cdots+N_k^2}}{(q)_{m_1}\cdots(q)_{m_k}} \beta_{m_k} = \frac{1}{(aq)_\infty} \sum_{n\geq 0} q^{kn^2} a^{kn} \alpha_n.$$

(G. Andrews, 1984)

Hint: use the $k$-fold iteration of Bailey's lemma.

3) Prove that the following sequences form the Bailey pair

$$i)\ \beta_n = \begin{cases} 1, & n = 0, \\ 0, & n > 0, \end{cases} \tag{15}$$

$$\alpha_n = \frac{(-1)^n q^{\frac{n(n-1)}{2}}(1-aq^{2n})(a)_n}{(1-a)(q)_n}.$$

Hint: use the Agarwal identity (see e.g. [An2]):

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})(q^{-n})_k(a)_k q^{nk}}{(1-a)(q)_k(aq^{n+1})_k} = \frac{(aq)_N q^{nN}(q^{1-n})_N}{(q)_N(aq^{n+1})_N}.$$

It is easy to see that the Watson-Andrews identity (9) (with $j = 1$ or $k+1$) follows from (14) and (15).
*ii*) Prove that if $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be a Bailey pair relative to $a$, then

$$\alpha'_n := \begin{cases} \alpha_0, & n = 0, \\ (1-a)a^n q^{n^2-n} \left\{ \dfrac{\alpha_n}{1-aq^{2n}} - \dfrac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}} \right\}, & n > 0, \end{cases}$$

$$\beta'_n := \sum_{k=0}^{n} \frac{a^k q^{k^2-k}}{(q)_{n-k}} \beta_k$$

is also a Bailey pair relative to $aq^{-1}$.

Proofs and further details see in [AAB], [Bre6], [P2], [GS], [FQ], [Q2], [Sl], [ML]. It is well-known (see e.g. [W1]) that the classical Rogers-Ramanujan identities (see e.g. Exercise 6 to Section 2.1) can be deduced from Watson's [W1] $q$-analog of Whipple's transformation formula (see e.g. [GR]). It seems very interesting to understand what kind of partition identitis correspond to the Milne (see e.g. [Ml3], [ML]) multidimensional generalization of Bailey's lemma and the Watson-Whipple transformation formula.

### 2.2. Dilogarithm and characters of the affine Kac-Moody algebras.

**Theorem E** (Kac-Wakimoto [KW2]). *Let* $\mathrm{ch}V_r^k$ *be the character of level* $k$ *representation* $V(k\Lambda_0)$ *of the affine Kac-Moody Lie algebra* $\widehat{sl_r}$. *Then*

$$\lim_{q\to 1}(1-q)\log \mathrm{ch}V_r^k = \frac{\pi^2}{6}\frac{(r^2-1)k}{r+k}.$$

**Theorem 10.** *We have*

$$\mathrm{ch}(V(k\Lambda_0)) = \sum_{\lambda\in\mathbf{Z}_+^{r-1}} \Theta_\lambda^k(z) c_\lambda^k(q),$$

*where*

$$\Theta_\lambda^k(z) := \Theta_\lambda^k(z_1,\ldots,z_{r-1}) = \sum_{m\in\mathbf{Z}^{r-1}} z^{km+\lambda} q^{\frac{1}{2}kmA_{r-1}m^t + mA_{r-1}\lambda^t}, \tag{2.19}$$

$$c_\lambda^k(q) = \frac{1}{(q)_\infty^{r-1}} \sum_{n=(n_i^a)} \frac{q^{\frac{1}{2}n(A_{r-1}\otimes T_{k-1}^{-1})n^t}}{\prod_{a,i}(q)_{n_i^a}}, \quad k\ge 2; \quad c_\lambda^{(1)}(q) = \frac{\delta_{\lambda,0}}{(q)_\infty^{r-1}} \tag{2.20}$$

*and summation in (2.20) is taken over the sequences of nonnegative integers* $n = (n_i^a)$, $1 \le a \le r-1$, $1 \le i \le k-1$ *under the following constraints*

$$\sum_{i=1}^{k-1} i n_i^a = \lambda_a, \quad 1 \le a \le r-1.$$

In (2.19) and (2.20) we used the Cartan matrices

$$A_l = \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & & \cdots & \\ & \cdots & \cdots & \\ & \cdots & & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}_{l\times l}$$

$$T_n = \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & & \cdots & \\ & \cdots & \cdots & \\ & \cdots & 2 & -1 \\ 0 & \cdots & -1 & 1 \end{pmatrix}_{n\times n} = (\min(i,j))^{-1}_{1\le i,\ j\le n}$$

**Corollary 5.** *The constant term of* $\mathrm{ch}(V(k\Lambda_0))$ *with respect to* $z$ *is equal to*

$$CT[\mathrm{ch}(k\Lambda_0)] = \frac{1}{(q)_\infty^{r-1}} \sum_{n\in(n_i^a)} \frac{q^{\frac{1}{2}n(A_{r-1}\otimes A_{k-1}^{-1})n^t}}{\prod_{i,a}(q)_{n_j^a}}, \tag{2.21}$$

*where summation is taken over the sequences of nonnegative integers* $n = (n_i^a)$, $1\le a\le r-1$, $1\le i\le k-1$, *such that*

$$\sum_{i=1}^{k-1} i n_i^a \equiv 0 \ (\mathrm{mod}\ k), \quad 1\le a\le r-1.$$

For $r=2$ the formula (2.21) is exactly the result of Lepowsky and Primc [LP] (see also [FeSt], [DKKMM], [KMM]). Analogously, for any weight $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{r-1} \ge 0)$, one can find the coefficient before $z^\lambda$ in the Laurent series $(\mathrm{ch}V_r^k)(z)$. The result is given by the RHS(2.21) under the following constraints

$$\sum_{i=1}^{k-1} i n_i^a \equiv \lambda_a (\mathrm{mod}\ k), \quad 1\le a\le r-1.$$

The last result about the constant term $\mathrm{CT}[z^{-\lambda}(\mathrm{ch}V_r^k)(z)]$ coincides with $A_r$-case of the Terhoeven and Kuniba-Nakanishi-Suzuki conjecture (see [Tr1] and [KNS], Section 2, (9)).

**2.3. Dilogarithm identities and algebraic $K$-theory** (A. Suslin, S. Bloch, D. Zagier, E. Frenkel, A. Szenes).

**2.3.1.** Bloch group.

For any field $F$ we consider the following exact sequence

$$0 \to C(F) \to D(F) \xrightarrow{\lambda} F^* \wedge F^* \xrightarrow{\chi} K_2(F) \to 0,$$

where

*i*) $K_2$ is the $K$-functor of Milnor. By the Theorem of Matsumoto (see e.g. [Mi1]) we have

$$K_2(F) = (F^* \otimes F^*)/I,$$

where $I$ is the subgroup of $F^* \otimes F^*$ generated by elements $x \otimes (1-x)$, $x \in F^* \setminus \{1\}$. In other words, $K_2F$ is generated by symbols $\{x, y\}$, $x, y \in F^*$, subject to the following relations:

$$\begin{aligned} &1)\ \ \{xy, z\} = \{x, z\}\{y, z\}, \quad \{x, yz\} = \{x, y\}\{x, z\}, \\ &2)\ \ \{x, 1-x\} = 1, \quad x \in F^* \setminus \{1\}. \end{aligned}$$

*ii*) $D(F)$ is the group, generated over $\mathbf{Z}$ by formal symbols $[x]$, $x \in F^* \setminus \{1\}$, where $F^*$ is the multiplicative group of $F$.

*iii*) $F^* \wedge F^*$ is the quotient of group $F^* \otimes_{\mathbf{Z}} F^*$ by the subgroup generated by the elements $x \otimes y + y \otimes x$. In other words, an abelian group $F^* \wedge F^*$ is generated by the elements $x \wedge y$ subject to the following relations

$$\begin{aligned} &1)\ \ x \wedge y = -y \wedge x, \\ &2)\ \ (xy) \wedge z = x \wedge z + y \wedge z. \end{aligned}$$

Consequently, we have $(\pm 1) \wedge x = 0$ in $F^* \wedge F^*$ for any $x \in F^*$.

*iv*) The homomorphism $\lambda$ is defined by

$$\lambda[x] = x \wedge (1-x), \quad x \in F^* \setminus \{1\}.$$

*v*) The homomorphism $\chi$ is defined by

$$\chi(x \wedge y) = \{x, y\}, \quad x, y \in F^*.$$

*vi*) $C(F) := \mathrm{Ker}\ \lambda$.

One can check that the elements of the form

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)], \quad x \neq y \in F^* \setminus \{1\} \qquad (2.22)$$

are contained in $C(F)$. The quotient $B(F)$ of $C(F)$ by the subgroup generated by the elements of this form is called the Bloch group [Bl], [Su].

Let us assume now that $F$ is a totally real field of algebraic numbers. The element $[x]+[1-x]$, $x \in F$, belongs to the Bloch group $B(F)$, does not depend on $x$, and has the order 6, [Su]. It is known that for the rational number field $\mathbf{Q}$ the group $B(\mathbf{Q})$ is generated by the element $[x]+[1-x]$ and is isomorphic to cyclic group $\mathbf{Z}/6$.

One can use the Rogers dilogarithm function to define a map $\mathcal{L} : B(\mathbf{R}) \to \mathbf{R}/(\mathbf{Z}\pi^2)$. Namely, let $\overline{\mathcal{L}}$ be a map $D(\mathbf{R}) \to \mathbf{R}$, which sends $[x]$ to $L(x)-\dfrac{\pi^2}{6}$, i.e. $\overline{\mathcal{L}}([x]) = L(x)-\dfrac{\pi^2}{6}$. We can restrict it to $C(\mathbf{R})$. Further, one can show that if $\alpha$ is an element of $C(\mathbf{R})$ of the form (2.22), then $\overline{\mathcal{L}}(\alpha) = 0 \pmod{\pi^2}$. More exactly, $\overline{\mathcal{L}}(\alpha) = 0$, except the case $x<0$ and $y>1$, when we have $\overline{\mathcal{L}}(\alpha) = -\pi^2$. Hence this map gives rise to a well-defined homomorphism $\overline{\mathcal{L}} : B(\mathbf{R}) \to \mathbf{R}/(\mathbf{Z}\pi^2)$. Following [FrSz2], one can use the homomorphism $\overline{\mathcal{L}}$ to study a torsion in the Bloch group $B(F)$ for totally real number fields, using the dilogarithm identities (1.16), or (1.28). So, let $\zeta_{k+2}$ be a primitive $k+2$-th root of unity and $\mathbf{Q}(\zeta_{k+2})^+$ be the maximal real subfield of the cyclotomic field $\mathbf{Q}(\zeta_{k+2})$. Consider the elements (see Section 1.4)

$$f_n = f_n^{(k)} := \frac{\sin^2 \dfrac{\pi}{k+2}}{\sin^2 \dfrac{\pi(n+1)}{(k+2)}}, \quad n = 1, \dots k.$$

It is clear that $f_n^{(k)} \in \mathbf{Q}(\zeta_{k+2})^+$. Let us define

$$\Delta_{k+2} = 2\sum_{n=1}^{k-1} \left[f_n^{(k)}\right].$$

Using the relations (see Section 1.4) $(1-f_m)^2 = \dfrac{f_n^2}{f_{n-1}f_{n+1}}$, $f_0 = f_k = 1$ (here $f_n := f_n^{(k)}$), one can show that $\Delta_{k+2} \in C(\mathbf{Q}(\zeta_{k+2})^+)$. Indeed, we have to check $\lambda(\Delta_{k+2}) = 0$. Using the properties of the elements $f_n$, one can find

$$\begin{aligned}\lambda(\Delta_{k+2}) &= \sum_{n=1}^{k-1} f_n \otimes (1-f_n)^2 = \sum_{n=1}^{l-3} f_n \otimes (f_n^2/f_{n-1}f_{n+1}) \\ &= \sum_{n=1}^{k-1} 2 f_n \otimes f_n - \sum_{n=1}^{k-2} (f_n \otimes f_{n+1} + f_{n+1} \otimes f_n) = 0,\end{aligned}$$

as we set out to prove.

Let $B'(F)$ be the quotient of $B(F)$ by the subgroup generated by $[x]+[1-x]$. A map sending $[x]$ to $L(x)$ gives rise to a well defined homomorphism $\mathcal{L}' : B'(\mathbf{R}) \to \mathbf{R}/\left(\mathbf{Z}\dfrac{\pi^2}{6}\right)$. Now we are able to formulate Frenkel-Szenes' result.

**Theorem F** ([FrSz2]). *Let $F$ be a totally real number field and $m_p$ be the maximal number $m \geq 0$ such that $F$ contains $\mathbf{Q}(\zeta_{p^m})^+$. Then*

*i) The symbols $\Delta_{p^{m_p}}$ generate the Bloch group $B(F)$.*

*ii) The symbol $\Delta_{k+2}$ generates the group $B'(\mathbf{Q}(\zeta_{k+2})^+)$.*

*iii) For a totally real number field $F$ the homomorphisms $\mathcal{L} : B(F) \to \mathbf{R}/(\mathbf{Z}\pi^2)$ and $\mathcal{L}' : B'(F) \to \mathbf{R}/(\mathbf{Z}\frac{\pi^2}{6})$ are injective.*

A proof of Theorem F uses the identity (1.16), with $j = 0$, and the description of the Bloch group $B(F)$ of a totally real number field, which is due to Merkuriev and Suslin [MS], and Levine [Lv]. According to this description, the group $B(F)$ is cyclic of order $b(F) = \frac{1}{2}\prod_p p^{m_p}$, where product is taken over all primes. Now, using the dilogarithm identity (1.16) (with $j = 0$), one can construct an element of $B(F)$ of order exactly $b(F)$. Namely, using the identity (1.16) one can find

$$\mathcal{L}(\Delta_{k+2}) - k\mathcal{L}(\Delta_6) = -\frac{2}{k+2}\pi^2 \pmod{\pi^2}.$$

The symbol $\Delta_6 = 4\left[\frac{1}{3}\right] + 2\left[\frac{1}{4}\right] \in B(\mathbf{Q})$ belongs to the Bloch group $B(F)$. The element

$$\Delta_{p^{m_p}} - (p^{m_p} - 2)\Delta_6 \in B(F)$$

under the homomorphism $\mathcal{L} : B(F) \to \mathbf{R}/\mathbf{Z}\pi^2$ gives an element of $\mathbf{R}/\mathbf{Z}\pi^2$ of the order exactly $p^{m_p}$, if $p \neq 2$, and $2^{m_2-1}$, if $p = 2$. So, these elements of $B(F)$ generate a cyclic group of order at least $b(F)$, hence they generate the whole group $B(F)$.

Now the group $B'(\mathbf{Q}(\zeta_{k+2})^+)$ is cyclic of order $(k+2)/\text{g.c.d.}(12, k+2)$, [MS]. On the other hand, $\Delta_{k+2}$ is an element $B'(\mathbf{Q}(\zeta_{k+2})^+)$ and according to (1.16),

$$\mathcal{L}'(\Delta_{k+2}) = -\frac{12}{k+2} \pmod{\frac{\pi^2}{6}}.$$

Thus, $\Delta_{k+2}$ generates a subgroup of $B'(\mathbf{Q}(\zeta_{k+2})^+)$ of order at least$(k+2)$ /g.c.d.$(12, k+2)$, and so it generates the whole group $B'(\mathbf{Q}/\zeta_{k+2})^+)$.

■

**2.3.2.** Goncharov's conjecture.

It is known that torsion subgroup of $B(\mathbf{R})$ is generated by the images of the groups $B(\mathbf{Q}(\zeta_l)^+)$ of real parts of cyclotomic fields, and consequently is isomorphic to $\mathbf{Q}/\mathbf{Z}$. It follows from Theorem F, that $B(\mathbf{R})_{\text{tor}}$ is generated by the symbols $\Delta_l$, and that the map $\mathcal{L}$ is injective on the torsion subgroup of $B(\mathbf{R})$.

**Conjecture 3** (Goncharov).

*i)* (Week form). *If $\alpha \in D(\mathbf{R})$ and $\mathcal{L}(\alpha) \equiv 0 \pmod{\pi^2}$, then $\lambda(\alpha) = 0$.*

*ii*) (Strong form). *If* $\alpha \in D(\mathbf{R})$ *and* $\mathcal{L}(\alpha) \equiv 0 \pmod{\pi^2}$, *then* $\alpha$ *is a linear combination of five-term elements of the form (2.22) (i.e. the map* $\mathcal{L}$ *is injective on the whole Bloch group* $B(\mathbf{R})$*).*

In other words, if we have a dilogarithm identity

$$\sum_i L(x_i) = c\frac{\pi^2}{6}, \quad \text{with} \quad c \in \mathbf{Q} \quad \text{and} \quad x_i \in \overline{\mathbf{Q}} \cap \mathbf{R},$$

then (hypothetically) we must have

*i*) (week form) $\lambda(\sum_i [x_i]) = 0$;

*ii*) (strong form) the relation $\sum_i L(x_i) = c\frac{\pi^2}{6}$ is a linear combination over $\mathbf{Q}$ of the five-term relations (1.4) with real-algebraic arguments.

Let us say now a few words about connection between dilogarithm identities (1.16) and (1.28) and torsion part of $B(\mathbf{R})$. The Galois group $G := \mathrm{Gal}(\mathbf{Q}(\zeta_{k+2})^+/\mathbf{Q})$ acts on the Bloch group $B(\mathbf{Q}(\zeta_{k+2})^+)$. Namely, let us consider an automorphism $\sigma_j$ of the field $\mathbf{Q}(\zeta_{k+2})^+/\mathbf{Q}$, which is given by

$$\zeta_{k+2} + \zeta_{k+2}^{-1} \to \zeta_{k+2}^{j+1} + \zeta_{k+2}^{-(j+1)}, \quad 0 \le j \le \left[\frac{k+2}{2}\right] - 1, \quad \text{g.c.d.}(j+1, k+2) = 1.$$

The elements $\sigma_j$ generate the Galois group $G$ and naturally act on the group $B(\mathbf{Q}(\zeta_{k+2})^+)$. It is clear that

$$\sigma_j(f_n^{(k)}) = \frac{\sin^2 \dfrac{(j+1)\pi}{k+2}}{\sin^2 \dfrac{(n+1)(j+1)\pi}{k+2}} \quad \text{and}$$

$$\Delta_{k+2}^{(j)} := \sigma_j(\Delta_{k+2}) = 2\sum_{n=1}^{k-1} \sigma_j(f_n^{(k)}) \in B(\mathbf{Q}(\zeta_{k+2})^+).$$

It follows from (1.16) that

$$\mathcal{L}(\Delta_{k+2}^{(j)}) = \frac{1}{3}(c_k - 24h_k^{(j)} - k)\pi^2 \pmod{\pi^2}, \tag{2.23}$$

where $c_k = \dfrac{3k}{k+2}$ is the central charge and $h_k^{(j)} = \dfrac{j(j+2)}{4(k+2)}$ is the conformal dimension of the primary field of spin $j/2$. From (2.23) we deduce

$$\mathcal{L}(\Delta_{k+2}^{(j_1)} - \Delta_{k+2}^{(j_2)}) = 8(h_k^{(j_2)} - h_k^{(j_1)})\pi^2 \pmod{\pi^2}.$$

Finally we are going to construct the elements $\Delta_{n,r}$ in $B(\mathbf{Q}(\zeta_{n+r})^+)$ using the dilogarithm identity (1.28). For this purpose let us denote

$$g_m^{(k)} := g_m^{(k)}(j) = \frac{\sin k\varphi \sin(n-k)\varphi}{\sin(m+k)\varphi \sin(m+n-k)\varphi}, \quad 1 \le k \le n-1,$$

where $\varphi = \dfrac{(j+1)\pi}{n+r}$, $0 \le j \le n+r-2$ and g.c.d.$(j+1, n+r) = 1$. Clearly, $g_m^{(k)} \in \mathbf{Q}(\zeta_{n+r})^+$. Let us introduce elements $\widetilde{\Delta}_{n,r} = 2\sum_{k=1}^{n-1}\sum_{m=1}^{r-1}[g_m^{(k)}(0)]$.

**Lemma 5.** $\widetilde{\Delta}_{n,r} \in C(\mathbf{Q}(\zeta_{n+r})^+)$.

Proof. We have to check $\lambda(\widetilde{\Delta}_{n,r}) = 0$. First of all, one can easily prove that for any $\varphi$ $(1 \le k \le n-1)$

$$\frac{(1-g_m^{(k)})^2}{(1-g_m^{(k-1)})(1-g_m^{(k+1)})} = \frac{(g_m^{(k)})^2}{g_{m-1}^{(k)} g_{m+1}^{(k)}}, \quad g_0^{(k)} := 1, \ g_m^{(n)} := 0. \tag{2.24}$$

If now $\varphi = \dfrac{(j+1)\pi}{r+n}$, then $g_r^{(k)} = 1$, $1 \le k \le n-1$, and it follows from (2.24) that the elements $g_a := g_m^{(k)}$, $a := (k,m)$, satisfy the Bethe-ansatz-like equations

$$g_a = \prod_b (1-g_b)^{2B_{a,b}}, \tag{2.25}$$

where $B = (B_{a,b}) = A_n \otimes A_r^{-1}$.

Consequently, $\lambda(\widetilde{\Delta}_{n,r}) = \sum_a B_{a,a} g_a \otimes g_a + \sum_{a<b} B_{a,b}(g_a \otimes g_b + g_b \otimes g_a) = 0.$

■

Let $\Delta_{n,r}$ be an element of $B(\mathbf{Q}(\zeta_{n+r})^+)$ corresponding to $\widetilde{\Delta}_{n,r} \in C(\mathbf{Q}(\zeta_{n+r})^+)$. It follows from (1.28) that

$$\mathcal{L}(\Delta_{n,r}) = -\frac{(n-1)r(r-1)}{3(n+r)}\pi^2 \pmod{\pi^2}, \quad \text{and} \tag{2.26}$$

$$\mathcal{L}(\Delta_{n,r} + \Delta_{r,n}) = -\frac{(r-1)(n-1)}{3}\pi^2 \pmod{\pi^2} \ \ (\text{level}-\text{rank duality}). \tag{2.27}$$

Furthermore, (2.27) admits a generalization. Namely, let us consider an automorphism $\sigma_j \in \mathrm{Gal}(\mathbf{Q}(\zeta_{n+r})^+/\mathbf{Q})$, which is given by

$$\sigma_j(\zeta+\zeta^{-1}) = \zeta^{j+1} + \zeta^{-(j+1)}, \quad \text{where}$$
$$\zeta := \zeta_{n+r}, \quad 0 \le j \le \left[\frac{n+r}{2}\right] \quad \text{and} \quad \text{g.c.d.}(j+1, n+r) = 1.$$

Let us introduce the elements

$$\Delta_{n,r}^{(j)} := \sigma_j(\Delta_{n,r}) = 2\sum_{k=1}^{n-1}\sum_{m=1}^{r-1} g_m^{(k)}(j),$$

which also belong to the Bloch group $B(\mathbf{Q}(\zeta_{n+r})^+)$. Then it follows from (1.28) that

*i)* $\mathcal{L}(\Delta_{n,r}^{(j)}) = \frac{1}{3}(c_r^{(n)} - 24h_j^{(r,n)} - (n-1)r)\pi^2 \pmod{\pi^2}$, where $c_r^{(n)} := \frac{(n^2-1)r}{n+r}$ is the central charge and $h_j^{(r,n)} := \frac{n(n^2-1)}{24}\cdot\frac{j(j+2)}{r+n}$, $0 \le j \le r+n-2$ is the conformal dimension of the primary field of "spin" $j/2$ for $sl_n$ level $r$ WZNW model.

*ii)* (Level-rank duality) $\mathcal{L}(\Delta_{n,r}^{(j)} + \Delta_{r,n}^{(j)}) = -\frac{(n-1)(r-1)}{3}\pi^2 \pmod{\pi^2}$.

(Hint: if g.c.d,$(j+1, n+r) = 1$, then $j(j+2)(n^2 - nr + r^2 - 1) \equiv 0 \pmod 6$)).

There exists a puzzling connection between the special linear combinations of symbols $\Delta_{n,r}^{(j)} \in B(\mathbf{R})$ and the central charges and conformal dimensions of primary fields of the coset Conformal Fields Theories obtained by the Goddard-Kent-Olive construction [GKO] (see e.g. [Kir7]).

**Remark.** In the paper [FzSz2], Section 5.2, for a totally real field of algebraic numbers $F$, a very interesting construction of elements in $K_3^{\mathrm{ind}}(F)$ (the indecomposable part of $K_3(F)$, i.e. the cokernel of the product map $K_1(F)^{\otimes 3} \to K_3(F)$) using the relative group $K_2$ of projective line over $F$ modulo two points and the solutions of the Bether-ansatz-like system of algebraic equations is given.

### 2.4. Connection with crystal basis.

**2.4.1.** Level 1 vacuum representation $\Lambda_0$ of the affine Lie algebra $\widehat{sl_n}$.

**Theorem 11.** *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a composition. Then*

$$\sum_{\lambda,\ l(\lambda)\le n} K_{\lambda,\mu}\cdot K_{\lambda,\ (1^{|\lambda|})}(q) = q^{n(\mu')}\begin{bmatrix} |\mu| \\ \mu_1,\ldots,\mu_n\end{bmatrix}_q, \tag{2.28}$$

$$\text{where}\quad \begin{bmatrix} N \\ m_1,\ldots,m_n\end{bmatrix}_q := \frac{(q)_N}{(q)_{m_1}\ldots(q)_{m_n}}$$

*is the q-analog of multinomial coefficient* $(N = m_1 + \cdots + m_n)$.

Let us explain notation. Here

*i)* $K_{\lambda,\mu}$ is the so-called Kostka number, which is equal to the number of (semi)standard Young tableaux of shape $\lambda$ and weight $\mu$, or, equivalently, to the dimension of weight $\mu$ subspace $V_\lambda(\mu)$ of the irreducible highest weight $\lambda$ representation $V_\lambda$ of the Lie algebra $\mathfrak{gl}_n$:

$$K_{\lambda,\mu} = \dim V_\lambda(\mu).$$

*ii)* $K_{\lambda,\mu}(q)$ is the so-called Kostka-Foulkes polynomial (see e.g. [Ma]), which can be defined from a decomposition of the Schur functions in terms of the Hall-Littlewood ones:

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(x;q).$$

It is well-known (see e.g. [Lu2] or [Ma]), that polynomial $\overline{K}_{\lambda,\mu}(q) := q^{n(\mu)-n(\lambda)} K_{\lambda,\mu}(q^{-1})$ coincides with Lusztig's $q$-analog of weight multiplicity dim $V_\lambda(\mu)$.

*iii)* $n(\lambda) := \sum_{i=1}^{n} (i-1)\lambda_i = \sum_{i=1}^{n} \binom{\lambda'_j}{2} = \sum_{1\le i<j\le n} \min(\lambda_i,\lambda_j)$, if $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbf{Z}_+^n$.

**The proof of Theorem 11** is based on the well-known results from the theory of symmetric functions. We will use the notation and terminology involving symmetric functions from Macdonald [Ma].

**Lemma 6.**

$$\sum_\lambda s_\lambda(x) K_{\lambda,\mu}(q) = Q_\mu \left( \frac{x}{1-q} \right).$$

Here we used the standard $\lambda$-ring theory notation, namely, for any symmetric function $f(x) := f(x_1, x_2, \ldots)$ and a new set of variables $y = (y_1, y_2, \ldots)$ one can define $f(xy) = f(\ldots, x_i y_j \ldots)$.

Proof. Let us remind that the Hall-Littlewood functions $P_\lambda$ and $Q_\lambda$ satisfy the following orthogonality condition (see [Ma], Chapter II, (4.4))

$$\sum_\lambda Q_\lambda(x;q) P_\lambda(y;q) = \prod_{i,j} \frac{1-qx_iy_j}{1-x_iy_j}.$$

Consequently,

$$\sum_\lambda Q_\lambda \left( \frac{x}{1-q} \right) P_\lambda(y,q) = \prod_{\substack{i,\,j \\ k\ge 0}} \frac{1-q^{k+1}x_iy_j}{1-q^kx_iy_j} = \prod_{i,j} (1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Here we used the orthogonality of Schur's functions ([Ma], Chapter I, (4.3)). It remains to remind the definition of Kostka-Foulkes polynomials:

$$s_\lambda(y) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(y;q).$$

■

To continue, let us remark that if $m \ge n$, then

$$Q_{(k^n)}(x_1,\ldots,x_m;q) = (q;q)_n (e_n(x))^k, \quad \text{see [Ma], Chapter III, (2.8).}$$

Therefore, in order to prove Theorem 9 we have to decompose the symmetric function

$$\sum_{\lambda} s_\lambda(x) K_{\lambda,(1^n)}(q) = Q_{(1^n)}\left(\frac{x}{1-q}\right) = (q;q)_n e_n\left(\frac{x}{1-q}\right)$$

in terms of the monomial symmetric ones $m_\lambda(x)$. For this purpose, let us remind that by definition ($CT$:= constant term)

$$e_n\left(\frac{x}{1-q}\right) = CT\ \left[t^{-n}\prod_j(-tx_j;q)_\infty\right].$$

Thus, using the Euler result

$$(-z;q)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q;q)_n},$$

one can obtain

$$e_n\left(\frac{x}{1-q}\right) = \sum_{\mu\vdash n} \frac{q^{n(\mu')}}{(q;q)_\mu} m_\mu(x),$$

where for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, we set $(q;q)_\mu := \prod_{j=1}^{n}(q;q)_{\mu_j}$. Consequently,

$$\sum_{\lambda} s_\lambda(x) K_{\lambda,(1^n)}(q) = \sum_{\mu\vdash n} q^{n(\mu')} \begin{bmatrix} n \\ \mu_1 \ldots \mu_n \end{bmatrix}_q m_\mu(x).$$

■

**Remark.** The identity (2.28) is implicitly contained in the Terada preprint [Te]. The proof given in [Te] is based on a study of the cohomology groups of the variety of $N$-stable flags. Our proof is pure algebraic and based on the theory of symmetric functions. Later B. Leclerc and J.-Y. Thibon gave almost the same proof (unpublished). It is possible to give a pure combinatorial proof of (2.28) using the properties of the Robinson-Schensted correspondence. Finally, the identity (2.28) can be extracted from [DJKMO1].

**Corollary 6** (of Theorem 11).

$$\sum_{\lambda,\ l(\lambda)\le n} s_\lambda(x) K_{\lambda,(1^{|\lambda|})}(q) = \sum_{m\in\mathbf{Z}_+^n} q^{\Sigma\binom{m_i}{2}} x^m \begin{bmatrix} |m| \\ m_1,\ldots,m_n \end{bmatrix}_q. \tag{2.29}$$

Now we are going to consider an appropriate limit $|m| \to \infty$ in (2.29). More exactly, let us assume that $|m| = nN$, and consider the following variant of (2.29):

$$q^{-\frac{(N^2-N)n}{2}} \sum_{\lambda, l(\lambda) \le n} \frac{s_\lambda(x_1, \ldots, x_n)}{(x_1 \ldots x_n)^N} K_{\lambda,(1^{|\lambda|})}(q)$$

$$= \sum_{\substack{k \in \mathbf{Z}^n, \\ |k|=0, k_i \ge -N, \ \forall i}} x_1^{k_1} \ldots x_n^{k_n} q^{\frac{1}{2}\sum k_i^2} \begin{bmatrix} nN \\ k_1+N, \ldots, k_n+N \end{bmatrix}_q . \qquad (2.30)_N$$

First of all, $\lim\limits_{N \to \infty} \mathrm{RHS}(2.30)_N =$

$$\frac{1}{(q)_\infty^{n-1}} \sum_{\substack{m \in \mathbf{Z}^n \\ |m|=0}} x^m q^{\frac{1}{2}\sum m_i^2} = \frac{1}{(q)_\infty^{n-1}} \sum_{k \in \mathbf{Z}^{n-1}} z_1^{k_1} \ldots z_{n-1}^{k_{n-1}} q^{\frac{1}{2} k A_{n-1} k^t},$$

where $z_i = \dfrac{x_i}{x_{i-1}}$, $1 \le i \le n-1$, $x_0 := x_n$. On the other hand,

$$\lim_{N \to \infty} \mathrm{LHS}(2.30)_N = \sum_{\substack{\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \in \mathbf{Z}^n \\ |\lambda|=0}} s_\lambda(x_1, \ldots, x_n) b_\lambda(q),$$

where for given weight $\lambda$ we set $(\lambda_N := \lambda + (N^n))$

$$b_\lambda(q) := \lim_{N \to \infty} q^{-\frac{n(N^2-N)}{2}} K_{\lambda_N,(1^{|\lambda_N|})}(q) = \frac{q^{n(\lambda')}}{(q)_\infty^{n-1}} \prod_{1 \le i \le j \le n} (1 - q^{\lambda_i - \lambda_j - i + j}). \qquad (2.31)$$

Finally, it is follow from $(2.30)_N$ that

$$\sum_{\lambda, \ l(\lambda) \le n} s_\lambda(x_1, \ldots, x_n) b_\lambda(q) = \frac{\Theta(x)}{(q)_\infty^{n-1}}, \quad \text{where}$$

$$\Theta(x) = \sum_{\substack{m = (m_1, \ldots, m_n) \in \mathbf{Z^n}, \\ |m|=0}} x^m q^{\frac{1}{2}(m_1^2 + \ldots + m_n^2)},$$

is the theta-function corresponding to the basic representation $V(\Lambda_0)$ of $\widehat{sl_n}$.

Hence, $b_\lambda(q)$ is the branching function for representation $\Lambda_0$ of the affine algebra $\widehat{sl_n}$.

**Remark.** It is interesting to compare the formula (2.31) with the following result of V. Kac ([Kac], (14.12.10)): for the infinite rank affine algebra of type $A_\infty$

$$\dim_q L(\Lambda_{s_1} + \Lambda_{s_2} + \cdots + \Lambda_{s_n}) = \frac{1}{(q)_\infty^n} \prod_{1 \le i < j \le n} (1 - q^{s_i - s_j + j - i}),$$

where $s_1 \ge s_2 \ge \cdots \ge s_n$ are arbitrary integers.

**2.4.2.** Kostka-Foulkes polynomials.

We start with reminding some basic properties of the Kostka polynomials (see e.g. [Ma], [Kir3]).

**Proposition F** (Hook-formula). *Let $\lambda \vdash n$ be a partition of natural number $n$. Then we have*

$$K_{\lambda,(1^n)}(q) = q^{n(\lambda')} \frac{(q)_n}{\prod_{x\in\lambda}\left(1-q^{h(x)}\right)}, \tag{2.32}$$

*where $h(x) := \lambda_i + \lambda'_j - i - j + 1$ is the hook-length corresponding to the box $x = (i,j) \in \lambda$.*

Before stating the next result about Kostka polynomials, let us explain some notation. Namely, let $\mathrm{STY}(\lambda,\mu)$ be a set of all (semi)standard Young tableaux of shape $\lambda$ and weight (or content, or evolution) $\mu$. It is well-known (see e.g. [Ma]) that $\#|\mathrm{STY}(\lambda,\mu)| = \dim\, V_\lambda(\mu)$, i.e. the number of (semi)standard Young tableaux of shape $\lambda$ and weight $\mu$ is equal to the dimension of weight $\mu$ subspace of the highest weight $\lambda$ irreducible representation $V_\lambda$ of the Lie algebra $\mathfrak{gl}_n$. Following A. Lascoux and M.-P. Schützenberger, for given partitions $\lambda$ and $\mu$ we denote by $c(T)$ (correspondently $\overline{c}(T)$) the charge (cocharge) of a tableau $T \in \mathrm{STY}(\lambda,\mu)$.

**Proposition G** (Lascoux-Schützenberger [LS]). *If $\lambda$ and $\mu$ are partitions, then*

$$K_{\lambda,\mu}(q) = \sum_{T\in\mathrm{STY}(\lambda,\mu)} q^{c(T)}.$$

In the remaining part of this section we are going to explain a new combinatorial formula for Kostka polynomial $K_{\lambda,\mu}(q)$ coming from the Bethe ansatz technique (see e.g. [Kir2], [Kir3]).

From the representation theoretic point of view, Bethe's ansatz method for the generalized Heisenberg magnet gives a very powerful and convenient algorithm for decomposing the tensor product of some special representations of semi-simple Lie algebra $\mathfrak{g}$ into irreducible parts (see e.g. [KR2], [Ku]). In the case of the Lie algebra $\mathfrak{gl}_n$, the Bethe ansatz method gives for a tensor product multiplicity $\mathrm{Mult}_{V_\lambda}(V_{\mu_1}\otimes\cdots\otimes V_{\mu_N})$an equivalent description as the number of solutions to some special system of algebraic equations (the so-called Bethe equations, see Exercise 2). Using the so-called "string conjecture" one can find the number of "string solutions" to the system of Bethe's equations. In spite of the well-known fact (see e.g. [EKK]) that solution to the Bethe equations does not have a "string nature" in general, the total number of solutions (with multiplicities) to the system of Bethe's equations given by using the "string conjecture", appears to be correct. The last statement (the so-called combinatorial completeness of Bethe's states) was proven in [Kir2] (see also [KL]) for the case when all weights $\mu_i$ have a rectangular shape (i.e. each weight $\mu_i$ is a proportional to some fundamental weight $w_a$).

A possibility to apply the Bethe ansatz technique to combinatorics of Young tableaux is based on the well-known fact (see e.g. [Lit] or [Stn]) that for the Lie algebra $\mathfrak{g}L_n$ (more generally, for the Lie superalgebra $\mathfrak{gl}(N/M))$ a weight multiplicity can be expressed in terms of tensor product multiplicity. Namely, if $l(\lambda) \le n$ and $\mu = (\mu_1, \ldots, \mu_N)$, then

$$\#|\mathrm{STY}(\lambda,\mu)| = K_{\lambda,\mu} = \dim V_\lambda^{(\mathfrak{gl}(N))}(\mu) = \mathrm{Mult}_{V_\lambda^{(\mathfrak{gl}(n))}} \left( V_{\mu_1 w_1}^{(\mathfrak{gl}(n))} \otimes \cdots \otimes V_{\mu_N w_1}^{(\mathfrak{gl}(n))} \right).$$

More generally (see e.g. [Se], [KR2]), if $\lambda, \mu$ as above and $\eta = (\eta_1, \ldots, \eta_M)$, then

$$\dim V_\lambda^{(\mathfrak{gl}(N/M))}(\mu|\eta) = \mathrm{Mult}_{V_\lambda^{(\mathfrak{gl}(n))}} \left( V_{\mu_1 w_1}^{(\mathfrak{gl}(n))} \otimes \cdots \otimes V_{\mu_N w_1}^{(\mathfrak{gl}(n))} \otimes V_{w_{\eta_1}}^{(\mathfrak{gl}(n))} \otimes \cdots \otimes V_{w_{\eta_M}}^{(\mathfrak{gl}(n))} \right).$$

Now we are going to state the main result about the tensor product multiplicities which follows from Bethe's ansatz technique. Thus, let $\mu_1, \ldots, \mu_N$ be the rectangular shape partitions (i.e, $\mu_j = m_{i,j} w_i$ for some $i$, $1 \le i \le n$, $1 \le j \le N$) and $\lambda$ be a partition such that $|\lambda| = \sum_j |\mu_j|$ and $l(\lambda) \le n$. Let us denote by $\mu^{(i)}$, $1 \le i \le n$, a composition composed by those $m \ne 0$ for which there exists $\mu_j$ $(1 \le j \le N)$ such that $\mu_j = m w_i$.

**Proposition 1** ([KR2]).

$$\mathrm{Mult}_{V_\lambda^{(\mathfrak{gl}(n))}} \left( V_{\mu_1}^{(\mathfrak{gl}(n))} \otimes \cdots \otimes V_{\mu_N}^{(\mathfrak{gl}(n))} \right) = \sum_{\{\nu\}} K_{\{\nu\}}, \tag{2.33}$$

*where summation is taken over all configurations* $\{\nu\}$ *of type* $(\lambda; \{\mu\})$.

Let us explain the notation in (2.33). By definition, a configuration $\{\nu\}$ of type $(\lambda; \{\mu\})$ is a collection of partitions (or Young diagrams) $\{\nu\} = \{\nu^{(1)}, \nu^{(2)}, \ldots\}$ such that

*i)* $|\nu^{(k)}| := \nu_1^{(k)} + \nu_2^{(k)} + \ldots = \sum\limits_{j \ge k+1} (\lambda_j - (j-k)|\mu^{(j)}|)$,

*ii)* $P_r^{(k)}(\nu; \{\mu\}) := Q_r(\mu^{(k)}) + Q_r(\nu^{(k-1)}) - 2Q_r(\nu^{(k)}) + Q_r(\nu^{(k+1)}) \ge 0$, for all $k, r \ge 1$, where $\nu^{(0)} := 0$ and $Q_r(\nu) := \sum\limits_{j \le r} \nu_j'$. Let us remind that for a partition $\lambda$ we denote by $\lambda' = (\lambda_1', \lambda_2', \ldots)$ the conjugate partition, i.e. $\lambda_i' = \{j \mid \lambda_j \ge i\}$.

We define the Kostka number $K_{\{\nu\}}$ corresponding to a configuration $\{\nu\}$ as follows

$$K_{\{\nu\}} := \prod_{k,r} \begin{pmatrix} P_r^{(k)}(\nu; \{\mu\}) + m_r(\nu^{(k)}) \\ m_r(\nu^{(k)}) \end{pmatrix}, \tag{2.34}$$

where $m_r(\nu^{(k)})$ is a number of length $r$ parts in partition $\nu^{(k)}$ and $\begin{pmatrix} N \\ n \end{pmatrix}$ is the binomial coefficient:

$$\begin{pmatrix} N \\ n \end{pmatrix} = \begin{cases} \dfrac{N!}{n!(N-n)!}, & \text{if } 0 \le n \le N; \\ 0 & \text{otherwise.} \end{cases}$$

An analytical proof of Proposition 1 is given in [Kir1] and [Kir2]. Another proof of Proposition 1 (see [Kir3] and [KK]) is based on a combinatorial interpretation of the both sides of equality (2.34). The left hand side of (2.34) admits an interpretation as the number of special kind of tableaux (see [KK]), whereas the right hand side admits that in terms of rigged configurations (see [Kir3]). The main step of a combinatorial proof of Proposition 1 is to construct a bijection between the set of rigged configurations and that of special kind of tableaux. The bijection constructed in [Kir3] has many intriguing and mysterious properties such as agreements with cocharge construction, the Schützenberger involution and so on. For example, let us define a $q$-analog of the RHS(2.34) by the following way

$$\overline{K}_{\lambda,\{\mu\}}(q) := \sum_{\{\nu\}} \overline{K}_{\{\nu\}}(q), \quad \overline{K}_{\{\nu\}}(q) := q^{\overline{c}(\nu)} \prod_{k,r} \begin{bmatrix} P_r^{(k)}(\nu;\{\mu\}) + m_r(\nu^{(k)}) \\ m_r(\nu^{(k)}) \end{bmatrix}_q, \tag{2.35}$$

where summation is taken over all configurations $\{\nu\}$ of type $(\lambda;\{\mu\})$ and the cocharge $\overline{c}(\nu)$ of a configuration $\{\nu\}$ is defined by

$$\overline{c}(\nu) := \sum_{k,r\geq 1} \binom{(\nu^{(k-1)})'_r - (\nu^{(k)})'_r}{2} - n(\lambda), \quad \nu^{(0)} := 0.$$

**Theorem 12** ([Kir3]). *i) Let us assume that $\mu := \mu^{(1)}$ is a partition, $l(\mu) \leq N$ and $\mu^{(i)} = 0$, if $2 \leq i \leq n$. Then*

$$\overline{K}_{\lambda,\{\mu\}}(q) = \overline{K}_{\lambda,\mu}(q),$$

*where $\overline{K}_{\lambda,\mu}(q)$ is the Lusztig $q$-analog of weight multiplicity for the Lie algebra $\mathfrak{gl}_N$.*

*ii) Let us assume that $\mu := \mu^{(1)}$ is a partition, $l(\mu) \leq N$ and $\mu_k^{(i)} = 0$ for all $k \geq 2$ and $2 \leq i \leq n$. Let us define the partition $\eta = (\mu_1^{(2)}, \mu_1^{(3)}, \ldots, \mu_1^{(n)})^+$ corresponding to a composition $(\mu_1^{(2)}, \mu_1^{(3)}, \ldots, \mu_1^{(n)})$. If $l(\eta') \leq M$ then*

$$\overline{K}_{\lambda,\{\mu\}}(q) = \overline{K}_{\lambda,\mu|\eta}(q),$$

*where $\overline{K}_{\lambda,\mu|\eta}(q)$ is the $q$-analog of weight multiplicity* $\dim V_\lambda^{(\mathfrak{gl}(N|M))}(\mu|\eta)$ *for the Lie superalgebra $\mathfrak{gl}(N|M)$ (see e.g. [Ser]).*

The formula (2.35) is very convenient in many combinatorial applications (see e.g. [Kir3], [Kir5], [Kir8], [F]). However, there exist other forms for (2.35), using the different parameterizations of the same type configurations set. For example, it is possible (and useful !) to rewrite the formula (2.35) for $\overline{K}_{\lambda,\{\mu\}}(q)$ using the vacancy numbers $\{P_r^{(k)}(\nu;\{\mu\})\}$ instead of parameters $\{\nu_r^{(k)}\}$. But having in mind the applications of our formula (2.35) for (dual) Kostka polynomials to a problem of computing the branching functions for integrable highest weight representations of the affine Lie algebras, it is more convenient to rewrite the RHS(2.35) in terms of parameters $m_r(\nu^{(k)})$. Namely, let us fix a natural number $l$ such that $m_p(\nu^{(k)}) = 0$, if $p \geq l$ and $1 \leq k \leq n$. We define a vector $m := m(\nu) =$

$$(m_1(\nu^{(1)}), \ldots, m_l(\nu^{(1)}), m_1(\nu^{(2)}), \ldots, m_l(\nu^{(2)}), \ldots, m_1(\nu^{(n)}), \ldots, m_l(\nu^{(n)})) \in \mathbf{Z}_+^{ln}.$$

Then it is almost trivial exercise to show that

1) $P_r^{(k)}(\nu;\{\mu\})+m_r(\nu^{(k)})=(m(I-B)+m_0)_r^{(k)}$, where

*i)* $B=C_n\otimes T_l^{-1}\in \mathrm{Mat}_{ln\times ln}(\mathbf{Z})$, and

$$C_n=\begin{pmatrix} 2 & -1 & \cdots & & 0\\ -1 & & \cdots & & \\ & \cdots & \cdots & \cdots & \\ & & \cdots & & -1\\ 0 & & \cdots & -1 & 2\end{pmatrix}_{n\times n},$$

$$T_l^{-1}=\begin{pmatrix} 2 & -1 & \cdots & & 0\\ -1 & 2 & \cdots & & \\ & \cdots & \cdots & \cdots & \\ & \cdots & -1 & 2 & -1\\ 0 & \cdots & & -1 & 1\end{pmatrix}_{l\times l}^{-1}=(\min(a,b))_{1\le a,b\le l}\,;$$

*ii)* vector $m_0:=(m_{0,r}^{(k)})$, where $m_{0,r}^{(k)}=\sum_{j\le r}(\mu^{(k)})'_j$, and $I:=I_{ln}=(\delta_{a,b})_{1\le a,b\le ln}$.

2) $\overline{c}(\nu)=\frac{1}{2}mBm^t-n(\lambda)$;

3) vector $m\in\mathbf{Z}_+^{ln}$ satisfies the following constraint $m\cdot Q=\gamma$, where

*i)* $\gamma:=(\gamma_r^{(k)})$, $\gamma_r^{(k)}=\sum_{j\ge k+1}\left\{\lambda_j-(j-k)\mid\mu^{(j)}|\right\}$;

*ii)* $Q:=I_n\otimes\begin{pmatrix}1 & \ldots & 1\\ & \ldots & \\ l & \ldots & l\end{pmatrix}_{l\times l}$.

Consequently, we have

$$q^{n(\lambda)}\overline{K}_{\lambda,\{\mu\}}(q)=\sum_{\substack{m\in\mathbf{Z}_+^{ln}\\ m\cdot Q=\gamma}}q^{\frac{1}{2}mBm^t}\prod_a\begin{bmatrix}(m(I-B)+m_0)_a\\ m_a\end{bmatrix}_q. \tag{2.36}$$

**Example** (Melzer's conjecture [Me1]). Let us consider a particular case of our formula (2.36) for (dual) Kostka polynomials when $\lambda=(\frac{1}{2}L+S,\ \frac{1}{2}L-S)$, $\mu^{(1)}=(1^L)$ and $\mu^{(k)}=0$, if $k\ge 2$. Here a "spin" $S$ is a half-integer such that $M:=\frac{1}{2}L-S\in\mathbf{Z}_+$.

First of all, in our case we have $B=2T_M^{-1}$, a vector $m_0=(L,\ldots,L)\in\mathbf{Z}_+^M$ and the vacancy numbers for given configuration $m\in\mathbf{Z}_+^M$ are the following ones $P_a(m;\{\mu\})=L-2(mT_M^{-1})_a$. Consequently (compare [Me1], (3.10)),

$$\mathcal{F}_S^{(L)}(q):=q^{-n(\lambda)}\sum_{\substack{m_a\in\mathbf{Z}_+\\ \sum_{a=1}^M am_a=M}}q^{\frac{1}{2}mBm^t}\prod_{a=1}^M\begin{bmatrix}L+(m(I-B))_a\\ m_a\end{bmatrix}_q=\overline{K}_{\lambda,(1^L)}(q). \tag{2.37}$$

On the other hand, using the hook-formula (see Proposition F) for Kostka polynomials, one can easily find

$$\overline{K}_{\lambda,(1^L)}(q) = \frac{1-q^{2S+1}}{1-q^{\frac{1}{2}L+S+1}} \begin{bmatrix} L \\ \frac{1}{2}L-S \end{bmatrix}_q \tag{2.38}$$

$$= \begin{bmatrix} L \\ \frac{1}{2}L-S \end{bmatrix}_q - q^{2S+1} \begin{bmatrix} L \\ \frac{1}{2}L-S-1 \end{bmatrix}_q \equiv \mathcal{B}_S^{(L)}(q).$$

Thus, $\mathcal{F}_S^{(L)}(q) = \mathcal{B}_S^{(L)}(q)$, as we set out to prove.

Now if $L \to \infty$, $S \to \infty$ and $M = \frac{1}{2}L - S$ is fixed (Thermodynamic limit), then it follows from (2.37) and (2.38) that

$$\sum_{\substack{m=(m_a)\in \mathbf{Z}_+^M \\ \sum_{a=1}^M am_a = M}} \frac{q^{\frac{1}{2}mBm^t}}{\prod_{a=1}^{M}(q)_{m_a}} = \frac{q^M}{(q)_M}.$$

Another interesting limit is $L \to \infty$, $M \to \infty$ and $S$ being fixed (Thermodynamic Bethe ansatz (TBA) limit). In the last case one can find

$$\frac{1-q^{2S+1}}{1-q} \sum_{\{m_a \in \mathbf{Z}_+\}} \frac{q^{\frac{1}{2}mBm^t+|m|}}{\prod_{a\geq 1}(q)_{m_a}} = \frac{1-q^{2S+1}}{(q)_\infty}, \tag{2.39}$$

where summation in (2.39) is taken over all sequences $\{m_a \in \mathbf{Z}_+\}_{a=1}^{\infty}$ such that $m_a = 0$ for almost all indices $a$, and $B := B_\infty = 2(\min(a,b))$, $1 \leq a, b < \infty$.

**A proof of the "limiting" identity (2.39).** First of all let us remark that if a partition $\lambda$ consists of only two parts, then for any weight $\mu$ and any configuration $\nu$ of type $(\lambda;\mu)$ one can compute the cocharge of $\nu$ by the following rules

$$\overline{c}(\nu) = mBm^t - n(\lambda) = 2n(\nu) := 2\sum_{k\geq 1} \binom{\nu_k'}{2}. \tag{2.40}$$

Now let us consider a sequence of configurations $\{\nu := \nu(M)\}$ of types $(\lambda := (M+2S, M); (1^{|\lambda|}))$, $M \geq 1$. It is clear from the formulae for cocharge (see (2.40)), that under the limit $M \to \infty$ and $S$ is fixed, the sequence of configurations $\{\nu(M)\}$ gives a non-zero contribution to the limit

$$\lim_{\substack{M\to\infty \\ S \text{ is fixed}}} \overline{K}_{\lambda=(M+2S,M),(1^{|\lambda|})}(q)$$

if and only if $\lim_{M\to\infty} \overline{c}(\nu(M)) < \infty$.

The last condition is equivalent to the following one: $\nu(M)=(\nu_1,\widetilde{\nu})$, where $\nu_1:=\nu_1(M)\to\infty$ if $M\to\infty$ and the partition $\widetilde{\nu}$ does not depend on $M$. It is clear that under these assumptions the cocharge $\overline{c}(\nu)=\sum_{k\ge1}(\widetilde{\nu}'_k)^2+|\widetilde{\nu}|$ does not depend on $M$ at all. It remains to consider the limit ($\nu:=\nu(M)$)

$$\lim_{M\to\infty}\overline{K}_{\{\nu(M)\}}(q)=\lim_{L\to\infty}q^{\overline{c}(\nu)}\prod_n\begin{bmatrix}L-2Q_n(\nu)+m_n(\nu)\\ m_n(\nu)\end{bmatrix}_q.$$

It is clear that if $n\le\widetilde{\nu}_1$, then $L-2Q_n(\nu(M))\to\infty$, if $M\to\infty$, and

$$\lim_{M\to\infty}\begin{bmatrix}L-2Q_n(\nu(M))+m_n(\nu(M))\\ m_n(\nu(M))\end{bmatrix}=\frac{1}{(q)_{m_n(\widetilde{\nu})}}.$$

Now, if $\widetilde{\nu}_1<n<\nu_1$, then $\nu_n(\nu)=0$. Finally, if $n=\nu_1$, then $L-2Q_n(\nu(M))=2S$ and $m_n(\nu(M))=1$. It remains to put $m_a:=m_n(\widetilde{\nu})$.

■

It is well-known (see e.g. [BS]), that the RHS of (2.39) is the character $\chi_S^{\mathrm{Vir}}=\mathrm{Tr}\,q^{L_0-S^2}$ of the irreducible highest weight representation of the Virasoro algebra at central charge $c=1$ and highest weight $h=S^2$, $S\in\frac12\mathbf{Z}_{\ge0}$.

More generally, let us consider the case when $\lambda$ is a partition $l(\lambda)\le n$ and $\mu^{(1)}=(1^{|\lambda|})$, $\mu^{(k)}=0$, if $k\ge2$. Then we have ($L:=|\lambda|,\widetilde{L}:=|\lambda|-\lambda_1$)

$$\overline{K}_{\lambda,(1^L)}(q)=\sum_m q^{\frac12 mBm^t}\prod_a\begin{bmatrix}(m(I-B)+m_0)_a\\ m_a\end{bmatrix}_q,\tag{2.41}$$

where $B=C_{n-1}\otimes T_{\widetilde{L}}^{-1}$, $m_0=(m_{0,r}^{(k)})$ and $m_{0,r}^{(k)}=L\cdot\delta_{k,1}$, $1\le k\le n-1$, $1\le r\le\widetilde{L}$. The summation in (2.41) is taken over all sequences $m\in\mathbf{Z}_+^{(n-1)\widetilde{L}}$ under the following constraint

$$\sum_{r=1}^{\widetilde{L}}rm_r^{(k)}=\sum_{j\ge k+1}\lambda_j,\quad 1\le k\le n-1.$$

It follows from the hook-formula (see Proposition F) that

$$\overline{K}_{\lambda,(1^L)}(q)=\frac{(q)_L}{\prod_{x\in\lambda}(1-q^{h(x)})}.$$

Now let us consider the limit $L\to\infty$, $\lambda_1\to\infty$, but the partition $\widetilde{\lambda}=(\lambda_2,\lambda_3,\dots,\lambda_n)$ is fixed (Thermodynamic limit). Then one can obtain the following identity

$$\sum_{m=(m_r^{(k)})}\frac{q^{\frac12 mBm^t}}{\prod_{r=1}^{\widetilde{L}}(q;q)_{m_r^{(1)}}}\prod_{k\ge2,\ r\ge1}\begin{bmatrix}(m(I-B))_r^{(k)}\\ m_r^{(k)}\end{bmatrix}_q=\frac{q^{n(\widetilde{\lambda})+|\widetilde{\lambda}|}}{\prod_{x\in\widetilde{\lambda}}(1-q^{h(x)})},\tag{2.42}$$

where summation on $m$ in (2.42) is the same as in (2.41). Another interesting limit is $L \to \infty$ all $\lambda_i \to \infty$ $(1 \le i \le n)$, but the differences $(\lambda_{n+1} := 0)$ $\lambda_i - \lambda_{i+1}$ $(1 \le i \le n)$ are fixed (TBA limit). In this case one can obtain the following result

$$\lim_{L\to\infty} \overline{K}_{\lambda_L,(1^{|\lambda_L|})}(q) = \frac{\prod\limits_{1\le i<j\le n} (1-q^{\lambda_i-\lambda_j-i+j})}{(q)_\infty^{n-1}}, \tag{2.43}$$

where $\lambda_L = \lambda + (L^n)$.

It is well-known (see e.g. [KP] or Section 2.4.1, (2.31)), that the RHS of (2.43) is the branching function for level 1 vacuum representation $\Lambda_0$ of $\widehat{sl_n}$.

**2.4.3.** Level $k$ vacuum representation $k\Lambda_0$ of the affine Lie algebra $\widehat{sl_2}$

**Theorem 13.** *Let $\mu = (\mu_1, \mu_2)$ be the length two composition, $|\mu| = 2N$. Then*

$$\sum_{\lambda,\ l(\lambda)\le 2} K_{\lambda,\mu} K_{\lambda,(2^N)}(q) = \sum_m \begin{bmatrix} N \\ \frac{\mu_1-m}{2}, \frac{\mu_2-m}{2}, m \end{bmatrix}_q \cdot q^{\frac{1}{2}m^2} q^{\frac{1}{2}(n(\mu')-N)}. \tag{2.44$_\mu$}$$

Let us deduce from Theorem 13 some combinatorial applications. First of all, it is clear that $K_{\lambda,\mu} = 1$ if $\lambda \succeq \mu$ with respect to the dominant order (see e.g. [Ma]) and $K_{\lambda,\mu} = 0$ otherwise. Hence $(\lambda = (\lambda_1 \ge \lambda_2))$,

$$K_{\lambda,(2^N)}(q) = \mathrm{RHS}(2.44)_{(\lambda_1,\lambda_2)} - \mathrm{RHS}(2.44)_{(\lambda_1+1,\lambda_2-1)}. \tag{2.45}$$

Using (2.45), one can find

$$\sum_{l=0}^{N} q^{-l} \frac{1-q^{2l+1}}{1-q} K_{(N+l,N-l),(2^N)}(q^2) \tag{2.46}$$
$$= q^{N^2-2N} \sum_k q^{k^2} \left\{ \sum_{l\ge 0}{}' (q^l + q^{-l}) q^{l^2} \begin{bmatrix} N-k \\ \frac{N-k-l}{2} \end{bmatrix}_{q^2} \right\} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2}.$$

The next Lemma makes it possible to simplify the RHS(2.46).

**Lemma 7.**

$$\mathrm{RHS}(2.46) = q^{N^2-2N} \sum_{k=0}^{N} q^{k^2+k} (1+q) \dots (1+q^{N-k}) \begin{bmatrix} N \\ k \end{bmatrix}_{q^2}. \tag{2.47}$$

Analytical proof of Lemma 7 will be given elsewhere.

**Corollary 7** (Milne's conjecture [Ml2]).

$$\sum_{l=0}^{m} q^{-l}\frac{1-q^{2l+1}}{1-q}K_{(m+l,m-l),(2^m)}(q^2) \tag{2.48}$$
$$=\sum_{k=0}^{m} q^{m^2-2m}q^{k^2+k}(1+q)\dots(1+q^{m-k})\begin{bmatrix} m \\ k \end{bmatrix}_{q^2}.$$

**Corollary 8** (of Theorem 13). *Let $\lambda=(\lambda_1,\lambda_2)$ be a partition, $|\lambda|\equiv 0 \pmod 2$, and $b_\lambda^{2\Lambda_0}(q)$ be the branching function for the level 2 vacuum representation $2\Lambda_0$ of the affine Lie algebra $\widehat{sl_2}$. Then*

$$b_\lambda^{2\Lambda_0}(q)=\lim_{N\to\infty} q^{-N}\overline{K}_{\lambda_N,\mu_N}(q),$$

*where $\lambda_N=(\lambda_1+2N,\lambda_2+2N)$ and $\mu_N=(2^M)$, $M=\dfrac{1}{2}|\lambda|+2N$.*

Finally, let us generalize the identity $(2.44)_\mu$ to the higher levels.

**Theorem 14.**

$$\sum_{l(\lambda)\le 2} K_{\lambda,\mu}K_{\lambda,(k^N)}(q) \tag{2.49}$$
$$=\sum_{m=(m_1,m_2,\dots,m_{k-1})\in\mathbf{Z}_+^{k-1}} q^{\frac{1}{k}n(\mu')-\frac{k-1}{2}N}q^{\frac{mA_{k-1}m^t}{k^2}}\begin{bmatrix} N \\ \frac{\mu_1-m_1}{k},\frac{\mu_2-m_{k-1}}{k},\frac{1}{k}(mA_{k-1})_a \end{bmatrix}_q.$$

**Corollary 9** (of Theorem 14). *Let $\lambda=(\lambda_1,\lambda_2)$ be a partition, $|\lambda|\equiv 0 \pmod{2k}$, and $b_\lambda^{k\Lambda_0}(q)$ be the branching function for the level $k$ vacuum representation $k\Lambda_0$ of the affine Lie algebra $\widehat{sl_2}$. Then*

$$\lim_{N\to\infty} q^{-k(N^2-N)}K_{\lambda_N,\mu_N}(q)=b_\lambda^{k\Lambda_0}(q),$$

*where $\lambda_N:=(\lambda_1+kN,\lambda_2+kN)$ and $\mu_N=(k^M)$, $M:=M(N)=\frac{1}{k}|\lambda|+2N$.*

Let us remind that, by definition, the partition $(k^M)$ is the one consisting of $M$ parts all equal to $k$.

Proof. Consider the following sum

$$\Sigma_N:=\sum_{\substack{\lambda\ ,\, l(\lambda)\le 2\\ |\lambda|=2kN}}\frac{s_\lambda(x_1,x_2)}{(x_1x_2)^{kN}}q^{-k(N^2-N)}K_{\lambda,(k^{2N})}(q). \tag{2.50}$$

It follows from Theorem 14 that

$$\Sigma_N=\sum_{\mu_1=0}^{2kN}(x_1/x_2)^{\mu_1-kN}\sum_{m=(m_1,\dots,m_{k-1})\in\mathbf{Z}_+^{k-1}} q^{\frac{1}{k}n(\mu')-(k-1)N-k(N^2-N)}q^{\frac{mA_{k-1}m^t}{k^2}}$$
$$\cdot\begin{bmatrix} 2N \\ \frac{\mu_1-m_1}{k},\frac{\mu_2-m_{k-1}}{k},\frac{1}{k}(mA_{k-1})_a \end{bmatrix}_q.$$

Let us put $\mu_1 = m_1 + (l+N)k$ (thus, $\mu_2 = (N-l)k - m_1$) and $z = x_1/x_2$. It is clear that

$$\frac{1}{k}n(\mu') - (k-1)N - k(N^2-N) = \frac{m_1^2}{k} + kl^2 + 2m_1 l.$$

Hence, we have

$$\Sigma_N = \sum_{m=(m_1,\ldots,m_{k-1})\in\mathbf{Z}_+^{k-1}} \sum_{\substack{l\in\mathbf{Z}\\ l\geq -N-\frac{m_1}{k}}} z^{kl+m_1} q^{kl^2+2m_1 l} q^{\frac{km_1^2+mA_{k-1}m^t}{k^2}} \cdot \begin{bmatrix} 2N \\ l+N, N-l-\dfrac{m_1+m_{k-1}}{k}, \dfrac{1}{k}(mA)_a \end{bmatrix}_q .$$

Consequently, $\lim\limits_{N\to\infty} \Sigma_N =$

$$\frac{1}{(q)_\infty^2} \sum_{m=(m_1,\ldots,m_{k-1})\in\mathbf{Z}_+^{k-1}} \sum_{l\in\mathbf{Z}} z^{kl+m_1} q^{kl^2+2m_1 l} q^{\frac{km_1^2+mA_{k-1}m^t}{k^2}} \prod_{a=1}^{k-1} (q)_{s_a}^{-1}, \tag{2.51}$$

where $s_a := \dfrac{1}{k}(mA_{k-1})_{k-a}$.

Now let us rewrite the summation in the RHS(2.51) in terms of parameters $s_a$, $a=1,\ldots,k-1$. One can easily check that

$$\sum_{a=1}^{k-1} a s_a = m_1, \quad \frac{km_1^2+mA_{k-1}m^t}{k^2} = sT_{k-1}^{-1}s^t.$$

Consequently, $\lim\limits_{N\to\infty} \Sigma_N =$

$$\frac{1}{(q)_\infty^2} \sum_{m_1=0}^{\infty} \sum_{\substack{s=(s_1,\ldots,s_{k-1})\in\mathbf{Z}_+^{k-1}\\ \sum_a a s_a = m_1}} \sum_{l\in\mathbf{Z}} z^{kl+m_1} q^{kl^2+2m_1 l} \frac{q^{sT_{k-1}^{-1}s^t}}{\prod_a (q)_{s_a}}$$

$$= \frac{1}{(q)_\infty^2} \sum_{m_1=0}^{\infty} \Theta_{m_1}^k(z) c_{m_1}^k(q) = \mathrm{ch}(V(k\Lambda_0)).$$

On the other hand,

$$\lim_{N\to\infty} \sum_{\substack{\lambda, l(\lambda)\leq 2\\ |\lambda|=2kN}} \frac{s_\lambda(x_1,x_2)}{(x_1x_2)^{kN}} q^{-k(N^2-N)} K_{\lambda_N,\mu_N}(q) = \sum_{\substack{\lambda=(\lambda_1\geq\lambda_2)\\ |\lambda|=0}} s_\lambda(x) b_\lambda(q),$$

where $b_\lambda(q) = \lim\limits_{N\to\infty} q^{-k(N^2-N)} K_{\lambda_N,(k^{2N})}(q)$.

Consequently, $b_\lambda(q)$ is the branching function of level $k$ vacuum-$\widehat{sl_2}$ representation $k\Lambda_0$.

∎

We assume that any branching function $b_\lambda^\Lambda(q)$ of integrable highest weight representation $V(\Lambda)$ of the affine Lie algebra $\widehat{sl_n}$ can be constructed as an appropriate limit of the Kostka-Foulkes polynomials. More exactly:

**Conjecture 4.** *Let $\Lambda = k\Lambda_0$ be the level $k$ vacuum representation of the affine Lie algebra $\widehat{sl_n}$. Then*

$$b_\lambda^\Lambda(q) = \lim_{N\to\infty} \overline{K}_{\lambda_N,(k^{Nn})}(q),$$

*where $\lambda_N := \lambda + ((kN)^n)$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ and $|\lambda| = 0$.*

**Remark.** There exists an interesting connection between the Milne polynomials ([Ml1], [Ml2], [Ga])

$$M_\mu(x;q) := \sum_\lambda s_\lambda(x) K_{\lambda,\mu}(q) = Q_\mu\left(\frac{x}{1-q}\right),$$

where $x = (x_1, \ldots, x_n)$, and the characters $\mathrm{ch}V(\Lambda)$ of integrable highest weight representations $V(\Lambda)$ of the affine Lie algebra $\widehat{sl}_n$. Roughly speaking, the character $\mathrm{ch}V(\Lambda)$ is an appropriate limit of Milne's polynomials (see the proofs of Theorems 11,13 and 14). It would be interesting to know the solutions to what kind of hierarchy (= a system of nonlinear partial differential equations) the characters of integrable highest weight representations of $\widehat{sl}_n$ are.

**2.4.4.** Crystal basis and Robinson-Schensted correspondence.

In this section we are going to discuss briefly a connection between a combinatorial description of the so-called crystal basis [Ka] of an integrable $\widehat{sl_n}$-module $V_n^k$ of level $k$, found by Jimbo, Misra, Miwa and Okado in [JMMO] (see also [MM], [KKNMMNN] and [DJKMO1], [DJKMO2]) and the Kostka-Foulkes polynomials via the Robinson-Schemsted-Knuth (RSK) correspondence (see e.g. [Sa], where a construction and basic properties of RSK are given).

The starting point for us is the following variant of the Jimbo-Misra-Miwa-Okado formula for the character $\mathrm{ch}(V_n^k)$ in the paths realization of the crystal basis, due to E. Frenkel and A. Szenes [FrSz2]. Thus (see [FrSz2], Section 6), let us introduce the set

$$S_n^k = \{\mathbf{a} = (a_1, \ldots, a_n) \mid a_i \geq 0,\ a_1 + \cdots + a_n = k\}.$$

Denote by $\tau : (a_1, a_2, \ldots, a_n) \to (a_n, a_1, \ldots, a_{n-1})$ the cyclic permutation acting on $S_n^k$ and define the element $\mathbf{a}^k \in S_n^k$ by $\mathbf{a}^k = (k, 0, \ldots, 0)$. Now the ground state path $\mu$ is the sequence $(\mathbf{a}^k, \tau\mathbf{a}^k, \tau^2\mathbf{a}^k, \ldots)$, and the set of restricted paths $\mathcal{R}_n^k$ is the set of sequences $\eta = (\eta_0, \eta_1, \ldots)$ of elements $\eta_j \in S_n^k$, which coincide with $\mu$ except for a finite set of indices. Define the weight function $w$ on $\mathcal{R}_n^k$ by the formula

$$w(\eta) = \sum_{m=1}^{\infty} m\left[H(\eta_{m-1}, \eta_m) - H(\mu_{m-1}, \mu_m)\right],$$

where $H_n(\mathbf{a},\mathbf{b}) := \max_{1\le i\le n}(h_i(\mathbf{a},\mathbf{b}))$ and

$$h_i(\mathbf{a},\mathbf{b}) = \sum_{j=1}^{i} a_j - \sum_{j=1}^{i-1} b_j.$$

**Theorem G** ([JMMO]). *As formal power series in $q$*

$$\mathrm{ch}\ V_n^k = \sum_{\eta\in\mathcal{R}_n^k} q^{w(\eta)}.$$

Now let us introduce the sets

$$W_N = \{\eta\in\mathcal{R}_n^k \mid \eta_j = \mu_j \ \text{ for } \ j\ge nN\},$$
$$W_N^{\mathbf{a}} = \{\eta\in\mathcal{R}_n^k \mid \eta_{nN-1} = \mathbf{a},\ \eta_j = \mu_j \ \text{ for } \ j\ge nN\}, \quad \mathbf{a}\in S_n^k,$$

and the polynomials (the characters of $W_N$ and $W_N^{\mathbf{a}}$)

$$w_N(q) = \sum_{\eta\in W_N} q^{w(\eta)}, \quad w_N^{\mathbf{a}}(q) = \sum_{\eta\in W_N^{\mathbf{a}}} q^{w(\eta)}.$$

Further, let us consider the column vector $\mathbf{w}_N$ of characters $\mathbf{w}_N^{\mathbf{a}}$ in a certain order, which is fixed once and for all. In the paper [FrSz2] the recurrence relation for vectors $\mathbf{w}_N$ was obtained. Namely,

$$\mathbf{w}_{N+1} = q^{-knN} M_k(q^{nN+n-1})\dots M_k(q^{nN+1})M_k(q^{nN})\mathbf{w}_N,$$

where $M_k(x)_{\mathbf{a},\mathbf{b}} = x^{H_n(\mathbf{a},\mathbf{b})}$, for $\mathbf{a},\mathbf{b}\in S_n^k$. This important recurrence relation gives a way to compute the polynomials $w_N(q)$.

Two particular cases when either $k=1$ and $n\ge 2$ is arbitrary or $n=2$ and $k\ge 1$ is arbitrary seem to be the most accessible from combinatorial point of view. Namely, let us consider at first the level $k=1$ case. Then we can imagine each finite path $\eta\in W_N$ as a word $w = a_1\dots a_{nN}$ of length $m = nN$, composed from the numbers $1,2,\dots n$. Let $\mu$ be the weight of $w$, i.e. $\mu_i = \{j \mid a_j\in w,\ a_j = i\}$. For given weight $\mu, l(\mu)\le n$, let us denote by $M(\mu)$ the set of all words of weight $\mu$. It is well-known (see e.g. [An1]) that

$$\mathrm{Card}\ M(\mu) = \frac{|\mu|!}{\mu_1!\dots\mu_n!} = \binom{|\mu|}{\mu_1,\dots,\mu_n}.$$

**Definition 7** ([Fo]). *A function $\varphi$ on the set $M(\mu)$ is called to be mahonian statistics, if*

$$\sum_{w\in M(\mu)} q^{\varphi(w)} = q^{d(\varphi)} \begin{bmatrix} |\mu| \\ \mu_1,\dots,\mu_n \end{bmatrix}_q,$$

*where a positive integer $d(\varphi)$ depends only on weight $\mu$.*

The classical examples of mahonian statistics (with $d(\mu)=0$) are the number of inversions ($\mathrm{inv}(w)$) and major (or greater) index ($\mathrm{Maj}(w)$) of a word $w\in M(\mu)$, see e.g. [An1], Section 3.4, or Exercise 1 to Section 2.4. The next result was proven in [DJKMO1] using the recurrence relations technique.

**Theorem H** (Date-Jimbo-Kuniba-Miwa-Okado). *The weight function $w(\eta)$ on the set restricted paths $\mathcal{R}_n^1$ defines on the weight $\mu$ subsets $M(\mu) \subset W_N$ the mahonian statistics with $d(\varphi) = 0$.In other words*

$$\sum_{\eta \in M(\mu)} q^{w(\eta)} = \begin{bmatrix} |\mu| \\ \mu_1, \ldots, \mu_n \end{bmatrix}_q .$$

Proof. First of all, we are going to replace a restricted path $\eta \in \mathcal{R}_n^1$ by some word. More exactly, let $\eta \in \mathcal{R}_n^1 \cap W_N$. The path $\eta$ has the form $\eta = (\eta_1, \eta_2, \ldots, \eta_{nN})$, where each $\eta_j \in S_n^1$. But $S_n^1 = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Thus one can replace the path $\eta$ by the word $w$ just changing $e_j$ on $j$. Let us assume that the word $w$ has a weight $\mu$. Our next task is to rewrite the weight function $w(\eta)$ given on the set of restricted paths, as a function on the set of words. The crucial observation is following: if $e_i$ and $e_j$ belong to $S_n^1$, then $H(e_i, e_j) = \chi(i \geq j)$, where for any statement $P$ the symbol $\chi(P)$ is equal to 1, if $P$ is true and $\chi(P) = 0$ otherwise. Using this observation, one can find

$$w(\eta) = \sum_{m=1}^{nN-1} m\chi(a_m \geq a_{m+1}) - \frac{nN(N-1)}{2},$$

where $w = a_1 a_2 \ldots a_{nN}$ is the word corresponding to the restricted path $\eta \in \mathcal{R}_n^1 \cap W_N$. To go further, let us introduce on the set of words $w = a_1 \ldots a_{nN}$ the statistics $\widetilde{\mathrm{Maj}}(w)$:

$$\widetilde{\mathrm{Maj}}(w) = \sum_{m=1}^{nN-1} m\chi(a_m \geq a_{m+1}).$$

Note that the classical major index of the word is defined as

$$\mathrm{Maj}(w) = \sum_{m=1}^{nN-1} m\chi(a_m > a_{m+1}).$$

It is the MacMahon Theorem (see e.g. [An1], Theorem 3.7) that

$$\sum_{w \in M(\mu)} q^{\mathrm{Maj}(w)} = \begin{bmatrix} |\mu| \\ \mu_1, \ldots, \mu_n \end{bmatrix}_q .$$

The proof given in [An1], Theorem 3.7, can be modified to show the statistics $\widetilde{\mathrm{Maj}}$ is also machonian with $d(\varphi) = n(\mu')$.

■

It is an interesting task to find a purely combinatorial proof of equidistribution for the statistics Maj and $\widetilde{\mathrm{Maj}} - n(\mu')$. We are going to consider this question in more details in a separate publication.

Another interesting question is a connection between the weight function $w(\eta)$ (or $\widetilde{\mathrm{Maj}}(w)$) and the Robinson-Schensted correspondence. Namely, applying the Robinson-Schensted correspondence to a given word $w \in M(\mu)$, we obtain the pair $(P, Q)$ of (semi)standard Young tableaux of the same shape, say $\lambda$, such that

$$P \in \mathrm{STY}(\lambda, \mu), \quad Q \in \mathrm{STY}(\lambda, (1^{nN})).$$

It is well-known (see e.g. [Sa]) that if $w \overset{\mathrm{RS}}{\longleftrightarrow} (P, Q)$, then $\mathrm{Maj}(w) = \mathrm{ind}Q$, where the index of a standard tableau $T \in \mathrm{STY}(\lambda, 1^{|\lambda|})$ is defined as $\mathrm{ind}(T) = \sum_{j \in \mathrm{Des}(T)} j$ and $\mathrm{Des}(T)$ is the so-called descent set of tableau $T$ ($j \in \mathrm{Des}(T)$ iff $(j+1)$ lies in tableau $T$ strictly below than $j$). This result allows to give a pure combinatorial proof of Theorem 9. Indeed, it is also well-known result (see e.g. [Ma]) that

$$K_{\lambda,(1^{|\lambda|})}(q) = \sum_{T \in \mathrm{STY}(\lambda, 1^{|\lambda|})} q^{\mathrm{ind}(T)}.$$

A relation between $\widetilde{\mathrm{Maj}}$-statistics and RS-correspondence is more complicated and will be considered elsewhere.

**Exercises to Section 2.4.**

**1.** Let $\mu$ be a composition, $l(\mu) = n$. Consider the set

$$M(\mu) = \left\{ w = a_1 a_2 \cdots a_{|\mu|} \mid a_k \in [1, n],\ \#\{i \mid a_i = j\} = \mu_j \right\}.$$

Let us define the number of inversions for a word $w$ as

$$\mathrm{inv}(w) = \sum_{1 \le i < j \le |\mu|} \chi(a_i > a_j).$$

*i*) Prove

$$\sum_{w \in M(\mu)} q^{\mathrm{inv}(w)} = \begin{bmatrix} |\mu| \\ \mu_1, \ldots \mu_n \end{bmatrix}_q. \qquad \text{(P. MacMahon, 1914)}$$

*ii*) Given $w \in M(\mu)$, let $w_{ij}$ be the subword of $w$ formed by deleting all letters $a_m$ such that $a_m \ne i$ or $j$. For example, if $w = 2411213144321 \in M(5,3,2,3)$ then $w_{12} = 21121121$, $w_{13} = 1113131$, $w_{14} = 41111441$, $w_{23} = 22332$, $w_{24} = 242442$, $w_{34} = 43443$.

The $Z$-index (Zeilberger's index) of $w$ is defined to be the sum of the major indices of all 2-letter subwords, $w_{ij}$, of $w$. That is

$$Z(w) = \sum_{1 \le i < j \le n} \mathrm{Maj}(w_{ij}).$$

Analogously, let us introduce the $\widetilde{Z}$-index of $w$ as follows

$$\widetilde{Z}(w) = \sum_{1 \le i < j \le n} \widetilde{\mathrm{Maj}}(w_{ij}) - n(\mu).$$

For our example, $\mathrm{inv}(w) = 29$, $\mathrm{Maj}(w) = 47$, $\widetilde{\mathrm{Maj}}(w) - n(\mu) = 59 - 15 = 44$, $Z(w) = 12 + 10 + 8 + 4 + 7 + 5 = 46$, $\widetilde{Z}(w) = 19 + 13 + 23 + 11 + 8 + 8 - 15 = 67$.

Prove

$$\sum_{w \in M(\mu)} q^{Z(w)} = \begin{bmatrix} |\mu| \\ \mu_1, \dots, \mu_n \end{bmatrix}_q. \qquad \text{(D. Zeilberger, D. Bressoud, 1985)}$$

*iii*) Give a combinatorial proof of equidistribution for the statistics

$$\mathrm{inv}, \quad \mathrm{Maj} \quad \text{and} \quad Z.$$

Further details and proofs one can find in [Bre3], [ZB], [Gre] and [Han].

**2.** (Tensor product multiplicities and Bethe's ansatz equations).

Let $\mu^{(j)} = (\mu_1^{(j)} \ge \dots \ge \mu_n^{(j)} \ge 0)$, $1 \le j \le N$, and $\lambda$ be the highest weights and $V_{\mu^{(j)}}$, $V_\lambda$ be the corresponding irreducible finite dimensional representations of the Lie algebra $\mathfrak{gl}_n$.

**Conjecture 5.** *The tensor product multiplicity*

$$\mathrm{Mult}_{V_\lambda}(V_{\mu^{(1)}} \otimes \dots \otimes V_{\lambda^{(N)}})$$

*is equal to the number of solutions to the following system of algebraic equations on variables* $v_1 = \{v_{1,1}, \dots, v_{1,m_1}\}$, $v_2 = \{v_{2,1}, \dots, v_{2,m_2}\}$, ... ,$v_n = \{v_{n,1}, \dots, v_{n,m_n}\}$ *(we assume that* $v_0 = v_{n+1} = \phi$*)*

$$\prod_{j=1}^{N} \frac{v_k - \lambda_k^{(j)} i}{v_k - \lambda_{k+1}^{(j)} i} = \prod_{v_{k-1}} \frac{v_k - v_{k-1} + i/2}{v_k - v_{k-1} - i/2} \prod_{v'_k \ne v_k} \frac{v_k - v'_k - i}{v_k - v'_k + i} \prod_{v_{k+1}} \frac{v_v - v_{k+1} + i/2}{v_k - v_{k+1} - i/2}.$$

*Here* $i = \sqrt{-1}$. *The number of equations and that of variables are the same and are equal to* $m_1 + \dots + m_n$. *The relation between the weights* $\lambda$ *and* $\mu^{(1)}, \dots, \mu^{(N)}$ *and the composition* $m = (m_1, m_2, \dots, m_n)$ *is the following*

$$2m_k - m_{k-1} - m_{k+1} = \sum_{l \ge k+1} \left\{ \sum_{j=1}^{N} \mu_l^{(j)} - \lambda_l \right\}.$$

**3.** Given two partitions $\lambda$ and $\mu$, $|\lambda| = |\mu|$, and a natural number $k$. Let us consider the new partitions

$$\lambda_N := (\lambda, (k^N))^+ \quad \text{and} \quad \mu_N := (\mu, (k^N))^+,$$

where for any composition $\nu$ we denote by $\nu^+$ the corresponding partition (for example, if $\nu = (4419583162)$, then $\nu^+ = (9865443211)$).

Prove that there exists the limit

$$\lim_{N \to \infty} K_{\lambda_N, \mu_N}(q) := G_{\lambda, \mu; k}(q),$$

and this limit is a rational function.

• Problem. Find an algebraic/representation theoretic interpretation of the rational functions $G_{\lambda,\mu;k}(q)$.

See [Stn2], [Kir3] and [Kir9] for proofs and further results about the rational functions $G_{\lambda,\mu;k}(q)$.

**4.** Prove the following relations

*i)* $\displaystyle\sum_{\lambda} \dim V_{\lambda}^{(n)} \cdot K_{\lambda,\mu}(1) = \prod_{j} \binom{n+\mu_j-1}{\mu_j}$,

*ii)* $\displaystyle\sum_{\lambda} \begin{bmatrix} n \\ \lambda' \end{bmatrix}_q \cdot K_{\lambda,\mu}(1) = \prod_{j} \begin{bmatrix} n+\mu_j-1 \\ \mu_j \end{bmatrix}_q$.

Here $\displaystyle\begin{bmatrix} n \\ \lambda \end{bmatrix} := \prod_{x=(i,j)\in\lambda} \frac{1-q^{n+i-j}}{1-q^{h(x)}}$ is the generalized $q$-binomial coefficient; $V_{\lambda}^{(n)}$ is the highest weight $\lambda$ irreducible representation of $\mathfrak{gl}(n)$ and $K_{\lambda,\mu}(1)$ is the Kostka number.

Hint: use the Littelwood formula

$$\sum_{\lambda} s_{\lambda}(x_1,\ldots,x_n)s_{\lambda}(y_1,\ldots,y_m) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{1-x_iy_j}.$$

**5.** (Construction of elements in $C(F)$). We use the notation from the Section 2.3.1.

**Definition 8.** *Every equation of the following form*

$$X^A \prod_{r\in I} (1-\zeta_m^{k_r} X^r)^{A_r} = 1, \tag{1}$$

*where* $A, k_r, A_r$ *are integers,* $\zeta_m = \exp\left(\dfrac{2\pi i}{m}\right) \in F$*, and* $I$ *is a finite set of natural numbers, is called a (generalized) cyclotomic equation.*

Now, suppose that $a \in F^*$, $a \neq 1$ satisfies the cyclotomic equation (1), and let a natural number $N$ satisfy

*i)* $2 \mid NA$,

*ii)* $r \mid NA$, $m \mid (NA_r/r)k_r$, for $r \in I$.Show that

$$b := \sum_{r\in I} (NA_r/r)[\zeta_m^{k_r} a^r] \in C(F).$$

**Remark.** Let $\overline{b}$ be the image of $b$ in the Bloch group $B(F)$. It is an interesting problem to find an effective criteria when $\overline{b} \in B(F)_{\text{tor}}$.

**2.5. Quantum dilogarithm.**

Let us remind that the Euler dilogarithm $Li_2(x)$ is defined for $0 \le x \le 1$ by

$$Li_2(x) = \sum_{n \ge 1} \frac{x^n}{n^2}.$$

**Definition 9** ($q$-analog of Euler's dilogarithm or quantum dilogarithm). *Quantum dilogarithm $Li_2(x;q)$ is the following formal series*

$$Li_2(x;q) = \sum_{n \ge 1} \frac{x^n}{n(1-q^n)}. \tag{2.52}$$

Before stating the basic properties of the quantum dilogarithm, let us remind the definition of the two $q$-exponential functions (see e.g. [GR], Appendix II)

$$e_q(x) = \sum_{n \ge 0} \frac{x^n}{(q)_n} = \frac{1}{(x)_\infty}, \tag{2.53}$$

$$E_q(x) = \sum_{n \ge 0} \frac{q^{\frac{n(n-1)}{2}} x^n}{(q)_n} = (-x)_\infty.$$

Here we use the standard $q$-analysis notation:

$$(x)_\infty := (x;q)_\infty = \prod_{n \ge 0} (1 - xq^n), \quad (x)_n := \frac{(x;q)_\infty}{(xq^n;q)_\infty}.$$

It is clear from the definition (2.53) that

$$(1+x)E_q(qx) = E_q(x), \quad e_q(qx) = (1-x)e_q(x), \quad e_{q^{-1}}(x) = E_q(-qx). \tag{2.54}$$

**Lemma 8.** *We have*

*i)* $Li_2(x;q) = \log(e_q(x))$,

*ii)* $Li_2(x;q) + Li_2(x;q^{-1}) = -\log(1-x)$.

Proof. it is clear from (2.54) that

$$Li_2(qx;q) = Li_2(x;q) + \log(1-x).$$

Hence, if $Li_2(x;q) = \sum_{n \ge 1} a_n x^n$, then $q^n a_n = a_n - \frac{1}{n}$.

■

**Proposition H** (Euler-Maclaurin's summation formula). *If $f \in C^{2n}([M,N])$, $M$ and $N$ integers, then*

$$\sum_{m=M}^{N} f(m) = \int_M^N f(t)dt + \frac{1}{2}(f(N)+f(M)) + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\left\{f^{(2k-1)}(N) - f^{(2k-1)}(M)\right\} - \int_M^N \frac{\overline{B}_{2n}(t)}{(2n)!} f^{(2n)}(t)dt. \tag{2.55}$$

Here $B_k$ are the Bernoulli numbers and $B_k(t)$ are the Bernoulli polynomials which are defined by

$$\frac{ze^{tz}}{e^z-1} = \sum_{k\geq 0} \frac{B_k(t)}{k!} z^k, \quad B_k := B_k(0).$$

$\overline{B}_n(t)$ is the so-called modified Bernoulli polynomial: $\overline{B}_n(t) = B_n(\{t\})$, where $\{t\} := t - [t]$ is the fractional part of a real number $t$.

As a simple corollary of the Euler-Maclauren summation formula, one can obtain

**Corollary 10.** *If $q = e^{-\epsilon}$ and $\epsilon \to 0$, then the function $e_q(x)$ has the following asymptotic expansion* $(0 < x < 1)$

$$e_q(x) = (1-x)^{-\frac{1}{2}} \exp\left(\frac{Li_2(x)}{\epsilon} + \frac{x\epsilon}{12(1-x)}\right)(1 + O(\epsilon^3)).$$

Indeed, it follows from (2.55) that (cf. [Mo], [UN]) if $0 < x < 1$ and $q = e^{-\epsilon}$, then as $\epsilon \to 0$,

$$\log((x;q)_\infty) \sim -\frac{Li_2(x)}{\epsilon} + \frac{1}{2}\log(1-x) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{P_{2k-1}(x)\epsilon^{2k-1}}{(1-x)^{2k-1}},$$

where $P_n(x)$ is a polynomial of degree $n$ satisfying

$$P_n(x) = (x - x^2)P'_{n-1}(x) + nxP_{n-1}(x), \quad P_0 = 1, \quad n = 1,2,3,\ldots.$$

Moreover, the error in terminating the series is less in absolute value than that of the first term neglected and has the same sign.

The most important and fundamental property of Euler's dilogarithm is the five-term relation (in Rogers' form, [Ro]) $Li_2(x) + Li_2(y) - Li_2(xy) =$

$$= Li_2\left(\frac{x(1-y)}{1-xy}\right) + Li_2\left(\frac{y(1-x)}{1-xy}\right) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right). \tag{2.56}$$

Before discussing the (quantum) analog of the five-term relation for the quantum dilogarithm $Li_2(x;q)$ (see also [FK]), let us consider in more detail the properties of $q$-exponent $e_q(x)$.

**Lemma 9.** *Assume that variables $a$ and $b$ satisfy the Weyl relation $ab = qab$. Then we have*

$$i) \quad e_q(b)e_q(a) = e_q(a+b), \tag{2.57}$$

$$ii) \quad e_q(a)e_q(b) = e_q(b-ba)e_q(a) \tag{2.58}$$

$$= e_q(a+b-ba) \tag{2.59}$$

$$= e_q(b)e_q(-ba)e_q(a) \tag{2.60}$$

$$= e_q(b)e_q(a-ba) \tag{2.61}$$

Proof. The first relation (2.57) is well-known and follows from the following result due to M.-P. Schützenberger (the middle of 50's)

$$\text{if} \quad ab = qab, \quad \text{then} \quad (a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

As for the identities of part *ii*), the crucial relation is (2.58). Both (2.59) and (2.60) follow from (2.58) and (2.57) (hint: $a(b-ba) = q(b-ba)a$, $bab = qb^2a$). Now let us prove the identity (2.58). We give two proofs.

*First proof.* Let us check that the both sides of (2.58) as the functions on $b$ satisfy the same functional equation. Indeed, $(e(\cdot) := e_q(\cdot))$

$$\begin{aligned}
&e(a)e(qb)e^{-1}(b)e^{-1}(a) = e(a)(1-b)e^{-1}(a) = 1 - e(a)be^{-1}(a) \\
&= 1 - be(qa)e^{-1}(a) = 1 - b(1-a) = 1 - b + ba; \\
&e(q(b-ba))e(a)e^{-1}(a)e^{-1}(b-ba) = 1 - b + ba.
\end{aligned}$$

*Second proof* (A. Volkov). We have

$$e(a)e(b)e^{-1}(a) = e(e(a)be(a^{-1})) = e(be(qa)e^{-1}(a)) = e(b(1-a)).$$

In a similar manner, the identity (2.61) may be derived from (2.54).

■

It is very surprising that (2.57) and (2.59) have the polynomial analogs.

**Lemma 10.** *Under the assumptions of Lemma 9, we have*

$$i) \quad (a;q)_n \cdot (b;q)_n = (a+b-q^n ba;q)_n, \tag{2.62}$$

$$ii) \quad (b;q)_n \cdot (a;q)_n = (a+b-ba;q)_n. \tag{2.63}$$

Proof. Let us prove (2.62) by induction. The case $n = 1$ is clear. Further,

$$\begin{aligned}
\text{LHS}(2.62) &= (1-q^{n-1}a)(a)_{n-1}(1-b)(qb)_{n-1} \\
&= (1-a)(qa)_{n-1}(qb)_{n-1} - (1-q^{n-1}a)b(qa)_{n-1}(qb)_{n-1} \\
&= (1-a-b+q^{n-1}ab)(qa)_{n-1}(qb)_{n-1} \\
&\overset{\text{induction}}{=} (1-a-b+q^n ba)(qa+qb-q^{n+1}ba)_{n-1} = (a+b-q^n ba)_n.
\end{aligned}$$

Analogously, $\text{LHS}(2.63) = (1-b)(qb)_{n-1}(1-q^{n-1}a)(a)_{n-1}$

$$= (1-q^{n-1}b)(b)_{n-1}(a)_{n-1} - q^{n-1}(1-b)(qb)_{n-1}a(a)_{n-1}$$
$$= (1-q^{n-1}b)(b)_{n-1}(a)_{n-1} - q^{n-1}(1-b)a(b)_{n-1}(a)_{n-1}$$
$$= (1-q^{n-1}b-q^{n-1}a+q^{n-1}ba)(b)_{n-1}(a)_{n-1} \overset{\text{induction}}{=} (a+b-ba)_n.$$

■

Now we are ready to give the quantum analog of the five-term relation (1.4) for the quantum dilogarithm $Li_2(x;q)$.

**Theorem I** (L.D. Faddeev and R.M. Kashaev, [FK]). *The five term or "pentagon" relation ($ab = qba$)*

$$e_q(a)e_q(b) = e_q(b)e_q(-ba)e_q(a) \tag{2.60}$$

*is a quantum analog of the Rogers five-term relation (1.4) or (2.56), i.e. an appropriate limit $q \to 1$ in (2.60) reduces the last to the relation (2.56).*

Let us rewrite the relations (2.57), (2.59) and (2.60) using the quantum dilogarithm. For this purpose we need the Baker-Campbell-Hausdorff series. Namely, for any Lie algebra with Lie bracket $[\cdot,\cdot]$, let us denote by $H(x,y)$ its Campbell-Hausdorff series, i.e.

$$e^x \cdot e^y = e^{H(x,y)},$$
$$H(x,y) = x+y+\frac{1}{2}[x,y]+\frac{1}{12}\{[x,[x,y]]+[y,[y,x]]\}+\frac{1}{24}[xyyx]$$
$$+\frac{1}{144}\{9[xxyyx]+9[yyxxy]+4[xyyyx]-2[xxxxy]-2[yyyyx]\}+\dots,$$

where $[a_1a_2\dots a_n]$ is the multiple commutator $[a_1,[a_2,\dots,[a_{n-1},a_n]\cdots]$.

**Proposition 2** (Functional equations for quantum dilogarithm). *We have ($ab = qba$)*

$$\begin{aligned} &i)\quad H(Li_2(a;q),Li_2(b;q)) = Li_2(a+b-ab;q),\\ &ii)\quad H(Li_2(b;q),Li_2(a;q)) = Li_2(a+b;q). \end{aligned} \tag{2.64}$$

**Corollary 11** (Five term relation for quantum dilogarithm).

$$\begin{aligned} H(Li_2(a;q),Li_2(b;q)) &= H(Li_2(b;q),H(Li_2(-ba;q),Li_2(a;q)))\\ &= H(H(Li_2(b;q),Li_2(-ba;q)),Li_2(a;q)). \end{aligned} \tag{2.65}$$

**Exercises to Section 2.5.**

**A** (Weyl algebra and $q$-exponential a function).

**1.** Let us consider a function $\varphi(a)$

$$\varphi(a) := e_{q^2}(-qa) = \sum_{n\geq 0}\frac{(-qa)^2}{(q^2;q^2)_n} = \frac{1}{\prod_{j\geq 0}(1+q^{2j+1}a)}.$$

Show that

*i)* $\varphi(qa) = (1+a)\varphi(q^{-1}a)$;

*ii)* If $a$ and $b$ satisfy the Weyl relation $ab = q^2 ba$, then

$$\varphi(b)\varphi(a) = \varphi(a+b), \tag{1}$$

$$\varphi(a)\varphi(b) = \varphi(b)\varphi(a+qb), \tag{2a}$$

$$= \varphi(a+b+qba), \tag{2b}$$

$$= \varphi(b)\varphi(qba)\varphi(a), \tag{2c}$$

Hint: use the relations (2.57)-(2.61).

**2.** Let us consider the theta-function $\theta(a) := [\varphi(a)\varphi(a^{-1})]^{-1}$. Show that

*i)* $\theta(qa) = a^{-1}\theta(q^{-1}a)$;

*ii)* $\theta(a) = \sum_{n\in\mathbf{Z}} q^{n^2} a^n$;

*iii)* (cf. [FV]) If $ab = q^2 ba$, then

$$\theta(a)\theta(b)\theta(a) = \theta(b)\theta(a)\theta(b). \tag{3}$$

Hint: use the following relation $\theta(a)\theta(b)a = b\theta(a)\theta(b)$.

**3.** Let us consider a function $f(a) := f(z,a) = \varphi(za)/\varphi(a)$. Show that

*i)* $\dfrac{f(qa)}{f(q^{-1}a)} = \dfrac{1+za}{1+a}, \quad \dfrac{f(qz,a)}{f(q^{-1}z,a)} = 1+az$;

*ii)* $f(z,a) = \sum_{n\geq 0} \dfrac{a^n q^{n^2}(z;q^{-2})_n}{(q^2;q^2)_n}$;

*iii)* (Cf. [FV]) If $ab = q^2 ba$, then $f(b)f(a) = f(a+b+qba)$.
Hint: use the relations

$$\varphi(za)\varphi(b) = \varphi(b)\varphi(za+zqba),$$

$$\varphi(zb)\varphi(za+zqba) = \varphi(za+zb+zqba).$$

**4.** Let us consider a function $\pi(a) := \pi(z,a) = \varphi(za)\varphi(za^{-1})$. Show that

*i)* $\pi(z,a) = \sum_{n\geq 0} \dfrac{(-qz)^n H_n(a)}{(q^2;q^2)_n}$, where $H_n(a) := H_n(a;q^2)$ is the so-called continuous $q$-Hermite polynomial. Prove the recurrence relation for the continuous $q$-Hermite polynomials:

$$H_n(a) = AH_{n-1}(a) - (1-q^{2n-2})H_{n-2}(a), \tag{4}$$

where $A = a + a^{-1}$, $H_0(a) = 1$, $H_1(a) = a + a^{-1}$.

It follows from (4), that

$$H_n(a;q^2) = \det \begin{vmatrix} A & 1 & \dots & 0 & 0 \\ 1-q^2 & A & \dots & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & A & 1 \\ 0 & 0 & \dots & 1-q^{2n-2} & A \end{vmatrix}_{n\times n}.$$

Hint: use the functional equation:

$$\frac{\pi(qa)}{\pi(q^{-1}a)} = \frac{1+za}{1+za^{-1}}. \tag{5}$$

*ii*) (Modifyed Yang-Baxter equation). If $ab = q^2ba$, then

$$\pi(z',b)\pi(zz',qba)\pi(z,a) = \pi(z,a)\pi(zz',qab^{-1})\pi(z',b). \tag{6}$$

Proof. Let us remark at first that ($ab = q^2ba$)

$$\begin{aligned} \pi(z,a)b(1+zq^{-1}a^{-1}) &= b(1+zqa)\pi(z,a), \\ \pi(z,a)b^{-1}(1+zq^{-1}a) &= b^{-1}(1+zqa^{-1})\pi(z,a). \end{aligned} \tag{7}$$

Indeed, using the functional equation (5), one can find

$$\pi(z,a)b(1+aq^{-1}a^{-1}) = b\pi(z,q^2a)(1+zq^{-1}a^{-1}) = b(1+zqa)\pi(z,a).$$

To go further, let us use the relations (1) and (2). One can check

$$\begin{aligned} \Phi := \pi(z',b)\pi(zz',qba) &= \varphi(z'b)\varphi(z'b^{-1})\varphi(zz'qba)\varphi(zz'qb^{-1}a^{-1}) \\ &\overset{(2c)}{=} \varphi(z'b)\varphi(zz'qba)\varphi(z(z')^2a)\varphi(z'b^{-1})\varphi(zz'qb^{-1}a^{-1}) \\ &\overset{(1)}{=} \varphi(z'b(1+qza))\varphi(z(z')^2a)\varphi(z'b^{-1}(1+qza^{-1})). \end{aligned}$$

Consequently (use the relation (7) !),

$$\begin{aligned} \Phi\cdot\pi(z,a) &= \pi(z,a)\varphi(z'b(1+zq^{-1}a^{-1}))\varphi(z(z')^2a)\varphi(z'b^{-1}(1+zq^{-1}a)) \\ &= \pi(z,a)\pi(zz',qab^{-1})\pi(z',b), \end{aligned}$$

as it was claimed.

■

**5** (Commutation relations for the continuous $q$-Hermite polynomials).

Show that ($ab = q^2ba$)

*i*) $q[B_1(a), B_n(b)] = (1-q^{2n})\left\{B_1(qab^{-1})B_{n-1}(b) - B_{n-1}(b)B_1(qba)\right\}$,

$$q[B_n(a),B_1(b)] = (1-q^{2n})\left\{B_{n-1}(a)B_1(qab^{-1}) - B_1(qba)B_{n-1}(a)\right\}.$$

*ii*) More generally, in the Weyl algebra $\{a,b \mid ab=q^2ba\}$ there exist the following relations $(m,n=1,2,3,\dots)$

$$\sum_{k=0}^{\min(n,m)} (-q^{-1})^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^2;q^2)_k B_{n-k}(b)B_k(qba)B_{m-k}(a) \tag{8}$$

$$= \sum_{k=0}^{\min(n,m)} (-q^{-1})^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^2;q^2)_k B_{m-k}(a)B_k(qab^{-1})B_{n-k}(b).$$

Hint: the relations (8) are equivalent to the modifyed Yang-Baxter equation (6).

**6.** Let us consider a function (cf. [FV]) $\quad r(z,a)=\theta(a)\pi(z,a)$. Show that

*i*) (functional equations)

$$\frac{r(z,qa)}{r(z,q^{-1}a)} = \frac{1+za}{a+z}, \quad \frac{r(qz,a)}{r(q^{-1}z,a)} = (1+za)(1+za^{-1}).$$

*ii*) $\displaystyle r(z,a) = 1+\sum_{n\ge 1} q^{n(n-1)}\frac{(z;q^{-2})_n}{(zq^2;q^2)_n}(a^n+a^{-n}).$

*iii*) (Yang-Baxter equation, L.D. Faddeev and A.Yu. Volkov [FV], A.N. Kirillov).

If $ab=q^2ba$ and $zz'=z'z$, then

$$r(z,a)r(zz',b)r(z',a) = r(z',b)r(zz',a)r(z,b). \tag{9}$$

Hint: using the commutation relations

$$\theta(b)^{\mp 1}g(a)\theta(b)^{\pm 1} = g(q^{\pm 1}ab^{\mp 1}),$$
$$\theta(a)^{\mp 1}g(b)\theta(a)^{\pm 1} = g(q^{\mp 1}a^{\pm 1}b),$$

to reduce a proof of (9) to that of (6).

*iv*) Using the functional equation $\quad (z+a)r(z,qa) = (1+za)r(z,q^{-1}a), \quad$ check that

$$\mathrm{LHS}(9)(az^{-1}+qba+z'b) = (z'a+qba+bz^{-1})\mathrm{LHS}(9),$$
$$\mathrm{RHS}(9)(az^{-1}+qba+z'b) = (z'a+qba+bz^{-1})\mathrm{RHS}(9).$$

Deduce from these relations that

$$\mathrm{LHS}(9) = \mathrm{RHS}(9).$$

**B.** (Miscellany).

**7.** Let $W_q$ be the algebra over the field of rational functions $\mathbf{C}(q)$ generated by $a$ and $b$ subject to the relation $ab=q^2ba$.

Prove that the maps $\quad \Delta\ (\widetilde{\Delta}):\ W_q \to W_q\otimes W_q$

$$\Delta(a) = 1\otimes a + a\otimes b, \quad (\widetilde{\Delta}(a) = a\otimes a),$$
$$\Delta(b) = b\otimes b, \quad (\widetilde{\Delta}(b) = 1\otimes b + b\otimes a)$$

define the comultiplications in $W_q$.

**8.** Consider an element $f(a,b)=\sum_{n,m} c_{n,m}(q)b^n a^m$ from the Weyl algebra $W_{q^{\frac{1}{2}}}$ and assume that there exists the limit

$$\lim_{q\to 1}(1-q)f(a,b)=\widetilde{f}(\widetilde{a},\widetilde{b}),$$

where $\widetilde{a}$ and $\widetilde{b}$ commutes.

Prove that

*i)* $\displaystyle \lim_{q\to 1}(1-q)\,[L(a),f(a,b)]=\log(1-\widetilde{a})\left(\widetilde{b}\frac{\partial}{\partial\widetilde{b}}\right)\widetilde{f}(\widetilde{a},\widetilde{b}),$

$$\lim_{q\to 1}(1-q)\,[f(a,b),L(b)]=\log(1-\widetilde{b})\left(\widetilde{a}\frac{\partial}{\partial\widetilde{a}}\right)\widetilde{f}(\widetilde{a},\widetilde{b}).$$

*ii)* $\displaystyle \lim_{q\to 1}(1-q)\sum_{n\geq 0}\frac{1}{n!}\mathrm{ad}^n_{L(a)}\left(f(a,b)\right)=\widetilde{f}(\widetilde{a},\widetilde{b}-\widetilde{a}\widetilde{b}),$

where $\mathrm{ad}^0_y(x)=x$ and $\mathrm{ad}^{n+1}_y(x)=[y,\mathrm{ad}^n_y(x)]$, $n\geq 0$.

Hint: use the Taylor formula $(xy=yx)$

$$\exp(\left(yx\frac{\partial}{\partial x}\right)f(x)=f\left(x\exp(y)\right).$$

**9** (Quantum polylogarithm). Let us define the quantum polylogarithm as the following formal series $(k\geq 1)$

$$Li_k(x;q):=\sum_{n\geq 1}\frac{x^n}{n(1-q^n)^{k-1}}. \tag{10}$$

- Prove that $(k\geq 2)$

$$\exp\left(Li_k(x;q)\right)=\prod_{m=(m_1,\ldots,m_{k-2})\in\mathbf{Z}_+^{k-2}} e_q(q^{|m|}x), \tag{11}$$

where $|m|:=m_1+\ldots+m_{k-2}$.

Hint: it is clear that $Li_k(x;q)-Li_k(qx;q)=Li_{k-1}(x;q)$. Consequently,

$$Li_k(x;q)=\sum_{m\geq 0}Li_{k-1}(q^m x;q),$$
$$\exp\left(Li_k(x;q)\right)=\prod_{m\geq 0}\exp\left(Li_{k-1}(q^m x;q)\right).$$

- (Functional equation for quantum trilogarithm). Prove that if $x$ and $y$ satisfy the Weyl relation $xy=qyx$, then

$$\begin{aligned}\exp\left(Li_3(x;q)\right)\exp\left(Li_3(y;q)\right)&=\exp\left(Li_3(y;q)\right)\exp\left(Li_3((y)_\infty x;q)\right)\\&=\exp\left(Li_3(y(x)_\infty;q)\right)\exp\left(Li_3(x;q)\right).\end{aligned} \tag{12}$$

It is a very intertesting task to study the asymptotic behavior of the quantum polylogarithm functional equations (see e.g. (2.64), (2.65) and (9), (11), (12)) when $q = \zeta e^{-\epsilon} \to 1$, $\epsilon \to 0$, where $\zeta = \exp\left(\frac{2\pi i}{l}\right)$. We intended to consider this interesting subject in [Kir10].

**Appendix.**

Proof of Proposition E, $n = 2$, $A = 1$. We have the following series of relations coming from the functional equation (1.4):

- $L(\beta_1/\alpha_2) + L(\alpha_2 x_1) - L(\beta_1 x_1) - L\left(\frac{\beta_1(1-\alpha_2 x_1)}{\alpha_2(1-\beta_1 x_1)}\right) - L\left(\frac{(\alpha_2-\beta_1)x_1}{1-\beta_1 x_1}\right) = 0,$
- $L(\beta_1/\alpha_1) + L(\alpha_1 x_2) - L(\beta_1 x_2) - L\left(\frac{\beta_1(1-\alpha_1 x_2)}{\alpha_1(1-\beta_1 x_2)}\right) - L\left(\frac{(\alpha_1-\beta_1)x_2}{1-\beta_1 x_2}\right) = 0,$
- $L(\beta_2/\alpha_1) + L(\alpha_1 x_1) - L(\beta_2 x_1) - L\left(\frac{\beta_2(1-\alpha_1 x_1)}{\alpha_1(1-\beta_2 x_1)}\right) - L\left(\frac{(\alpha_1-\beta_2)x_1}{1-\beta_2 x_1}\right) = 0,$
- $L(\beta_2/\alpha_2) + L(\alpha_2 x_2) - L(\beta_2 x_2) - L\left(\frac{\beta_2(1-\alpha_2 x_2)}{\alpha_2(1-\beta_2 x_2)}\right) - L\left(\frac{(\alpha_2-\beta_2)x_2}{1-\beta_2 x_2}\right) = 0,$
- $L\left(\frac{(\alpha_1-\beta_1)x_2}{1-\beta_1 x_2}\right) + L\left(\frac{(\alpha_2-\beta_1)x_1}{1-\beta_1 x_1}\right) - L(t) - L\left(\frac{x_1(\alpha_2-\beta_1)(1-\beta_2 x_2)}{(1-\beta_1 x_1)(1-\alpha_2 x_2)}\right)$
  $- L\left(\frac{x_2(\alpha_1-\beta_1)(1-\beta_2 x_1)}{(1-\beta_1 x_2)(1-\alpha_1 x_1)}\right) = 0,$
- $L\left(\frac{(\alpha_1-\beta_2)x_1}{1-\beta_2 x_1}\right) + L\left(\frac{(\alpha_2-\beta_2)x_2}{1-\beta_2 x_2}\right) - L(t) - L\left(\frac{x_1(\alpha_1-\beta_2)(1-\beta_1 x_2)}{(1-\beta_2 x_1)(1-\alpha_1 x_2)}\right)$
  $- L\left(\frac{x_2(\alpha_2-\beta_2)(1-\beta_1 x_1)}{(1-\beta_2 x_2)(1-\alpha_2 x_1)}\right) = 0,$
- $L\left(\frac{\beta_1(1-\alpha_2 x_1)}{\alpha_2(1-\beta_1 x_1)}\right) + L\left(\frac{\beta_2(1-\alpha_1 x_1)}{\alpha_1(1-\beta_2 x_1)}\right) - L\left(\frac{\beta_1\beta_2(1-t)}{\alpha_1\alpha_2}\right)$
  $- L\left(\frac{\beta_1(1-\alpha_2 x_1)(1-\beta_2 x_2)}{(1-\beta_1 x_1)(\alpha_2-\beta_2)}\right) - L\left(\frac{\beta_2(1-\alpha_1 x_1)(1-\beta_1 x_2)}{(1-\beta_2 x_1)(\alpha_1-\beta_1)}\right) = 0,$
- $L\left(\frac{\beta_1(1-\alpha_1 x_2)}{\alpha_1(1-\beta_1 x_2)}\right) + L\left(\frac{\beta_2(1-\alpha_2 x_2)}{\alpha_2(1-\beta_2 x_2)}\right) - L\left(\frac{\beta_1\beta_2(1-t)}{\alpha_1\alpha_2}\right)$
  $- L\left(\frac{\beta_1(1-\alpha_1 x_2)(1-\beta_2 x_1)}{(1-\beta_1 x_2)(\alpha_1-\beta_2)}\right) - L\left(\frac{\beta_2(1-\alpha_2 x_2)(1-\beta_1 x_1)}{(1-\beta_2 x_2)(\alpha_2-\beta_1)}\right) = 0,$
- $L\left(\frac{(\alpha_2-\beta_2)x_2(1-\beta_1 x_1)}{(1-\alpha_2 x_1)(1-\beta_2 x_2)}\right) + L\left(\frac{\beta_1(1-\alpha_2 x_1)(1-\beta_2 x_2)}{(\alpha_2-\beta_2)(1-\beta_1 x_1)}\right) - L(x_2\beta_1)$
  $- L\left(\frac{-\beta_1/\alpha_1}{1-\beta_1/\alpha_1}\right) - L\left(\frac{-\alpha_1 x_2}{1-\alpha_1 x_2}\right) = 0,$
- $L\left(\frac{(\alpha_1-\beta_1)x_2(1-\beta_2 x_1)}{(1-\alpha_1 x_1)(1-\beta_1 x_2)}\right) + L\left(\frac{\beta_2(1-\alpha_1 x_1)(1-\beta_1 x_2)}{(\alpha_1-\beta_1)(1-\beta_2 x_1)}\right) - L(x_2\beta_2)$

$$- L\left(\frac{-\beta_2/\alpha_2}{1-\beta_2/\alpha_2}\right) - L\left(\frac{-\alpha_2 x_2}{1-\alpha_2 x_2}\right) = 0,$$

- $$L\left(\frac{(\alpha_2-\beta_1)x_1(1-\beta_2 x_2)}{(1-\alpha_2 x_2)(1-\beta_1 x_1)}\right) + L\left(\frac{\beta_2(1-\alpha_2 x_2)(1-\beta_1 x_1)}{(\alpha_2-\beta_1)(1-\beta_2 x_2)}\right) - L(x_1\beta_2) - L\left(\frac{-\beta_2/\alpha_1}{1-\beta_2/\alpha_1}\right) - L\left(\frac{-\alpha_1 x_1}{1-\alpha_1 x_1}\right) = 0,$$
- $$L\left(\frac{(\alpha_1-\beta_2)x_1(1-\beta_1 x_2)}{(1-\alpha_1 x_2)(1-\beta_2 x_1)}\right) + L\left(\frac{\beta_1(1-\alpha_1 x_2)(1-\beta_2 x_1)}{(\alpha_1-\beta_2)(1-\beta_1 x_2)}\right) - L(x_1\beta_1) - L\left(\frac{-\beta_1/\alpha_2}{1-\beta_1/\alpha_2}\right) - L\left(\frac{-\alpha_2 x_1}{1-\alpha_2 x_1}\right) = 0.$$

Let us denote by $S$ the LHS of the sum of all of these relations. Let $D$ be a difference between the LHS and the RHS of the identity from Proposition E. Then one can check

$$S = -2D - \sum_{i,j}\left\{L(\alpha_i x_j) + L\left(\frac{-\alpha_i x_j}{1-\alpha_i x_j}\right) - 2L(\alpha_i/\alpha_j)\right\} + 2L(1) = 0.$$

But it follows from the integral representation (1.2) that a function

$$l(t) := \sum_{i,j}\left\{L(\alpha_i x_j(t)) + L\left(\frac{-\alpha_i x_j(t)}{1-\alpha_i x_j(t)}\right)\right\}$$

does not depend on $t$. Taking $t=1$ one can find this constant. The result is

$$l(t) = 2\sum L(\alpha_i/\alpha_j) + 2L(1),$$

consequently, $D = 0$.

Note that the next Lemma and identities (A1)-(A2) had played the key role in the proving of Proposition E.

**Lemma A1.** *Let $r(x,t)$ be a polynomial as in Proposition E. Then we have*

*i)* $\left(1 - \frac{A\beta_1\dots\beta_n}{\alpha_1\dots\alpha_n}(1-t)\right)\prod_{l=1}^{n}(1-\beta x_l) = \prod_{l=1}^{n}\left(1-\frac{\beta}{\alpha_l}\right)$, *where $\beta = \beta_k$ for some $k$, $1 \le k \le n$,*

*ii)* $\left(1 - \frac{A\beta_1\dots\beta_n}{\alpha_1\dots\alpha_n}(1-t)\right)\prod_{l=1}^{n}(1-\alpha x_l) = -(1-t)\prod_{l=1}^{n}\left(1-\frac{\beta_l}{\alpha}\right)$, *where $\alpha = \alpha_k$ for some $k$, $1 \le k \le n$.*

**Corollary** (of Lemma A1, with $n=2$). *We have*

$$1 - \frac{(\alpha_1-\beta_1)x_2(1-\beta_2 x_1)}{(1-\beta_1 x_2)(1-\alpha_1 x_1)} = -\frac{(\alpha_1-\beta_1)(\alpha_1-\beta_2)}{(1-\beta_1 x_2)(1-\alpha_1 x_1)(\alpha_1\alpha_2-\beta_1\beta_2(1-t))}, \qquad (A1)$$

$$1 - \frac{\beta_1(1-\beta_2 x_2)(1-\alpha_2 x_1)}{(\alpha_2-\beta_2)(1-\beta_1 x_1)} = \frac{\alpha_1(\alpha_2-\beta_1)}{(1-\beta_1 x_1)(\alpha_1\alpha_2-\beta_1\beta_2(1-t))}. \qquad (A2)$$

■

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