Trigonometrical identities and geometrical inequalities for links and knots *

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Abstract

In the present paper links and knots are investigated as a singular set of geometric cone–manifolds with the three-sphere as underlying space. Trigonometrical identities between lengths of singular components and cone angles of these cone–manifolds (Sine, Cosine, and Tangent rules) are obtained. Geometrical inequalities between volumes and singular geodesic lengths of the cone–manifolds are also given. They can be considered as a sort of isoperimetric inequalities well-known for convex polyhedra.

Keywords: *hyperbolic orbifold, hyperbolic cone-manifold, complex length, Tangent Rule, Sine Rule, Cosine Rule, isoperimetric inequalities*

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0. Introduction

Knot theory was born around the year 1867 in Scotland from the imagination of three phisicists: J. C. Maxwell, P. G. Tait, and W. Thomson (Lord Kelvin). For more details see [Kn], [HKW]. Maxwell's interest for knots came from his theory of electromagnetism. For instance, he gave in [Ma] an important interpretation of Gauss integral formula for the linking coefficient of two knots in the 3-space: it is equal to the work required to move a magnet pole along one knot while the other knot is run by an electric current. Another curious fact is that Seifert surface whose boundary is a given knot being introduced by Tait via pure phisical arguments. Due to afforts of J. B. Listing, K. Reidemeister, and M. Dehn knot theory was gradually embodied in the more general theory of 3-dimensional manifolds. The notion of the fundamental group was introduced and the group theory became one of the most powerful tools in the knot theory. In 1975 R. Riley [R] had found examples of hyperbolic structures on some knot and link complements in the three-sphere. Later, in the spring of 1977 W. P. Thurston had announced an existence

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theorem for Riemannian metrics of constant negative curvature on 3-manifolds. In particular, it turned out that knot complement of a simple knot (excepting torical and satellite) admits a hyperbolic structure. This fact allowed to consider knot theory as a part of geometry and Kleinian group theory. Starting from Alexander's works polynomial invariants became a convenient insrument for knot investigation. A lot of different kinds of such polynomials were investigated in the last twenty years. Among these there are Jones-, Kaufmann-, HOMFLY-, A–polynomials and others ([Kauf], [CCGLS], [HLM2]). This relates the knot theory with algebra and algebraic geometry.

In the present paper we investigate knots and links as singular subsets of the 3-sphere endowed by Riemannian metric of constant curvature (negative, positive, or zero). More precisely, our aim is to investigate the structure of geometrical cone–manifolds whose underlying space is the three-sphere and the singular set is a given knot or link.

Section 1 contains a list of trigonometrical identities (Sine, Cosine, and Tangent rules) relating the lengths of singular geodesics of geometrical cone-manifolds with their cone–angles. Cone-manifolds are supposed to be hyperbolic, spherical, or Euclidean. Similar results are known for the right–angled hexagons in the hyperbolic 3-space which can be considered as triangles with complex lengths and angles [Fench]. Related results can be also obtained for a class of knotted graphs. For example, they take place for the rational knots with bridges through their tunnels.

Section 2 is devoted to explicite calculation of volume of some cone-manifolds in hyperbolic and spherical geometries. In particular, simple volume formulas will be obtained for the figure-eight cone–manifold. Partially, these results are well-known and were given earlier in [HLM3], [MV], and [Kj].

Section 3 gives inequalities between volumes and singular geodesic lengths of the cone–manifolds under investigation. They can be consider as a sort of isoperimetric inequalities well-known for convex polyhedra [BZ].

1. Trigonometrical identities for knots and links

1.1 Cone–manifolds, complex distances and lengths

We start with the definition of cone–manifold modeled in hyperbolic, spherical or Euclidian structure.

Definition 1.1.1. A 3–dimensional *hyperbolic cone–manifold* is a Riemannian 3–dimensional manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics. To each component of singular set we associate a real number $n \geq 1$ such that the cone-angle around the component is $\alpha = 2\pi/n$. The concept of the hyperbolic cone-manifold generalizes the hyperbolic manifold which appears in the partial case when all cone-angles are 2π. The hyperbolic cone-manifold is also a generalization of the hyperbolic 3–orbifold which arises when all associated numbers n are integers. Euclidean and spherical cone–manifolds are defined similarly.

In the present paper hyperbolic, spherical or Euclidean cone-manifolds C are considered whose underlying space is the three-dimensional sphere and the singular set $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ is a link consisting of components $\Sigma^j = \Sigma_{\alpha_j}$, $j = 1, 2, \ldots, k$ with cone-angles $\alpha_1, \ldots, \alpha_k$ respectively.

Recall a few well-known facts from the hyperbolic geometry.

Let $\mathbb{H}^3 = \{(z, \xi) \in \mathbb{C} \times \mathbb{R} : \xi > 0\}$ be the upper half model of the 3 -dimensional hyperbolic space endowed by the Riemannian metric $ds^2 = \dfrac{dz d\bar{z} + d\xi^2}{\xi^2}$. We identify the group of orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL(2, \mathbb{C})$ consisting of linear fractional transformations

$$A : z \in \mathbb{C} \to \frac{az + b}{cz + d}.$$

By the canonical procedure the linear transformation A can be uniquely extended to the isometry of $\mathbb{H}^3$. We prefer to deal with the matrix $\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ rather than the element $A \in PSL(2, \mathbb{C})$. The matrix $\widetilde{A}$ is uniquely determined by the element A up to a sign. If there will be no confusions we shall use the same letter A for both A and $\widetilde{A}$.

Let C be a hyperbolic cone–manifold with the singular set Σ. Then C defines a nonsingular but incomplete hyperbolic manifold $N = C - \Sigma$. Denote by Φ the fundamental group of the manifold N.

The hyperbolic structure of N defines, up to congugation in $PSL(2, \mathbb{C})$, a holonomy homomorphism

$$\hat{h} : \Phi \to PSL(2, \mathbb{C}).$$

It is shown in [Zhou] that the monodromy homomorphism of a compact orientable cone-orbifold can be lifted to $SL(2, \mathbb{C})$. Denote by $h : \Phi \to SL(2, \mathbb{C})$ this lifting homomorphism. Chose an orientation on the link $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ and fix a meridian-longitude pair $\{m_j, l_j\}$ for each component $\Sigma_j = \Sigma_{\alpha_j}$. Then the matricies $M_j = h(m_j)$ and $L_j = h(l_j)$ satisfy the following properties:

$$\mathrm{tr}(M_j) = 2\cos(\alpha_j/2), \quad M_j L_j = L_j M_j,\ j = 1, 2, \ldots, k.$$

Definition 1.1.2. A *complex length* γ_j of the singular component Σ^j of the cone-manifold C is defined as displacement of the isometry L_j of $\mathbb{H}^3$, where $L_j = h(l_j)$ is represented by the longitude l_j of Σ^j.

Immediately from the definition we get [Fench, p.46]

$$2\cosh\gamma_j = \mathrm{tr}\,(L_j^2) \tag{1.1.1}$$

We note [BZie, p.38] that the meridian-longitude pair $\{m_j, l_j\}$ of the oriented link is uniquely determined up to a common conjugating element of the group Φ. Hence, the complex length $\gamma_j = l_j + i\ \varphi_j$ is uniquely determined up to a sign and $\pmod{2\pi i}$ by the above definition.

We need two conventions to choose correctly real and imaginary parts of γ_j. The first convention is the following. Since Σ^j does not shrink to a point, $l_j \neq 0$. Hence, we choose γ_j in such a way that $l_j = \Re\,\gamma_j > 0$. The second convention is concerned with the imaginary part $\varphi_j = \Im\,\gamma_j$. We want to choose φ_j such that the following identity holds

$$\cosh\frac{\gamma_j}{2} = -\frac{1}{2}\,\mathrm{tr}(L_j) \tag{1.1.2}$$

By virtue of identity $\mathrm{tr}(L_j)^2 - 2 = \mathrm{tr}(L_j^2)$ equality (1.1.1) is a consiquence of (1.1.2). The converse, in general, is true only up to a sign. Under the second convention (1.1.1) and (1.1.2) are equivalent. The two above conventions lead to convenient analytic formulas for calculation of γ_j and l_j. More precisely, there are simlple relations between these numbers and eigenvalues of matrix L_j. Recall that $\det L_j = 1$. Since matrix L_j is loxodromic it has two eigenvalues f_j and $1/f_j$. We choose f_j so that $|f_j| > 1$. The case $|f_j| = 1$ is impossible because in this case the matrix L_j is elliptic and $l_j = 0$. Hence

$$f_j = -e^{\frac{\gamma_j}{2}},\ \ |f_j| = e^{\frac{l_j}{2}}. \tag{1.1.3}$$

1.2. Whitehead link cone–manifold

Denote by $W(\alpha,\beta)$ the cone-manifold whose underlying space is the 3-sphere and whose singular set consists of two components of the Whitehead link with cone angles $\alpha = 2\pi/m$ and $\beta = 2\pi/n$ (see Fig.1). It follows from Thurston's theorem that $W(\alpha,\beta)$ admits a hypebolic structure for all sufficiently small α and β. The region of hyperbolicity of $W(\alpha,\beta)$ was investigated in [HLM2] and [KM]. In particular, this cone–manifold is hyperbolic for $m, n > 2.507..$ The following theorems have been obtained in [M].

Theorem 1.2.1 (The Tangent Rule). *Suppose that cone–manifold $W(\alpha,\beta)$ is hyperbolic. Denote by γ_α and γ_β complex lengths of the singular geodesics of $W(\alpha,\beta)$ with cone angles α and β respectively. Then*

$$\frac{\tanh\frac{\gamma_\alpha}{4}}{\tanh\frac{\gamma_\beta}{4}} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}.$$

Theorem 1.2.2 (The Sine Rule). *Let $\gamma_\alpha = l_\alpha + i\,\varphi_\alpha$ (resp.γ_β) be a complex length of the singular geodesic of a hyperbolic cone-manifold $W(\alpha,\beta)$ with cone angle α (resp. β). Then*

$$\frac{\sin\frac{\varphi_\alpha}{2}}{\sinh\frac{l_\alpha}{2}} = \frac{\sin\frac{\varphi_\beta}{2}}{\sinh\frac{l_\beta}{2}}.$$

The Whitehead link cone–manifold $W(\alpha,\beta)$. Fig. 1

Euclidean analoges of Theorems 1.2.1 and 1.2.2 were abtained by R.N. Shmatkov (1999). The similar results can be stated also for spherical cone–manifold $W(\alpha,\beta)$.

1.3. Cone–manifold $6_2^2(\alpha,\beta)$

According to [Rolf] denote by 6_2^2 two-bridge link with the ratinal slope $10/3$. Consider a cone–manifold $6_2^2(\alpha,\beta)$ whose underlying space is the three-sphere and singular set is formed by two components of the link $6_2^2(\alpha,\beta)$ with cone angles α and β (Fig.2). A canonical fundamental set for two-bridge cone–manifolds admitting hyperbolic, Euclidean, or spherical structure has been constructed in [MR1]. Applying this construction to the cone–manifold under consideration we get the following proposition.

Proposition 1.3.1 *The cone manifold $6_2^2(\alpha,\beta)$ admits a hyperbolic, Euclidean, or spherical structure in regions R_h, R_e and R_s respectively, where*

$$(i) \quad R_h = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 \le \alpha,\beta < 4\pi/3, \cos\alpha/2 + \cos\beta/2 > 1/2\}$$

$$(ii) \quad R_e = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 \le \alpha,\beta < 4\pi/3, \cos\alpha/2 + \cos\beta/2 = 1/2\}$$

$$(iii) \; R_h = \{(\alpha,\beta) \in \mathbb{R}^2 : 2\pi/3 < \alpha,\beta < 4\pi/3, |\cos\alpha/2 \pm \cos\beta/2| < 1/2\,\}$$

We recall that in the case $\alpha = 0$ or $\beta = 0$ the corresponding component of the cone–manifold $6_2^2(\alpha,\beta)$ should be replace by complete hyperbolic cusp.

The cone–manifold $6_2^2(\alpha,\beta)$. Fig. 2

The next proposition gives an explicit formula for a real lenght of the singular component of the cone–manifold $6_2^2(\alpha,\beta)$.

Proposition 1.3.2. *Suppose that cone–manifold $6_2^2(\alpha,\beta), (\alpha,\beta) \in R_h$ is hyperbolic. Then the length l_α of the singular geodesic of $6_2^2(\alpha,\beta)$ with cone angle α is defined by the formula*

$$\cosh\frac{l_\alpha}{2} = 2\frac{N + M\sqrt{1+K^2}}{(1+M^2)\sqrt{1+N^2}},$$

where $M = \cot\frac{\alpha}{2}, N = \cot\frac{\beta}{2},$ *and* $K^2 = (1+M^2)(1+N^2)/4.$

Proof is based on the following considerations.

Let $\Sigma = \Sigma^1 \cup \Sigma^2$ be the singular set of cone–manifold $C = 6_2^2(\alpha,\beta)$. Consider a nonsingular noncomplete hyperbolic manifold $C - \Sigma$ and denote by $\Phi = \pi_1(C-\Sigma)$ its fundamental group. Then Φ has the following presentation

$$\Phi = < s_\alpha, s_\beta : s_\alpha l_\alpha = l_\alpha s_\alpha,\, l_\alpha = s_\beta s_\alpha s_\beta {s_\alpha}^{-1} {s_\beta}^{-1} {s_\alpha}^{-1} s_\beta s_\alpha s_\beta >,$$

where s_α and s_β are meridians of the components Σ_α and Σ_β respectively, and l_α is a longitude of Σ_α. Let $h : \Phi \to SL(2,\mathbb{C})$ be a holonomy homomorphism, $S = h(s_\alpha)$, and $T = h(l_\beta.)$ After a suitable conjugation in the group $SL(2,\mathbb{C})$ homomorphism h can be chosen in such a way that

$$S = \begin{pmatrix} \cos\mu & i\,e^{\frac{\rho}{2}}\sin\mu \\ i\,e^{-\frac{\rho}{2}}\sin\mu & \cos\mu \end{pmatrix}, \quad T = \begin{pmatrix} \cos\nu & i\,e^{-\frac{\rho}{2}}\sin\nu \\ i\,e^{\frac{\rho}{2}}\sin\nu & \cos\nu \end{pmatrix}, \qquad (1.3.1)$$

where $\mu = \alpha/2$, $\nu = \beta/2$, ρ is a complex distance between axis of S and T in the hyperbolic space $\mathbb{H}^3$.

Set $L = h(l_\alpha)$. Then the matrix equation $SL = LS$ and a restriction that L is loxodromic for all $\alpha > 0$ gives the following quadratic equation for $u = \cosh \rho$

$$u^2 - (MN + \sqrt{1+K^2})u + K^2 + MN\sqrt{1+K^2} = 0, \tag{1.3.2}$$

where M, N and K are the same as in the statement of the proposition.

Routine calculation of the trace of the element $L = TSTS^{-1}T^{-1}S^{-1}TST$ modulo equaition (1.3.2) gives

$$\mathrm{tr}(L) = 4\frac{N + M\sqrt{1+K^2}}{(1+M^2)\sqrt{1+N^2}}. \tag{1.3.3}$$

Inside of region of hyperbolicity R_h we have $\mathrm{tr}(L) > 2$. From(1.1.1) the complex length $l_\alpha = l_\alpha + i\varphi_\alpha$ is given by $\cosh \frac{\gamma_\alpha}{2} = -\frac{1}{2}\,\mathrm{tr}(L)$. Hence $\cosh \frac{\gamma_\alpha}{2} < -1$, $\frac{\gamma_\alpha}{2} = \frac{l_\alpha}{2} + i\pi$ and

$$\cosh \frac{l_\alpha}{2} = \frac{1}{2}\,\mathrm{tr}(L) = 2\frac{N + M\sqrt{1+K^2}}{(1+M^2)\sqrt{1+N^2}}.$$

Theorem 1.3.3 (The Sine Rule). *Let l_α and l_β be lengths of the singular geodesics of a hyperbolic cone–manifold $6_2^2(\alpha,\beta)$ with cone angles α and β respectively. Then*

$$\frac{\sin \frac{\alpha}{2}}{\sinh \frac{l_\alpha}{2}} = \frac{\sin \frac{\beta}{2}}{\sinh \frac{l_\beta}{2}}.$$

Proof. Immediately from Proposion 1.3.2 we get

$$\sinh \frac{l_\alpha}{2} = \frac{\sqrt{\Delta}}{(1+M^2)\sqrt{1+N^2}} = \sin\mu \frac{\sqrt{\Delta}}{\sqrt{(1+M^2)(1+N^2)}},$$

where $\Delta = 4M^2 + 4N^2 + 4MM\sqrt{4 + (1+M^2)(1+N^2)} - (1+M^2)(1+N^2)$ is a symmetric function of M and N. Hence

$$\frac{\sinh \frac{l_\alpha}{2}}{\sin\mu} = \frac{\sqrt{\Delta}}{\sqrt{(1+M^2)(1+N^2)}} = \frac{\sinh \frac{l_\beta}{2}}{\sin\nu}$$

and the result follows.

Theorem 1.3.4 (The Cosine Rule). *Let l_α and l_β be lengths of the singular geodesics of a hyperbolic cone–manifold $6_2^2(\alpha,\beta)$ with cone angles α and β respectively. Then*

$$\frac{\cos \frac{\alpha}{2} \cosh \frac{l_\beta}{2} - \cos \frac{\beta}{2} \cosh \frac{l_\alpha}{2}}{\cos\alpha - \cos\beta} = -1.$$

Proof. By Proposition 1.3.2 we get

$$\cosh\frac{l_\alpha}{2}=2\frac{N+M\sqrt{1+K^2}}{(1+M^2)\sqrt{1+N^2}}$$

and

$$\cosh\frac{l_\beta}{2}=2\frac{M+N\sqrt{1+K^2}}{(1+N^2)\sqrt{1+M^2}}.$$

Hence

$$\frac{M}{\sqrt{1+M^2}}\cosh\frac{l_\beta}{2}-\frac{N}{\sqrt{1+N^2}}\cosh\frac{l_\alpha}{2}=2\frac{M^2-N^2}{(1+M^2)(1+N^2)}=\cos\beta-\cos\alpha.$$

Since $\frac{M}{\sqrt{1+M^2}}=\cos\frac{\alpha}{2}$ and $\frac{N}{\sqrt{1+N^2}}=\cos\frac{\beta}{2}$ the theorem is proved.

1.4 The Borromean cone–manifold

In this subsection we investigate geometric properties of a cone–manifold $B(\alpha,\beta,\gamma)$ with singular set the Borromean rings (Fig. 3). The cone angles of three components of the singular set are α,β,γ. As above, the corresponding lengths of the singular set components will be denoted by l_α, l_β, and l_γ.

The Borromean cone–manifold $B(\alpha,\beta,\gamma)$. Fig. 3

It is well-known fact that $B(\alpha,\beta,\gamma)$ can be obtained by glueing together eight copies of the Lambert cube $Q(\alpha/2,\beta/2,\gamma/2)$ with essential dihedral angles $\alpha/2,\beta/2,\gamma/2$. See [T] and [HLM1] for details. In particular, it is shown in [T] that $Q(\alpha/2,\beta/2,\gamma/2)$ (and hence $B(\alpha,\beta,\gamma)$) is hyperbolic if $0\le\alpha,\beta,\gamma<\pi$ and Euclidean if $\alpha/2=\beta/2=\gamma/2=\pi$.

Moreover, if $L_\alpha, L_\beta, L_\gamma$ denote the edge lengths of $Q(\alpha/2,\beta/2,\gamma/2)$ with dihedral angles $\alpha/2,\beta/2,\gamma/2$ we get

$$L_\alpha=\frac{l_\alpha}{4},\quad L_\beta=\frac{l_\beta}{4},\quad L_\gamma=\frac{l_\gamma}{4}. \qquad (1.4.1)$$

The Lambert cube $Q(\alpha/2,\beta/2,\gamma/2)$. Fig. 4

As in the case of cone–manilolds $W(\alpha,\beta)$ and $6_2^2(\alpha,\beta)$ there are simple trigonometrical identities relating the lengths $l_\alpha, l_\beta, l_\gamma$ of $B(\alpha,\beta,\gamma)$ with its cone angles α,β,γ. We start with the following

Theorem 1.4.1 (The Tangent Rule). *Let $B(\alpha,\beta,\gamma)$, be a hyperbolic Borromean cone–manifold with cone angles $0<\alpha,\beta,\gamma<\pi$ and the singular geodesic lengths $l_\alpha, l_\beta, l_\gamma$. Then*

$$\frac{\tan\frac{\alpha}{2}}{\tanh\frac{l_\alpha}{4}}=\frac{\tan\frac{\beta}{2}}{\tanh\frac{l_\beta}{4}}=\frac{\tan\frac{\gamma}{2}}{\tanh\frac{l_\gamma}{4}}=T,$$

where T is a positive number defined by $T^2 = K + \sqrt{K^2 + L^2M^2N^2}$, $L = \tan\frac{\alpha}{2}$, $M = \tan\frac{\beta}{2}$, $N = \tan\frac{\gamma}{2}$, *and* $K = (L^2 + M^2 + N^2 + 1)/2$.

Proof. We prefer to deal with the Lambert cube $Q(\alpha/2, \beta/2, \gamma/2)$ rather then cone–manifold $B(\alpha, \beta, \gamma)$. It follows from the result of [K] that the edge lengths L_α, L_β and L_γ are related with its angles by

$$\frac{\tan\frac{\alpha}{2}}{\tanh L_\alpha} = \frac{\tan\frac{\beta}{2}}{\tanh L_\beta} = \frac{\tan\frac{\gamma}{2}}{\tanh L_\gamma} = T, \tag{1.4.2}$$

where $T = \tan\theta$ for some angle θ such that $\alpha, \beta, \gamma \leq 2\theta \leq \pi$. The simple proof of this formula by means of Gram matrix techniques can be find also in [V]. The following equation for T was obtained in ([K], p.564, eq. (II)) and ([HLM1], eq. (A.2)) in slightly different terms

$$T^2 = \frac{T^2 - L^2}{1 + L^2}\frac{T^2 - M^2}{1 + M^2}\frac{T^2 - N^2}{1 + N^2}, \tag{1.4.3}$$

The last equation is equivalent to

$$(T^2 + 1)(T^4 - (L^2 + M^2 + N^2 + 1)T^2 - L^2M^2N^2) = 0.$$

Since T is a positive number we get

$$T^4 - (L^2 + M^2 + N^2 + 1)T^2 - L^2M^2N^2 = 0. \tag{1.4.4}$$

Hence $T^2 = K + \sqrt{K^2 + L^2M^2N^2}$, and $K = (L^2 + M^2 + N^2 + 1)/2$. Taking into acount (1.4.1) and (1.4.2) we finish the proof.

The next three theorems can be considered as a consequences of the Tangent Rule.

Theorem 1.4.2 (The Sine Rule). *Let $B(\alpha, \beta, \gamma)$, be a hyperbolic Borromean cone–manifold with cone angles $0 < \alpha, \beta, \gamma < \pi$ and the singular geodesic lengths $l_\alpha, l_\beta, l_\gamma$. Then*

$$\frac{\sin\frac{\alpha}{2}}{\sinh\frac{l_\alpha}{4}}\frac{\sin\frac{\beta}{2}}{\sinh\frac{l_\beta}{4}}\frac{\sin\frac{\gamma}{2}}{\sinh\frac{l_\gamma}{4}} = T,$$

where T is a positive number defined by $T^2 = K + \sqrt{K^2 + L^2M^2N^2}$, $L = \tan\frac{\alpha}{2}$, $M = \tan\frac{\beta}{2}$, $N = \tan\frac{\gamma}{2}$, *and* $K = (L^2 + M^2 + N^2 + 1)/2$.

Proof. We rewrite the statement of the Tangent Rule in the form

$$\sinh^2 L_\alpha = \frac{L^2}{T^2 - L^2}, \ \sinh^2 L_\beta = \frac{M^2}{T^2 - M^2}, \ \sinh^2 L_\gamma = \frac{N^2}{T^2 - N^2}, \tag{1.4.5}$$

We get also

$$\sin^2\frac{\alpha}{2} = \frac{L^2}{1 + L^2}, \ \ \sin^2\frac{\beta}{2} = \frac{M^2}{1 + M^2}, \ \ \sin^2\frac{\gamma}{2} = \frac{N^2}{1 + N^2}. \tag{1.4.6}$$

By virtue of (1.4.3) we have from (1.4.5) and (1.4.6)

$$\frac{\sin^2 \frac{\alpha}{2}}{\sinh^2 L_\alpha} \frac{\sin^2 \frac{\beta}{2}}{\sinh^2 L_\beta} \frac{\sin^2 \frac{\gamma}{2}}{\sinh^2 L_\gamma} = \frac{T^2 - L^2}{1 + L^2} \frac{T^2 - M^2}{1 + M^2} \frac{T^2 - N^2}{1 + N^2} = T^2.$$

By taking the square root we obtain the statement of the theorem.

By similar arguments the following theorems can be proved.

Theorem 1.4.3 (The Cosine Rule).

$$\frac{\cos \frac{\alpha}{2}}{\cosh \frac{l_\alpha}{4}} \frac{\cos \frac{\beta}{2}}{\cosh \frac{l_\beta}{4}} \frac{\cos \frac{\gamma}{2}}{\cosh \frac{l_\gamma}{4}} = \frac{1}{T^2},$$

Theorem 1.4.4 (The Sine-Cosine Rule).

$$\frac{\sin \frac{\alpha}{2}}{\sinh \frac{l_\alpha}{4}} \frac{\sin \frac{\beta}{2}}{\sinh \frac{l_\beta}{4}} \frac{\cos \frac{\gamma}{2}}{\cosh \frac{l_\gamma}{4}} = 1,$$

2. Explicit volume calculation

2.1. The Schläfli formula

In this section we will obtain explicit formulas for volume of some special cone–manifolds in the hyperbolic and spherical geometries. In the case of complete hyperbolic structure on the simplest knot and link complements such formulas in terms of Lobachevsky function are well-known and widely represented in [T]. In general situation, a hyperbolic cone–manifold can be obtained by completion of non-complete hyperbolic structure on a suitable knot or link complenent. If the cone–manifold is compact explicit formulas are know just in a few cases [Hds] , [HLM3], [MV], [Kj]. In all these cases the starting point for the volume calculation is the Schläfli formula (see, for example [Hds])

Theorem 2.1.1. (The Schläfli volume formula) *Suppose that C_t is a smooth 1–parameter family of (curvature K) cone–manifold structures on a n-manifold, with singular locus Σ of a fixed topological type. Then the derivative of volume of C_t satisfies*

$$(n-1)K dV(C_t) = \sum_\sigma V_{n-2}(\sigma) d\theta(\sigma)$$

where the sum is over all components σ of the singular locus Σ, and $\theta(\sigma)$ is the cone angle along σ.

In the present paper we will deal mostly with three-dimensional cone–manifold structures of negative constant curvature $K = -1$. The Schläfli formula in this case reduces to

$$dV = -\frac{1}{2} \sum_i l_{\theta_i} d\theta_i, \qquad (2.1.1)$$

where the sum is taken over all components of the singular set Σ with lengths l_{θ_i} and cone angles θ_i.

Our aim is to obtain the volume formulas for cone–manifolds $W(\alpha,\beta), 6_2^2(\alpha,\beta)$ and $B(\alpha,\beta,\gamma)$ described in the above section. Since the figure eight cone–manifold $4_1(\alpha)$ is the two-fold covering of $6_2^2(\alpha,\pi)$ its volume is twice the volume of $6_2^2(\alpha,\pi)$. This leads to a simple volume formula for the figure eight cone–manifold obtained earlier in more complicated form in [HLM3], [MV] and [Kj].

2.2. Volume of the Whitehead link cone–manifold

First of all we consider the case of the hyperbolic Whitehead link with one complete cusp.

Theoren 2.2.2. Let $W(0,\alpha)$ be a hyperbolic Whitehead link cone–manifold with a complete hyperbolic structure on one cusp and cone angle α, $0 \le \alpha < \pi$ on the another. Then the volume of $W(0,\alpha)$ is given by the formula

$$Vol\, W(0,\alpha) = \frac{1}{2}\int_{\alpha}^{\pi} arcosh\,(8 - 8\cos t + \cos 2t)dt.$$

Proof. By [KM] cone–manifold $W(0,\alpha)$ is hyperbolic for all $0 \le \alpha < \pi$. Denote by V_α the hyperbolic volume of $W(0,\alpha)$. By Schläfli formula we have $dV_\alpha = -\frac{1}{2}l_\alpha d\alpha$. By calculation produced in [M] we obtain

$$\cosh l_\alpha = \frac{M^4 + 10M^2 + 17}{(M^2+1)^2}, \qquad (2.2.1)$$

where $M = \cot\frac{\alpha}{2}$. Simplifying (2.2.1) we get $\cosh l_\alpha = 8 - 8\cos\alpha + \cos 2\alpha$ and

$$l_\alpha = arcosh\,(8 - 8\cos\alpha + \cos 2\alpha). \qquad (2.2.2)$$

By integrating the Schläfli formula we have

$$V_\alpha = -\frac{1}{2}\int_{\theta}^{\alpha} arcosh\,(8 - 8\cos t + \cos 2t)dt + V_\theta, \qquad (2.2.3)$$

for an arbitrary θ, $0 \le \theta < \pi$. We note that the geometrical limit $W(0,\pi)$ of the cone–manifolds $W(0,\theta)$ as $\theta \to \pi - 0$ is not hyperbolic, since its two-fold covering branched over the π-component is the torus link 4/1. Also, $W(0,\pi)$ contains no two-dimensional suborbilolds of the type $S^2(\pi,\pi,\pi)$. Hence, by Theorem 7.1.2 of [Kj] we have $\lim_{\theta\to\pi-0} V_\theta = 0$. Going over to the limit we immediately get from (2.2.3) the statement of the theorem.

In the case of closed cone–manifold $W(\alpha,\beta)$ the volume function becomes more complicated and can be expressed in terms of roots of a cubic equation. See [M] and [KM] for details.

2.3. Volumes of the $6_2^2(\alpha,\beta)$, the figure eight, and the Borromean rings cone–manifolds

This subsection will be organized in the following way. First of all, by making use of length formula for a singular geodesic of the cone–manifold $6_2^2(\alpha,\beta)$

from section 1 and the Schläfli variation formula we obtain a simple expression for $Vol\, 6_2^2(\alpha,\beta)$. Then taking into account that the figure eight cone–manifold $4_1(\alpha)$ and the Borromean rings cone–manifold $B(\alpha,\alpha,\alpha)$ are, respectively two-fold and three-fold coverings of $6_2^2(\alpha,\beta)$ for $\beta=\pi$ and $\beta=\frac{2\pi}{3}$, we find the volume formulas for both of them. These formulas turn out to be simplier then the corresponding formulas obtained earlier in [HLM3], [MV], and [K].

Theorem 2.3.1. *Suppose that cone–manifold $6_2^2(\alpha,\beta)$, $(\alpha,\beta)\in R_e$ is hyperbolic. Then its volume is defined by the formula*

$$Vol\, 6_2^2(\alpha,\beta)=\int_{\alpha}^{\alpha^*} E(\frac{\alpha}{2},\frac{\beta}{2})d\alpha,$$

where $E(\mu,\nu)=arcosh\,(2\sin^2\mu\cos\nu+\cos\mu\sqrt{4\sin^2\mu\sin^2\nu+1})$ and α^*, $0\le\alpha^*<\frac{2\pi}{3}$ is uniquely determined by the equation $\cos\frac{\alpha^*}{2}+\cos\frac{\beta}{2}=\frac{1}{2}$.

Proof. Recall (Proposition 1.3.1) that $6_2^2(\alpha,\beta)$ is hyperbolic for $(\alpha,\beta)\in R_h$, where the region R_h is bounded by the coordinate axes and by the curve $R_e=\{(\alpha,\beta)\in\mathbb{R}^2:0\le\alpha,\beta<4\pi/3,\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}=\frac{1}{2}\}$. Moreover, for all points $(\alpha,\beta)\in R_e$ cone–manifold $6_2^2(\alpha,\beta)$ admits Euclidean structure. If $(\alpha,\beta)\in R_h$ then the lengths l_α and l_β of are defined by Proposition 1.3.2. Hence

$$\cosh\frac{l_\alpha}{2}=2\sin^2\mu\cos\nu+\cos\mu\sqrt{4\sin^2\mu\sin^2\nu+1}, \qquad (2.3.1)$$

where $\mu=\frac{\alpha}{2}.\,\nu=\frac{\beta}{2}$ and the similar formula takes place for $\cosh\frac{l_\beta}{2}$. By the Schläfli formula for $V=V(\alpha,\beta)=Vol\,6_2^2(\alpha,\beta)$ we get

$$dV=-\frac{l_\alpha}{2}d\alpha-\frac{l_\beta}{2}d\beta. \qquad (2.3.2)$$

Choose a path of integration γ to be a segment with terminal points (α,β) and (α^*,β), where $(\alpha^*,\beta)\in R_e$ and note that $6_2^2(\alpha^*,\beta)$ is the Euclidean cone–manifold. Along the path γ we have $\beta\equiv const$ and (2.3.2) reduces to $dV=-\frac{l_\alpha}{2}d\alpha$. By Theorem 7.1.2 in [Kj] we obtain $V(\alpha,\beta)\to0$ as $\alpha\to\alpha^*$. Hence by (2.3.1)

$$V(\alpha,\beta)=\int_{\alpha^*}^{\alpha}-\frac{l_\alpha}{2}d\alpha=\int_{\alpha}^{\alpha^*}E(\frac{\alpha}{2},\frac{\beta}{2})d\alpha,$$

where $E(\mu,\nu)$ is the same as in the statement of the theorem.

Corollary 2.3.2. The figure eight cone–manifold $4_1(\alpha)$ is hyperbolic for $0\le\alpha<\frac{2\pi}{3}$. The hyperbolic volume of $4_1(\alpha)$ is given by the formula

$$Vol\,4_1(\alpha)=\int_{\alpha}^{\frac{2\pi}{3}} arcosh\,(1+\cos t-\cos 2t)dt.$$

Proof. We have $2E(\frac{\alpha}{2},\frac{\pi}{2}) = arcosh\,(1+\cos\alpha-\cos 2\alpha)$ and $V(\alpha,\pi) = \int_\alpha^{\frac{2\pi}{3}} E(\frac{\alpha}{2},\frac{\pi}{2})d\alpha$. Since the figure eight cone–manifold $4_1(\alpha)$ is two-fold covering of $V(\alpha,\pi)$, by Proposition 1.3.1 it is hyperbolic for $0 \le \alpha < \frac{2\pi}{3}$. We get

$$Vol\, 4_1(\alpha) = 2V(\alpha,\pi) = \int_\alpha^{\frac{2\pi}{3}} arcosh\,(1+\cos t - \cos 2t)dt.$$

We remark that equivalent but more complicated formulas for $Vol\, 4_1(\alpha)$ were obtained in [HLM3], [MV], and [Kj].

Corollary 2.3.3. The hyperbolic volume of the Borromean rings cone–manifold $B(\alpha,\alpha,\alpha)$, $0 \le \alpha < \pi$ is given by the formula

$$Vol\, B(\alpha,\alpha,\alpha) = 12\int_0^{\cos\frac{\alpha}{2}} arcosh\,\frac{u+\sqrt{4-3u^2}}{2}\,\frac{du}{\sqrt{1-u^2}}.$$

Proof is based on the fact that $B(\alpha,\alpha,\alpha)$ is the three-fold covering of the cone–manifold $6_2^2(\alpha,\frac{2\pi}{3})$ and on equality $E(\frac{\alpha}{2},\frac{2\pi}{3}) = 2\,arcosh\,\frac{u+\sqrt{4-3u^2}}{2}$, where $u = \cos\frac{\alpha}{2}$.

It was noted in the subsection 1.4 that the volume of the hyperbolic cone–manifold $B(\alpha,\beta,\gamma)$, $0 \le \alpha,\beta,\gamma < \pi$ is eight times the volume of the Lambert cube $L(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2})$. Hence, according to [K], $Vol\, B(\alpha,\beta,\gamma)$ can be obtain as a linear combination of eight Lobachevsky functions. The analoges of Theorem 2.3.1 and its corollaries can be obtained also in the spherical geometry. We restrict ourself by the statement of a spherical analog of the Corollary 2.3.2 ([MR2]).

Theorem 2.3.4. The figure eight cone–manifold $4_1(\alpha)$ is spherical for $\frac{2\pi}{3} < \alpha < \frac{4\pi}{3}$. The spherical volume of $4_1(\alpha)$ is given by the formula

$$Vol\, 4_1(\alpha) = \int_{\frac{2\pi}{3}}^{\alpha} arccos\,(1+\cos t - \cos 2t)dt,\ \frac{2\pi}{3} < \alpha \le \pi$$

and

$$Vol\, 4_1(\alpha) = 2\pi(\alpha - 0.9\pi) - \int_\pi^\alpha arccos\,(1+\cos t-\cos 2t)dt,\ \pi < \alpha < \frac{4\pi}{3}.$$

Earlier [HLM3] the existence of the spherical structure on $4_1(\alpha)$ was established only for $\frac{2\pi}{3} < \alpha \le \pi$.

3. Geometrical inequalities

As above denote by $V(\alpha)$ and $l(\alpha)$ the volume and the singular geodesic length of the figure eight cone–manifold $4_1(\alpha)$. It follows from Theorem 2.3.4 and Corallary 2.3.2 (see also [HLM3]) that $V(\alpha) \to 0$ as $\alpha \to \frac{2\pi}{3} \pm 0$. Hence "the Euclidean" volume $V(\frac{2\pi}{3}) = 0$. Certainly, this contradicts the geometric intuition. To avoid this phenomenon we introduce the notion of *specific volume* $v(\alpha)$. By definition

$v(\alpha) = \frac{V(\alpha)}{l^3(\alpha)}$. In particular, in the Euclidean case for $\alpha = \alpha_0 = \frac{2\pi}{3}$ we get from [MR1]

$$v_0 = v(\alpha_0) = \frac{\sqrt{3}}{108}.$$

Explicit volume formulas obtained in Section 2 ensure that the specific volume function $v(\alpha)$ is continuous for all $0 < \alpha < \frac{4\pi}{3}$.

Theorem 3.1. *Let $4_1(\alpha)$ be the figure eight cone–manifold with cone angle α. Denote by $V(\alpha)$ the volume and by $l(\alpha)$ the length of the singular geodesic of $4_1(\alpha)$. Then*

$$(i) \quad V(\alpha) > v_0\, l^3(\alpha), \ if \ 0 < \alpha < \alpha_0 = \frac{2\pi}{3} \ (hyperbolic \ case)$$

$$(ii) \quad V(\alpha) = v_0\, l^3(\alpha), \quad if \ \alpha = \alpha_0 \qquad (Euclidean \ case)$$

$$(iii) \quad V(\alpha) < v_0\, l^3(\alpha), \ if \ \alpha_0 < \alpha < 2\alpha_0 \quad (spherical \ case),$$

where $v_0 = \frac{\sqrt{3}}{108}$ is the specific volume of $4_1(\alpha_0)$.

Proof. Case (ii) immediately follows from the definition of $v_0 = v(\alpha_0)$.

To prove (i) we will show that

$$(V(\alpha) - v_0\, l^3(\alpha))' < 0, \ 0 < \alpha < \alpha_0.$$

Since $l(\alpha) > 0$ and by the Schläfli formula $V'(\alpha) = -\frac{1}{2}l(\alpha)$, the last inequality is equivalent to

$$1 + 6v_0\, l(\alpha)l'(\alpha) > 0, \ 0 < \alpha < \alpha_0.$$

By Corollary 2.3.2 we have $l(\alpha) = 2\, arcosh\,(1+\cos\alpha - \cos 2\alpha)$, and the inequality is verified by straightforward calculation.

In case (iii) we need to prove

$$(V(\alpha) - v_0\, l(\alpha))' > 0, \ \alpha_0 < \alpha < 2\alpha_0.$$

By the Schläfli formula and Proposition 2.3.4 we have

$$V'(\alpha) = \frac{1}{2}l(\alpha) = arccosh\,(1 + \cos\alpha - \cos 2\alpha).$$

Again, the inequality is a routine consiquence of these formulas.

The next theorem can be proved by similar arguments.

Theorem 3.2. *The Borromean rings cone–manifold $B(\alpha, \alpha, \alpha)$ is hyperbolic for $0 < \alpha < \pi$, Euclidean for $\alpha = \pi$, and spherical for $\pi < \alpha < 2\pi$. Denote by $V(\alpha)$ the volume and by $L(\alpha)$ the half length of a singular geodesic of $B(\alpha, \alpha, \alpha)$. Then*

$$(i) \quad V(\alpha) > L^3(\alpha), \ 0 < \alpha < \pi$$

$$(ii) \quad V(\alpha) = L^3(\alpha), \qquad \alpha = \pi$$

$$(iii) \;\; V(\alpha) < L^3(\alpha), \; \pi < \alpha < 2\pi.$$

Different arguments are needed to obtain the following result

Theorem 3.3. *Let $B(\alpha, \beta, \gamma)$, $0 < \alpha, \beta, \gamma < \pi$ be a hyperbolic Borromean rings cone–manifold. Denote by $V(\alpha, \beta, \gamma)$ the volume and by $L(\alpha), L(\beta), L(\gamma)$ half lengths of singular geodesics of $B(\alpha, \beta, \gamma)$ with cone angles α, β, γ respectively. Then*

$$V(0,0,0) > V(\alpha, \beta, \gamma) > L(\alpha)L(\beta)L(\gamma).$$

Proof. In 1928 Grötsch [G] has proved the following theorem: *Let $\mathcal{S}$ be the conformal image of a square and let A, B, C, D be the images of the sides of the square traversed in a clockwise direction. Suppose the distance between the "opposite" sides A and C of $\mathcal{S}$ is a, and between B and D is b. Then the area of $\mathcal{S}$ can be no smaller then ab.*

The $n-$dimensional analog of the Grötsch theorem for an arbitrary Riemannian metric was obtained in ([BZ], Theorem 8.2.1). In particular, it follows from [BZ] that hyperbolic volume of the Lambert cube $L(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ with essential edges $L_\alpha, L_\beta, L_\gamma$ satisfies the inequality

$$Vol\, L(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}) > L_\alpha L_\beta L_\gamma. \tag{3.1}$$

In the Euclidean case $\alpha = \beta = \gamma = \pi$ we get the equality $Vol\, L(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = L_\pi^3$.

Since by the Schläfli theorem $\frac{\partial}{\partial \alpha} V(\alpha, \beta, \gamma) = -L(\alpha) < 0$, $0 < \alpha < \pi$, the upper bound $V(\alpha, \beta, \gamma) < V(0,0,0)$ is established. We recall from section 2.4 that

$$V(\alpha, \beta, \gamma) = 8\, Vol\, L(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}),\; L_\alpha = 2L(\alpha),\; L_\beta = 2L(\beta),\; L_\gamma = 2L(\gamma).$$

Hence, the lower bound follows from inequalty (3.1).

No doubt the spherical analog of the theorem takes place too. It will be obtain anywhere more.

Remark 3.4. We note that the length $L(\alpha)$ in Theorem 3.2 is bounded above by $\log 3$. The equality $L(\alpha) = \log 3$ holds for $\cos \alpha = -\frac{1}{3}$. Contrary, the length $L(\alpha)$ in Theorem 3.3 is unbounded. More precisely, $L(\alpha) \to +\infty$, $L(\beta) \to 0$, $L(\gamma) \to 0$ as $\alpha \to \frac{\pi}{2}$, $\beta \to 0$, $\gamma \to 0$. The proof of these properties immediately follows from the Tangent Rule (Theorem 1.4.1).

References

[BZ] Yu. D. Burago, V. A. Zalgaller, *Geometric inequalities,* Grundlehren der Mathematischen Wissenschaften, 285, Berlin ets., Springer-Verlag, 1988.

[BZie] G. Burde, H. Zieschang, *Knots,* De Gruyter Studies in Mathematics, Berlin - New-York, 1985.

[CCGLS] D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, *Plane curves associated to character varieties of 3-manifolds,* Invent. Math., vol. 118 (1994), 47-84.

[G] H. Grötsch, *Extremalprobleme der Konformen Abbildung I,* Berichte Akad. Leipzig, 80, 1928, 367-376.

[Fench] W. Fenchel, *Elementary Geometry in Hyperbolic Space,* Walter de Gruyter, Berlin–New York 1989.

[HKW] P. de la Harpe, M. Kervaire and C. Weber, *On the Jones Polynomial,* L'Enseignement Mathématique, Vol.32 (1986), 271–335.

[HLM1] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On the Borromean orbifolds: geometry and arithmetic,* in: Topology'90, eds. B. A. Apanasov, W. Newmann, A. Reid, L. Siebenmann, de Gruyter, Berlin, 1992, 133–167.

[HLM2] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant,* Journal of Knots and its Ramification, vol.4., no.1, 1995, 81-114.

[HLM3] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On a remarkable polyhedron geometrizing the figure eight cone manifolds,* J. Math. Sci. Univ. Tokyo, vol. 2, 1995, 501–561.

[Hds] C. D. Hodgson, *Schläfli revisited: Variation of volume in constant curvature spaces,* Preprint.

[Kauf] L. Kauffman, *New Invariants in the Theory of Knots,* Amer. Math. Monthly, vol. 95, no. 3, 195-242.

[K] R. Kellerhals, *On the volume of hyperbolic polyhedra,* Math. Ann., vol. 28 (1989), 245–270.

[KM] A.-Ch. Kim, A. Mednykh, *On the hyperbolic structure on the Whitehead link cone–manifold,* Preprint, 1999.

[Kj] S. Kojima, *Deformation of hyperbolic 3-cone-manifolds,* J. Differential Geometry, Vol. ?? (1999) ???-???.

[Kn] C. G. Knott, *Life and Scientific Work of P.G.Tait,* Cambridge University Press (1911).

[Ma] J. C. Maxwell *A treatise on eletricity and magmetism,* Oxford (1883).

[M] A. Mednykh, *On the the Remarkable Properties of the Hyperbolic Whitehead Link Cone–Manifolds,* Universität Bielefeld, Sonderforschungsbereich 343, "Discrete Structuren in der Mathematik", Preprint, 99–039.

[MR1] A. Mednykh, A. Rasskazov, *On the structure of the canonical fundamental set for the 2-bridge link orbifolds,* Universität Bielefeld, Sonderforschungsbereich 343, "Discrete Structuren in der Mathematik", Preprint, 98–062.

[MR2] A. Mednykh, A. Rasskazov, *On the degenaration of geometric structures on the figure eight cone–manifold,* Preprint, 1999

[MV] A. Mednykh, A. Vesnin, *Hyperbolic volumes of Fibonacci manifolds,* Siberian Math. J.36, 1995, 235–245.

[R] R. Riley, *An elliptical path from parabolic representations to hyperbolic structure,* in: Topology of Low-Dimension manifolds, LNM, 722, Springer-Verlag, 1979, 99-133.

[Rolf] D. Rolfsen, *Knots and links,* Publish of Perish Inc., Berkeley Ca., 1976.

[T] W. P. Thurston, *The geometry and topology of three-manifolds,* Princeton University Press (1979).

[Zhou] Qing Zhou, *The Moduli Space of Hyperbolic Cone Structures,* J. Diferential Geometry, Vol. 51 (1999), 517-550.

[V] E. B. Vinberg, *Geometry II,* Encyclopeadia of Mathematical Sciences, Vol.29, Springer Verlag, 1992.