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On the volume of symmetric tetrahedron [1]

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Abstract

An elementary formula is obtained for the volume of symmetric tetrahedron in hyperbolic and spherical spaces.

Mathematical Subject Classification: 30F40, 51M10, 57M50

Key words: hyperbolic tetrahedron, spherical tetrahedron, volume formula, Gram matrix

1 Introduction

The calculation of volume of polyhedron is very old and difficult problem. A few years ago it was shown by I.H. Sabitov [Sb] that the volume of Euclidean polyhedron is a root of algebraic equation whose coefficient are function combinatorial type and lengths of polyhedra. In hyperbolic and spherical spaces the situation is march more complicated. Since Lobachevsky and Schläfli (see [L] and [Sh] respectively) the volume formula for biorthogonal tetrahedron (orthoscheme) is know. The volume of the Lambert cube and some other polyhedron were calculated by R. Kellerhals [K], D. A. Derevnin, A. D. Mednykh [DM], A. D. Mednykh, J. Parker, A. Yu. Vesnin [MPV] and other. The volume of regular polyhedron was obtained by G. Martin [M]. The volume of ideal hyperbolic polyhedron in many important particular cases was found by E. Vinberg [V].

The volume formula for arbitrary hyperbolic and spherical tetrahedron for a long time was unknown. Some attempt to obtain such a formula contains in Wu–Yi Hsiang [H]. Just recently simple volume formula for tetrahedron was obtained by Yu. Cho, H. Kim [ChK] and J. Murakami, U. Yano [MY]. Easy proof for this formula which covers also the volume of truncated tetrahedron can be find in A. Ushijima [U].

The aim of this paper is to find an elementary formula for volume of symmetric tetrahedron both in hyperbolic and spherical spaces.

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2 Preliminary results

Denote by $\mathbb{X}^n$ the Euclidean, hyperbolic or spherical n-space. Let compact tetrahedron $T = (A, B, C, D, E, F) \in \mathbb{X}^3$ have vertices v_1, v_2, v_3, v_4 and dihedral angles A, B, C, D, E, F with edge lengths $l_A, l_B, l_C, l_D, l_E, l_F$ respectively (see Fig. 1).

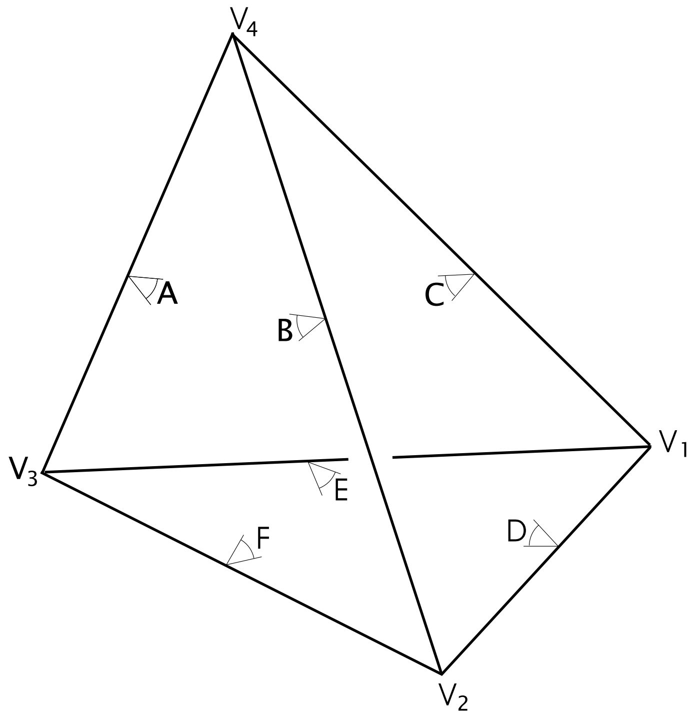

Figure 1: The tetrahedron T

We will call tetrahedron T *symmetric* if $A = D$, $B = E$, $C = F$.

Our calculation of volume of tetrahedron can be based on the following Schläfli formula (see, for instance [Sh], [Hd], [K]).

Theorem 1 (The Schläfli volume formula). *Let compact simplex $S \in \mathbb{X}^n$ $(n \geq 2)$ have vertices $P_1, ..., P_{n+1}$ and dihedral angles $\alpha_{jk} = \angle(S_j, S_k)$, $1 \leq j < k \leq n+1$, of order $n-1$ formed by the faces S_j, S_k of S with apex $S_{jk} := S_j \cap S_k$. Then the differential of the volume function V_n on the set of all simplices in $\mathbb{X}^n$ can be represented by*

$$K dV_n(S) = \frac{1}{n-1} \sum_{\substack{j,k=1 \\ j<k}}^{n+1} V_{n-2}(S_{jk}) d\alpha_{jk} \qquad (V_o(S_{jk}) := 1),$$

where K is the curvature of $\mathbb{X}^n$.

In the present paper we set $K=-1$ for hyperbolic space and $K=1$ for spherical space. The Schläfli formula for hyperbolic and spherical 3-spaces can be reduced to

$$KdV = \frac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^{4} l_{jk}\,d\alpha_{jk},$$

where l_{jk} are the lengths of the correspondent edges of S.

3 The volume of hyperbolic tetrahedron

Let T be a hyperbolic tetrahedron. Denote by

$$G = \langle -\cos\alpha_{ij}\rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

the Gram matrix of T and by $H = \langle c_{ij}\rangle_{i,j=1,2,3,4}$ the associated with G matrix form by $c_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is $(i,j)-th$ minor of G. Following arguments by A. Ushijima [U] we obtain

Proposition 1. *Let T be a proper hyperbolic tetrahedron. Then*

$$\begin{aligned} &\text{(i)} && \det G < 0 \\ &\text{(ii)} && c_{ii} > 0,\ i = 1,2,3,4 \\ &\text{(iii)} && \frac{\sin A}{\sinh l_A} = \frac{\sqrt{c_{33}c_{44}}}{\sqrt{-\det G}} \end{aligned} \tag{3.1}$$

As an immediate consequence of this proposition we have the following result

Proposition 2. *Let T be a proper hyperbolic tetrahedron. Then*

$$\frac{\sin A \sin D}{\sinh l_A \sinh l_D} = \frac{\sin B \sin E}{\sinh l_B \sinh l_E} = \frac{\sin C \sin F}{\sinh l_C \sinh l_F} = \frac{\sqrt{P}}{\Delta} \tag{3.2}$$

where $P = c_{11}c_{22}c_{33}c_{44}$ and $\Delta = -\det G$

From now on we suppose that the tetrahedron T is symmetric. By direct straightforward calculation we obtain $c_{11} = c_{22} = c_{33} = c_{44} = \gamma$, where

$$\gamma = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C. \tag{3.3}$$

And also

$$\Delta = -\det G = (1-a+b+c)(1+a-b+c)(1+a+b-c)(-1+a+b+c), \quad (3.4)$$

where $a = \cos A,\ b = \cos B,\ c = \cos C$.

Putting this calculations into Proposition 2 we have

Proposition 3 (The Sine Rule). *Let T be a symmetric hyperbolic tetrahedron. Then*

$$\frac{\sin A}{\sinh l_A} = \frac{\sin B}{\sinh l_B} = \frac{\sin C}{\sinh l_C} = u, \quad (3.5)$$

where $u = \dfrac{\gamma}{\sqrt{\Delta}}$ and γ, Δ are defined by (3.3) and (3.4) respectively.

We note the following useful identity

$$u^2 + 1 = \frac{4(\cos A + \cos B \cos C)(\cos B + \cos A \cos C)(\cos C + \cos B \cos A)}{\Delta}. \quad (3.6)$$

The following lemma can be obtained by elementary calculations

Lemma 1. *Let t is defined by equality*

$$t^2 = \frac{4(a+bc)(b+ac)(c+ab)}{(1-a+b+c)(1+a-b+c)(1+a+b-c)(-1+a+b+c)}, \quad (3.7)$$

where $a = \cos A,\ b = \cos B,\ c = \cos C$ and A, B, C are the dihedral angles of a symmetric hyperbolic tetrahedron T. Then

$$\arcsin\frac{a}{t} + \arcsin\frac{b}{t} + \arcsin\frac{c}{t} = \arcsin\frac{1}{t}. \quad (3.8)$$

Proof. Notice first that from (3.6) follows $t^2 = u^2 + 1 > 1$. Show that t defined by equality (3.7) satisfies to equality (3.8). By the basic formula

$$\arcsin(x \pm y) = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$$

we transform (3.8) to

$$\arcsin\left(\frac{a}{t}\sqrt{1-\frac{b^2}{t^2}} + \frac{b}{t}\sqrt{1-\frac{a^2}{t^2}}\right) = \arcsin\left(\frac{1}{t}\sqrt{1-\frac{c^2}{t^2}} - \frac{c}{t}\sqrt{1-\frac{1}{t^2}}\right).$$

Hence, (3.8) is equivalent to

$$a\sqrt{t^2-b^2} + b\sqrt{t^2-a^2} = \sqrt{t^2-c^2} - c\sqrt{t^2-1}. \quad (3.9)$$

From the other side the straightforward calculation shows that (3.7) implies

$$\begin{aligned} t^2 - a^2 &= \frac{\Big(a(1 - a^2 + b^2 + c^2) + 2bc\Big)^2}{\Delta}, \\ t^2 - b^2 &= \frac{\Big(b(1 - b^2 + a^2 + c^2) + 2ac\Big)^2}{\Delta}, \\ t^2 - c^2 &= \frac{\Big(c(1 - c^2 + a^2 + b^2) + 2ab\Big)^2}{\Delta}, \\ t^2 - 1 &= \frac{\Big(1 - a^2 - b^2 - c^2 - 2abc\Big)^2}{\Delta}. \end{aligned}$$

By (3.1) (ii) we have

$$1 - a^2 - b^2 - c^2 - 2abc = c_{11} > 0$$

and it is not difficult to see that

$$\begin{aligned} \sqrt{t^2 - a^2} &= \frac{a(1 - a^2 + b^2 + c^2) + 2bc}{\sqrt{\Delta}}, \\ \sqrt{t^2 - b^2} &= \frac{b(1 - b^2 + a^2 + c^2) + 2ac}{\sqrt{\Delta}}, \\ \sqrt{t^2 - c^2} &= \frac{c(1 - c^2 + a^2 + b^2) + 2ab}{\sqrt{\Delta}}, \\ \sqrt{t^2 - 1} &= \frac{1 - a^2 - b^2 - c^2 - 2abc}{\sqrt{\Delta}}. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9) we have the identity. □

Let T be a symmetric hyperbolic tetrahedron. Denote by $V = V(A, B, C)$ the hyperbolic volume of T. Since $A = D$, $B = E$, $C = F$, $l_A = l_D$, $l_B = l_E$, $l_C = l_F$, by Theorem 1 we have

$$dV = -l_A dA - l_B dB - l_C dC. \tag{3.11}$$

Hence

$$\frac{\partial V}{\partial A} = -l_A, \quad \frac{\partial V}{\partial B} = -l_B, \quad \frac{\partial V}{\partial C} = -l_C. \tag{3.12}$$

We note that if $A, B, C \to \arccos \frac{1}{3}$ then T is going to regular Euclidean tetrahedron. In this case $\Delta \to 0$ and $u \to +\infty$. By the Sine Rule, $l_A, l_B, l_C \to 0$ and, consequently $V \to 0$. So, we have

$$V(\arccos \frac{1}{3}, \arccos \frac{1}{3}, \arccos \frac{1}{3}) = 0. \tag{3.13}$$

Now we are able to prove the following

Theorem 2. *Let T be a symmetric hyperbolic tetrahedron whose dihedral angles corresponding to pairs of opposite edge are A, B, C. The hyperbolic volume of T is given by the formula*

$$V = \int\limits_u^{+\infty} \Big(\arcsin \frac{\cos A}{\sqrt{\nu^2+1}} + \arcsin \frac{\cos B}{\sqrt{\nu^2+1}} + \arcsin \frac{\cos C}{\sqrt{\nu^2+1}} - \arcsin \frac{1}{\sqrt{\nu^2+1}} \Big) \frac{d\nu}{\nu} \tag{3.14}$$

where $u = \dfrac{1-a^2-b^2-c^2-2abc}{\sqrt{(1-a+b+c)(1+a-b+c)(1+a+b-c)(-1+a+b+c)}}$,

$$a = \cos A, \quad b = \cos B, \quad c = \cos C.$$

Proof. We set

$$F(A,B,C,\nu) = \arcsin \frac{a}{\sqrt{\nu^2+1}} + \arcsin \frac{b}{\sqrt{\nu^2+1}} + \arcsin \frac{c}{\sqrt{\nu^2+1}} - \arcsin \frac{1}{\sqrt{\nu^2+1}}$$

and $\widetilde{V}(A,B,C) = \int\limits_u^{+\infty} F(A,B,C,\nu)d\nu$. To prove the theorem is sufficient to show that $\widetilde{V}$ satisfies (3.12) with initial data (3.13). By the Leibnitz Rule we have

$$\frac{\partial \widetilde{V}}{\partial A} = -F(A,B,C,u)\frac{\partial u}{\partial A} + \int\limits_u^{+\infty} \frac{\partial F(A,B,C,\nu)}{\partial A} d\nu. \tag{3.15}$$

For $t^2 = u^2+1$ in (3.6) by Lemma 1 we have $F(A,B,C,u) = 0$. Hence, taking into account that $\dfrac{\partial F(A,B,C,u)}{\partial A} = \dfrac{\sin A}{u\sqrt{u^2+\sin^2 A}}$ and by the Sine Rule $l_A = \operatorname{arcsinh} \dfrac{\sin A}{u}$ we obtain

$$\frac{\partial \widetilde{V}}{\partial A} = \int\limits_u^{+\infty} \frac{\partial F(A,B,C,\nu)}{\partial A} d\nu = \int\limits_u^{+\infty} \frac{\sin A}{\nu\sqrt{\nu^2+\sin^2 A}} d\nu = -l_A. \tag{3.16}$$

The equalities

$$\frac{\partial \widetilde{V}}{\partial B} = -l_B, \quad \frac{\partial \widetilde{V}}{\partial C} = -l_C \tag{3.17}$$

can be obtained by the similar way. Let $A, B, C \to \arccos \frac{1}{3}$, then $u \to +\infty$ and the relation

$$\widetilde{V}(\arccos \frac{1}{3}, \arccos \frac{1}{3}, \arccos \frac{1}{3}) = 0 \tag{3.18}$$

follows from the convergence of integral $\int\limits_u^{+\infty} F(A,B,C,\nu)d\nu$. □

Substituting $\nu = \tan t$ in the above proposition we have

Theorem 3. *Let T be a symmetric hyperbolic tetrahedron whose dihedral angles corresponding to pairs of opposite edge are A, B, C. Then the hyperbolic volume of T is given by the formula*

$$2 \int_{\theta}^{\frac{\pi}{2}} (\arcsin(\cos A \cos t) + \arcsin(\cos B \cos t) + \arcsin(\cos C \cos t) - \arcsin(\cos t)) \frac{dt}{\sin 2t} \tag{3.19}$$

where $\theta \in [0, \frac{\pi}{2}]$, is defined by

$$\tan^2 \theta = \frac{1 - a^2 - b^2 - c^2 - 2abc}{\sqrt{(1 - a + b + c)(1 + a - b + c)(1 + a + b - c)(-1 + a + b + c)}},$$

$$a = \cos A, \quad b = \cos B, \quad c = \cos C.$$

The obtained result numerically coincides with a result obtained early by Yu. Cho, H. Kim [ChK] and can be expressed in term of the Lobachevsky function $\Lambda(x) = -\int_0^x \ln |2 \sin \xi| d\xi$.

4 The volume of spherical tetrahedron

Let T be a spherical tetrahedron with Gram matrix

$$G = \langle -\cos \alpha_{ij} \rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

and associated matrix $H = \langle c_{ij} \rangle_{i,j=1,2,3,4}$. The next proposition essentially follows from [L].

Proposition 4. *Let T be a spherical tetrahedron. Then*

$$\begin{array}{ll} \text{(i)} & \det G > 0 \\ \text{(ii)} & c_{ii} > 0,\ i = 1, 2, 3, 4 \\ \text{(iii)} & \dfrac{\sin A}{\sin l_A} = \dfrac{\sqrt{c_{33} c_{44}}}{\sqrt{\det G}} \end{array} \tag{4.20}$$

Proof. Conditions (i) and (ii) follows from existence of spherical tetrahedron whose Gram matrix is G (see [L], [V]). To prove (iii) we used the following assertion due to Jacobi ([P], Theorem 2.5.1, p.12). ☐

Lemma 2 (Jacobi). *Let $A = \langle a_{ij} \rangle_{i,j=1,\dots,n}$ be a matrix and $\Delta = \det A$ is determinant of A. Denote by $C = \langle c_{ij} \rangle_{i,j=1,\dots,n}$ the matrix formed by elements $c_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} is $(n-1) \times (n-1)$ minor obtained by removing $i-th$ line and $j-th$ column of the matrix A. Then for any k, $1 \leq k \leq n-1$ we have*

$$det\langle c_{ij} \rangle_{i,j=1,\dots,k} = \Delta^{k-1} det\langle a_{ij} \rangle_{i,j=k+1,\dots,n} \tag{4.21}$$

By applying Lemma 2 to matrices G and H for $k = 2$ we obtain $1 - \cos^2 A = \det G\,(c_{33}c_{44} - c_{34}^2)$. Since $\cos l_A = \dfrac{c_{34}}{\sqrt{c_{33}c_{44}}}$, the relation (iii) follows (compare A. Ushijima [U]).

As a consequence of Proposition 4 similar to hyperbolic case we have

Proposition 5. *Let T be a spherical tetrahedron. Then*

$$\frac{\sin A \sin D}{\sin l_A \sin l_D} = \frac{\sin B \sin E}{\sin l_B \sin l_E} = \frac{\sin C \sin F}{\sin l_C \sin l_F} = \frac{\sqrt{P}}{\det G} \tag{4.22}$$

where $P = c_{11}c_{22}c_{33}c_{44}$.

Now we apply the obtained result to symmetric tetrahedron.

Proposition 6 (The Sine Rule). *Let $T = T(A, B, C, A, B, C)$ be a symmetric spherical tetrahedron. Then*

$$\frac{\sin A}{\sin l_A} = \frac{\sin B}{\sin l_B} = \frac{\sin C}{\sin l_C} = v, \tag{4.23}$$

where $v = \frac{\gamma}{\sqrt{\Delta}}$,

$$\gamma = c_{11} = c_{22} = c_{33} = c_{44} = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C,$$

$$\Delta = \det G = (1 - a + b + c)(1 + a - b + c)(1 + a + b - c)(1 - a - b - c),$$

and $\quad a = \cos A, \quad b = \cos B, \quad c = \cos C.$

We note also that

$$v^2 - 1 = \frac{4(a + bc)(b + ac)(c + ab)}{\Delta}. \tag{4.24}$$

The following lemma can be obtained by the same arguments as Lemma 2

Lemma 3. *Let p is defined by equality*

$$p^2 = \frac{4(a+bc)(b+ac)(c+ab)}{(1-a+b+c)(1+a-b+c)(1+a+b-c)(1-a-b-c)}.$$

where $a = \cos A,\ b = \cos B,\ c = \cos C$ and A, B, C are the dihedral angles of a symmetric spherical tetrahedron T. Then

$$\operatorname{arcsinh}\frac{a}{p} + \operatorname{arcsinh}\frac{b}{p} + \operatorname{arcsinh}\frac{c}{p} = \operatorname{arcsinh}\frac{1}{p}.$$

Denote by $V = V(A, B, C)$ the spherical volume of tetrahedron $T(A, B, C, A, B, C)$. Then by Theorem 1 (The Schläfli volume formula) we have

$$dV = l_A dA + l_B dB + l_C dC$$

Hence

$$\frac{\partial V}{\partial A} = l_A, \quad \frac{\partial V}{\partial B} = l_B, \quad \frac{\partial V}{\partial C} = l_C. \tag{4.25}$$

As in hyperbolic case we note that T collapsed to a point as $\Delta \to 0$ or $V \to +\infty$. In particular for $A, B, C \to \arccos\frac{1}{3}$, we obtain

$$V(\arccos\frac{1}{3}, \arccos\frac{1}{3}, \arccos\frac{1}{3}) = 0 \tag{4.26}$$

Theorem 4. *Let $T = T(A, B, C, A, B, C)$ be a symmetric spherical tetrahedron whose dihedral angled corresponding to pairs of opposite edges are A, B, C. Then the spherical volume of T is given by the formula*

$$\begin{aligned} V \;=\; -\int\limits_{v}^{+\infty} \Big(&\operatorname{arcsinh}\frac{\cos A}{\sqrt{\nu^2-1}} + \operatorname{arcsinh}\frac{\cos B}{\sqrt{\nu^2-1}} + \\ &\operatorname{arcsinh}\frac{\cos C}{\sqrt{\nu^2-1}} - \operatorname{arcsinh}\frac{1}{\sqrt{\nu^2-1}}\Big)\frac{d\nu}{\nu}, \end{aligned} \tag{4.27}$$

where $v = \dfrac{1-a^2-b^2-c^2-2abc}{\sqrt{(1-a+b+c)(1+a-b+c)(1+a+b-c)(1-a-b-c)}}$,

$$a = \cos A, \quad b = \cos B, \quad c = \cos C.$$

Proof. We set $\widetilde{V}(A, B, C) = -\int\limits_{v}^{+\infty} \hat{F}(A, B, C, \nu)d\nu$, where

$$\hat{F}(A, B, C, \nu) = \operatorname{arcsinh}\frac{\cos A}{\sqrt{\nu^2-1}} + \operatorname{arcsinh}\frac{\cos B}{\sqrt{\nu^2-1}} + \operatorname{arcsinh}\frac{\cos C}{\sqrt{\nu^2-1}} - \operatorname{arcsinh}\frac{1}{\sqrt{\nu^2-1}}.$$

We have to show that $\widetilde{V}$ satisfy (4.25) and (4.26). Then we have $\widetilde{V}(A,B,C) = V(A,B,C)$ By the Leibnitz Rule

$$\frac{\partial \widetilde{V}}{\partial A} = \hat{F}(A,B,C,v)\frac{\partial v}{\partial A} - \int\limits_{v}^{+\infty} \frac{\partial \hat{F}(A,B,C,\nu)}{\partial A} d\nu.$$

By Lemma 3 for $p^2 = v^2 - 1$ we have $\hat{F}(A,B,C,v) = 0$.
Since

$$\frac{\partial \hat{F}(A,B,C,\nu)}{\partial A} = \frac{-\sin A}{\nu\sqrt{\nu^2 - \sin^2 A}}$$

and, by the Sine Rule

$$l_A = \arcsin \frac{\sin A}{v},$$

we obtain

$$\frac{\partial \widetilde{V}}{\partial A} = -\int\limits_{v}^{+\infty} \frac{\partial \hat{F}(A,B,C,\nu)}{\partial A} d\nu = \int\limits_{v}^{+\infty} \frac{\sin A}{\nu\sqrt{\nu^2 + \sin^2 A}} d\nu = l_A. \tag{4.28}$$

The equalities

$$\frac{\partial \widetilde{V}}{\partial B} = l_B, \quad \frac{\partial \widetilde{V}}{\partial C} = l_C$$

can be obtained by the similar way. In the case $A,B,C \to \arccos\frac{1}{3}$, we have $v \to +\infty$. The relation $\widetilde{V}(\arccos\frac{1}{3}, \arccos\frac{1}{3}, \arccos\frac{1}{3}) = 0$ follows from the convergence of the integral $\int\limits_{v}^{+\infty} \hat{F}(A,B,C,\nu)d\nu$. □

Corollary 1. *Let $T = T(A,B,C,A,B,C)$ be a symmetric spherical tetrahedron. Suppose that $\pi - A$, $\pi - B$ and $\pi - C$ are sides of a right angled spherical triangle, that is one of the three conditions $\cos A + \cos B\cos C = 0$, $\cos B + \cos A\cos C = 0$ or $\cos C + \cos B\cos A = 0$ is satisfied. Then the spherical volume of T is equal to*

$$\frac{A^2 + B^2 + C^2}{2} - \frac{\pi^2}{4}.$$

Proof. Since the spherical space $\mathbb{S}^3$ is tessellated by sixteen copies of tetrahedron $T = T(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ we have $V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = \frac{1}{16}Vol(\mathbb{S}^3) = \frac{\pi^2}{8}$. Hence, by Theorem 4 we get

$$V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = \int\limits_{1}^{+\infty} \operatorname{arcsinh}\frac{1}{\sqrt{\nu^2 - 1}}\frac{d\nu}{\nu} = \frac{\pi^2}{8}. \tag{4.29}$$

Suppose that A,B,C satisfying the condition of the theorem. Then, by (4.24), $v^2 - 1 = 0$. Hence $v = 1$.

Lemma 4.

$$I(A) = -\int_1^{+\infty} \operatorname{arcsinh}\frac{\cos A}{\sqrt{\nu^2-1}}\frac{d\nu}{\nu} = \frac{A^2}{2} - \frac{\pi^2}{8}, 0 \le A \le \frac{\pi}{2}.$$

Proof. Indeed,

$$I^{'}(A) = \int_1^{+\infty} \frac{\sin A}{\sqrt{1+\frac{\cos^2 A}{\nu^2-1}}}\frac{d\nu}{\nu\sqrt{\nu^2-1}} = \int_1^{+\infty} \frac{\sin A}{\sqrt{\nu^2-\sin^2 A}}\frac{d\nu}{\nu} = A$$

and, verified by (4.29), $I(0) = -\frac{\pi^2}{8}$. Hence $I(A) = \frac{A^2}{2} - \frac{\pi^2}{8}$. □

By Theorem 4 we obtain $V(A,B,C) = I(A)+I(B)+I(C)-I(0) = \frac{A^2+B^2+C^2}{2} - \frac{\pi^2}{4}$. □

Substituting $\nu = \coth t$ in the statement of Theorem 4 we obtain

Theorem 5. *Let $T = T(A,B,C,A,B,C)$ be a symmetric spherical tetrahedron whose dihedral angles corresponding to pairs of opposite edges are A,B,C. Then the spherical volume of T is given by the formula*

$$-2\int_0^{\tau} (\operatorname{arcsinh}(\cos A\sinh t)+\operatorname{arcsinh}(\cos B\sinh t)+\operatorname{arcsinh}(\cos C\sinh t)-t)\frac{dt}{\sinh 2t} \tag{4.30}$$

where τ is a positive number defined by

$$\coth^2\tau = \frac{1-a^2-b^2-c^2-2abc}{\sqrt{(1-a+b+c)(1+a-b+c)(1+a+b-c)(1-a-b-c)}},$$

$$a = \cos A, \quad b = \cos B, \quad c = \cos C.$$

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