On the volume of spherical Lambert cube *

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Abstract

The calculation of volumes of polyhedra in the three-dimensional Euclidean, spherical and hyperbolic spaces is very old and difficult problem. In particular, an elementary formula for volume of non-euclidean simplex is still unknown. One of the simplest polyhedra is the Lambert cube $Q(\alpha,\beta,\gamma)$. By definition, $Q(\alpha,\beta,\gamma)$ is a combinatorial cube, with dihedral angles α,β and γ assigned to the three mutually non-coplanar edges and right angles to the remaining. The hyperbolic volume of Lambert cube was found by Ruth Kellerhals (1989) in terms of the Lobachevsky function $\Lambda(x)$. In the present paper the spherical volume of $Q(\alpha,\beta,\gamma)$ is defined in the terms of the function

$$\delta\,(\alpha,\theta)=\int\limits_{\theta}^{\frac{\pi}{2}}\log\left(1-\cos 2\alpha\cos 2\tau\right)\frac{d\tau}{\cos 2\tau}$$

which can be considered as a spherical analog of the function

$$\Delta\,(\alpha,\theta)=\Lambda\,(\alpha+\theta)-\Lambda\,(\alpha-\theta)\,.$$

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1 Introduction

The calculation of volumes of polyhedra in Euclidean, spherical and hyperbolic spaces is very old and difficult problem. The main principles for volume calculations in non-euclidean geometries were given in 1836 by Lobachevsky [L] and in 1852 by Schläfli [Sh]. In particular, they have found the volumes of the orthogonal three-dimensional simlicies (orthoschemes).

In general, every polyhedron can be decomposed into a finite number of orthoschemes. But, in spite of this, an elementary formula for volume of non-euclidean simplex is still unknown[1].

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[1] Just recently, such a formula was obtained in [CK] and [MY]. Simple proof of the formula can be found in [U].

About 1935 Coxeter [C] revived interest in the work of these two authors by developing an integration method for non-euclidean orthoschemes of dimension three. This method was generalized by Böhm [B] for spaces of constant non-vanishing curvature of arbitrary dimension. Further advance in this direction was achieved in the papers by Vinberg [V], Milnor [M] and Ruth Kellerhals [K]. The simplest generalization of notion of three-dimensional orthoscheme is the Lambert cube $Q(\alpha,\beta,\gamma)$. Recall that $Q(\alpha,\beta,\gamma)$ is a combinatorial cube, with dihedral angles α,β and γ assigned to the three mutually non-coplanar edges and right angles to the remaining.

The Lambert cube $Q(\alpha,\beta,\gamma)$ can be realized in the hyperbolic space ([K, HLM]) if $0<\alpha,\beta,\gamma<\frac{\pi}{2}$ and in the spherical space [D] if $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$.

The hyperbolic volume of $Q(\alpha,\beta,\gamma)$ was found in [K] in terms of the Lobachevsky function

$$\Lambda(x)=-\int\limits_0^x \log|2\sin t|\,dt.$$

In the present paper the spherical volume of $Q(\alpha,\beta,\gamma)$ is defined in the terms of the function

$$\delta\left(\alpha,\theta\right)=\int\limits_{\theta}^{\frac{\pi}{2}}\log\left(1-\cos 2\alpha\cos 2\tau\right)\frac{d\tau}{\cos 2\tau}$$

which can be considered as a spherical analog of the function

$$\Delta\left(\alpha,\theta\right)=\Lambda\left(\alpha+\theta\right)-\Lambda\left(\alpha-\theta\right).$$

The main result of the work is the following theorem.

Theorem 1. *The volume of a spherical Lambert cube $Q(\alpha,\beta,\gamma)$ with essential angles α,β and γ, $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$ is given by the formula*

$$V(\alpha,\beta,\gamma)=\frac{1}{4}\left(\delta\left(\alpha,\theta\right)+\delta\left(\beta,\theta\right)+\delta\left(\gamma,\theta\right)-2\delta\left(\frac{\pi}{2},\theta\right)-\delta\left(0,\theta\right)\right),$$

where

$$\delta\left(\alpha,\theta\right)=\int\limits_{\theta}^{\frac{\pi}{2}}\log\left(1-\cos 2\alpha\cos 2\tau\right)\frac{d\tau}{\cos 2\tau}$$

and θ, $\frac{\pi}{2}<\theta<\pi$ is the principal parameter defined by

$$\begin{aligned}\tan^2\theta&=-p+\sqrt{p^2+L^2M^2N^2},\\ p&=\frac{L^2+M^2+N^2+1}{2},\\ L&=\tan\alpha, M=\tan\beta, N=\tan\gamma.\end{aligned}$$

It is interesting to compare properties of the function $\delta(\alpha, \theta)$ with the properties of the function $\Delta(\alpha, \theta)$. Notice that the hyperbolic volumes of knots, orbifolds and cone manifolds in many cases can be expressed in terms of $\Delta(\alpha, \theta)$ (see [Th, MV, V, K]). In particular, the main result of the work [K] can be rewrited in the following form.

Theorem 2. *The volume of a hyperbolic Lambert cube $Q(\alpha, \beta, \gamma)$ with essential angles α, β and γ , $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ is given by the formula*

$$V(\alpha, \beta, \gamma) = \frac{1}{4}\left(\Delta(\alpha, \theta) + \Delta(\beta, \theta) + \Delta(\gamma, \theta) - 2\Delta\left(\frac{\pi}{2}, \theta\right) - \Delta(0, \theta)\right),$$

where θ, $0 < \theta < \frac{\pi}{2}$ is the principal parameter defined by conditions

$$\tan^2\theta = p + \sqrt{p^2 + L^2M^2N^2},$$
$$p = \frac{L^2 + M^2 + N^2 + 1}{2},$$
$$L = \tan\alpha, M = \tan\beta, N = \tan\gamma.$$

2 Metric properties of Lambert cube

Consider the Euclidean space $\mathbb{R}^4 = \left(R^4, d\sigma^2\right)$ with the metric

$$d\sigma^2 = \frac{dx^2}{A^2} + \frac{dy^2}{B^2} + \frac{dz^2}{C^2} + dt^2$$

induced by the scalar product

$$((x, y, z, t), (x', y', z', t')) = \frac{xx'}{A^2} + \frac{yy'}{B^2} + \frac{zz'}{C^2} + tt',$$

where A,B and C are given positive numbers.

Define

$$S^3 = \{v = (x, y, z, t) \in \mathbb{R}^4 : (v, v) = 1\}.$$

The sphere S^3 endowed with the metric $d\sigma^2$ becomes the spherical space

$$\mathbb{S}^3 = (S^3, d\sigma^2)$$

with the constant Gaussian curvature $k = 1$. To see that we set $ds^2 = dx^2 + dy^2 + dz^2 + dt^2$. Then the mapping $\left(R^4, ds^2\right) \longrightarrow \left(R^4, d\sigma^2\right)$ defined by $(x, y, z, t) \rightarrow (Ax, By, Cz, t)$ is an isometry sending the unit sphere in $\left(R^4, ds^2\right)$ onto $\mathbb{S}^3$.

At the same time we consider R^4 as a projective space RP^3 with the homogeneous coordinates $(x : y : z : t)$.

As the Klein model $\mathbb{K}$ of the spherical space $\mathbb{S}^3$ choose the hyperplane $\mathbb{K} = \{(x, y, z, t) \in \mathbb{R}^4 : t = 1\}$, which will be identified further with the Euclidean space $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in R\}$.

There exists one-to-one correspondence between $\mathbb{K}$ and the upper semisphere $\mathbb{S}^3_+ = \{(x, y, z, t) \in \mathbb{S}^3 : t > 0\}$ formed by projection from the origin.

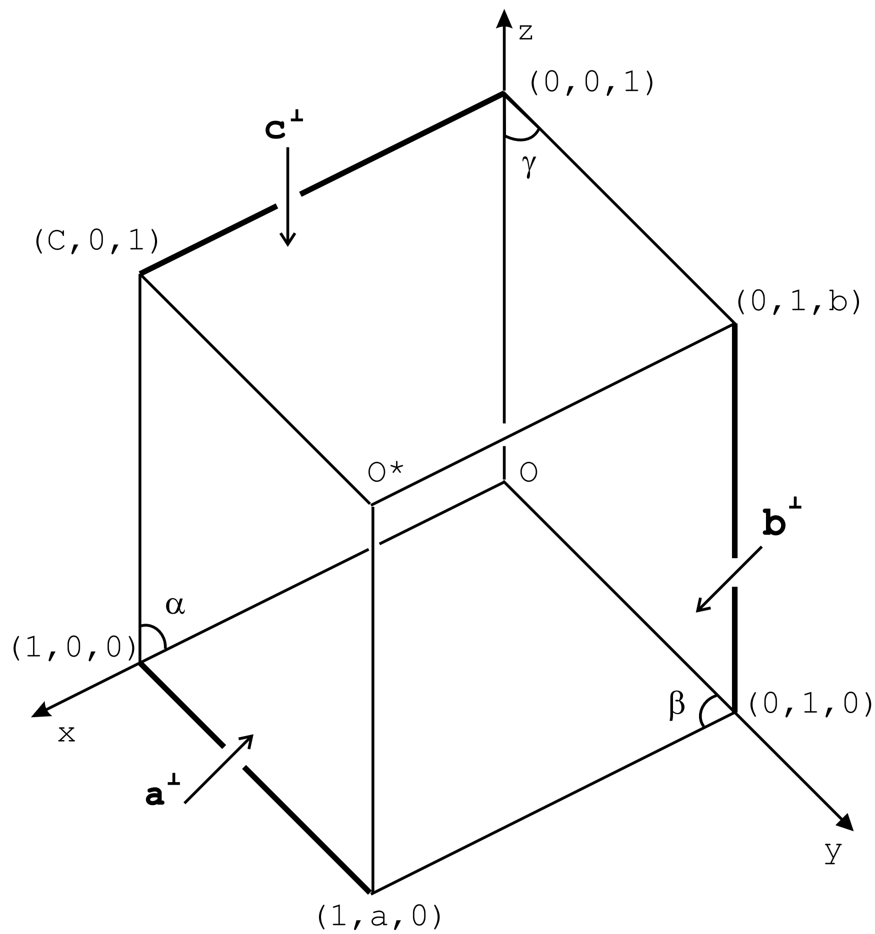

Figure 1:

Reflections in coordinate planes and rotations in coordinate lines of $\mathbb{R}^4$ are both Euclidean and spherical isometries.

Following to [HLM] we realize the spherical Lambert cube $Q(\alpha, \beta, \gamma)$ as a projection of the Euclidean polyhedron $P(a, b, c)$ represented on the Fig. 1 .

We can assume that the essential angles α, β and γ are formed by the pairs of the planes $\{a^{\perp}, z = 0\}$, $\{b^{\perp}, x = 0\}$ and $\{c^{\perp}, y = 0\}$. The projective equations of the planes $a^{\perp}$, $b^{\perp}$ and $c^{\perp}$ are the following

$$a^{\perp} = \{(x : y : z : t) \mid x + (1 - c)z - t = 0\},$$
$$b^{\perp} = \{(x : y : z : t) \mid y + (1 - a)x - t = 0\},$$
$$c^{\perp} = \{(x : y : z : t) \mid z + (1 - b)y - t = 0\}.$$

The poles of these planes P_a, P_b and P_c are given by

$$P_a = (-A^2 : 0 : (c-1)C^2 : 1),$$
$$P_b = ((a-1)A^2 : -B^2 : 0 : 1),$$
$$P_c = (0 : (b-1)B^2 : -C^2 : 1)$$

respectively. Then the conditions of the orthogonality

$$\cos(a^\perp, b^\perp) = \cos(b^\perp, c^\perp) = \cos(c^\perp, a^\perp) = 0,$$

are equivalent to the equalities

$$(P_a, P_b) = (P_b, P_c) = (P_c, P_a) = 0.$$

As a result we have the following relations

$$(1-a)A^2 + 1 = (1-b)B^2 + 1 = (1-c)C^2 + 1 = 0$$

or

$$a = 1 + \frac{1}{A^2},\ b = 1 + \frac{1}{B^2},\ c = 1 + \frac{1}{C^2}.$$

Hence

$$P_a = (-A^2 : 0 : 1 : 1),$$
$$P_b = (1 : -B^2 : 0 : 1),$$
$$P_c = (0 : 1 : -C^2 : 1).$$

The poles a', b' and c' of the planes $a'^\perp$, $b'^\perp$ and $c'^\perp$ symmetric to the planes $a^\perp$, $b^\perp$ and $c^\perp$ with respect to $z = 0$, $x = 0$ and $y = 0$ respectively are given by

$$P_{a'} = (-A^2 : 0 : -1 : 1),$$
$$P_{b'} = (-1 : -B^2 : 0 : 1),$$
$$P_{c'} = (0 : -1 : -C^2 : 1).$$

The angle between $a^\perp$ and $a'^\perp$ is equal 2α and we have

$$\cos 2\alpha = -\frac{(P_a, P_{a'})}{\sqrt{(P_a, P_a)(P_{a'}, P_{a'})}} = -\frac{A^2 - \frac{1}{C^2} + 1}{A^2 + \frac{1}{C^2} + 1}.$$

Hence

$$\tan^2 \alpha = C^2(A^2 + 1)$$

and by analogy we obtain

$$\tan^2 \beta = A^2(B^2 + 1),$$
$$\tan^2 \gamma = B^2(C^2 + 1).$$

As a result we have the following

Lemma 1. *The polyhedron $P(a, b, c)$ in the Klein model $\mathbb{K}$ of the spherical space $\mathbb{S}^3 = (S^3, d\sigma^2)$ has the right dihedral angles between the faces intersecting at the point O^* if and only if*

$$a = 1 + \frac{1}{A^2}, \ b = 1 + \frac{1}{B^2}, \ c = 1 + \frac{1}{C^2}.$$

Moreover, the dihedral angles α, β and γ of $P(a, b, c)$ satisfy the relations

$$\begin{aligned} \tan^2 \alpha &= C^2(A^2 + 1), \\ \tan^2 \beta &= A^2(B^2 + 1), \\ \tan^2 \gamma &= B^2(C^2 + 1). \end{aligned}$$

Set $L = \tan \alpha$, $M = \tan \beta$, $N = \tan \gamma$ and find an algebraic equation for variable

$$T = -ABC$$

in terms of L, M, N. By Lemma 1, we have

$$\begin{aligned} C^2A^2 + C^2 &= L^2, \\ A^2B^2 + A^2 &= M^2, \\ B^2C^2 + B^2 &= N^2. \end{aligned} \tag{2.1}$$

Multiplying the first of equations by B^2 and subtracting the third, we express B^2 by means of $A^2B^2C^2$. Similarly, expressing A^2 and C^2, we have

$$\begin{aligned} A^2B^2C^2 + N^2 &= (1 + L^2)B^2, \\ A^2B^2C^2 + L^2 &= (1 + M^2)C^2, \\ A^2B^2C^2 + M^2 &= (1 + N^2)A^2. \end{aligned} \tag{2.2}$$

By multiplying the equations, we obtain

$$(T^2 + L^2)(T^2 + M^2)(T^2 + N^2) = (1 + L^2)(1 + M^2)(1 + N^2)T^2,$$

where $T = -ABC$.

Note that $T = \pm 1$ is a root of the above equation of order six. Rewrite the equation in the following equivalent form

$$(T^2 - 1)(T^4 + (L^2 + M^2 + N^2 + 1)T^2 - L^2M^2N^2) = 0.$$

In the case $T^2 = 1$ we have $A^2B^2C^2 = 1$. Then from (2.2) and (2.1) we deduce the identity

$$1^4 + (L^2 + M^2 + N^2 + 1)1^2 - L^2M^2N^2 = 0.$$

Hence, the equation under consideration is equivalent to

$$T^4 + (L^2 + M^2 + N^2 + 1)T^2 - L^2M^2N^2 = 0.$$

By definition, $T = -ABC$, where $A, B, C > 0$. Hence, T as a negative root of the equation

$$T^2 = -\left(\frac{L^2+M^2+N^2+1}{2}\right) + \sqrt{\left(\frac{L^2+M^2+N^2+1}{2}\right)^2 + L^2M^2N^2}.$$

Notice that (2.2) can be rewritten in the form

$$A^2 = \frac{T^2+M^2}{1+N^2},\ B^2 = \frac{T^2+N^2}{1+L^2},\ C^2 = \frac{T^2+L^2}{1+M^2}.$$

As a result we obtain the following

Lemma 2. *The values A, B, C and $T = -ABC$ satisfy the following relations*

$$T^4 + (L^2+M^2+N^2+1)T^2 - L^2M^2N^2 = 0,$$
$$T^2 = -p + \sqrt{p^2 + L^2M^2N^2},$$

where

$$p = \frac{L^2+M^2+N^2+1}{2},$$
$$A^2 = \frac{T^2+M^2}{1+N^2},\ B^2 = \frac{T^2+N^2}{1+L^2},\ C^2 = \frac{T^2+L^2}{1+M^2}.$$

Here $A, B, C > 0$ and $T = -ABC < 0$ are uniquely determined by the values $L = \tan\alpha, M = \tan\beta$ and $N = \tan\gamma$.

Now we shall find the lengths of edges for spherical polyhedron $Q(\alpha, \beta, \gamma)$. First we find the length L_α of the edge with dihedral angle α. The projective coordinates of the terminal points of the edge are given by

$$u_{100} = (1:0:0:1), \qquad u_{1a0} = (1:a:0:1).$$

Let s_{100} and s_{1a0} be their projections on the upper hemisphere

$$S^3{}_+ = \{v = (x,y,z,t) : (v,v) = 1, t > 0\}.$$

We have

$$s_{100} = (\lambda : 0 : 0 : \lambda),$$

where $\frac{1}{A^2}\lambda^2 + \lambda^2 = 1$. Hence

$$\lambda = \frac{A}{\sqrt{1+A^2}}.$$

By analogy, for

$$s_{1a0} = (\tilde{\lambda} : \tilde{\lambda}a : 0 : \tilde{\lambda}) \in S^3{}_+$$

we have

$$\frac{\tilde{\lambda}^2}{A^2} + \frac{\tilde{\lambda}^2 a^2}{B^2} + \tilde{\lambda}^2 = 1,$$

where $a = 1 + \frac{1}{A^2}$ is defined by Lemma 1. It gives

$$\tilde{\lambda} = \frac{AB}{\sqrt{A^2(B^2+1)+1}} \frac{A}{\sqrt{A^2+1}}.$$

We obtain

$$\cos L_\alpha = (s_{100}, s_{1a0}) = \frac{\tilde{\lambda}\lambda}{A^2} + \tilde{\lambda}\lambda = \frac{AB}{\sqrt{A^2(B^2+1)+1}}.$$

We notice that the polyhedron $P(a,b,c)$ is contained in the first octant of the Euclidean space $\mathbb{R}^3$(see Fig. 1). Hence, the spherical length L_α of the edge whose dihedral angle α satisfies the inequality $0 < L_\alpha < \frac{\pi}{2}$. Thus

$$\sin L_\alpha = \frac{\sqrt{A^2+1}}{\sqrt{A^2(B^2+1)+1}}$$

and

$$\tan L_\alpha = \frac{\sqrt{A^2+1}}{AB}.$$

By analogy we obtain

$$\tan L_\beta = \frac{\sqrt{B^2+1}}{BC},$$

$$\tan L_\gamma = \frac{\sqrt{C^2+1}}{AC},$$

So we prove the following

Lemma 3. *The spherical lengths L_α, L_β and L_γ are given by formulas*

$$\tan L_\alpha = \frac{\sqrt{A^2+1}}{AB}, \quad \tan L_\beta = \frac{\sqrt{B^2+1}}{BC}, \quad \tan L_\gamma = \frac{\sqrt{C^2+1}}{AC}.$$

As an immediate corollary of Lemma 1 and Lemma 2 we obtain the following theorem (see also [D]).

Theorem 3. *Let α, β and γ are such that $\frac{\pi}{2} < \alpha, \beta, \gamma < \pi$. Then there exists a spherical Lambert cube with the essential angles α, β and γ .*

Proof. By Lemma 2, the values A^2, B^2 and C^2 uniquely determined by the equalities

$$A^2 = \frac{T^2+M^2}{1+N^2}, \quad B^2 = \frac{T^2+N^2}{1+L^2}, \quad C^2 = \frac{T^2+L^2}{1+M^2}.$$

Then by Lemma 1, we conclude the existence of the Euclidean polyhedron $P(a,b,c)$.

The spherical cube $Q(\alpha,\beta,\gamma)$ with the essential angles α, β and γ is a result of projection of $P(a,b,c)$ into S^3. □

For our purposes we need metrical relations between essential angles and lengths of Lambert cube $Q(\alpha,\beta,\gamma)$. They are given by the following two theorems.

Theorem 4 (The Tangent Rule). *Let $Q(\alpha,\beta,\gamma)$, $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$ be a spherical Lambert cube. Denote by L_α, L_β and L_γ lengths of edges whose dihedral angles are α,β and γ respectively. Then*

$$\frac{\tan\alpha}{\tan L_\alpha}=\frac{\tan\beta}{\tan L_\beta}=\frac{\tan\gamma}{\tan L_\gamma}=T.$$

where T is a negative root of the equation

$$T^4+(L^2+M^2+N^2+1)T^2-L^2M^2N^2=0$$

and $L=\tan\alpha,\ M=\tan\beta,\ N=\tan\gamma$.

Proof. By Lemma 1, Lemma 3 and the condition $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$, we have

$$\tan\alpha=-C\sqrt{A^2+1}$$

and

$$\tan L_\alpha=\frac{\sqrt{A^2+1}}{AB}.$$

Then

$$\frac{\tan\alpha}{\tan L_\alpha}=-ABC=T,$$

where T is the same as in Lemma 2. By the same way, we obtain the other equalities of the theorem. □

The parameter T from Theorem 4 can be presented in the form $T=\tan\theta$, where $\frac{\pi}{2}<\theta<\pi$. The value θ plays the significant role in the studying of metric structure of Lambert cube and is called a *principal parameter.*

The following theorem is a direct corollary of Theorem 4.

Theorem 5 (The Sine-Cosine Rule). *Let $Q(\alpha,\beta,\gamma)$, $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$ be a spherical Lambert cube. Denote by L_α, L_β and L_γ lengths of edges whose dihedral angles α,β and γ, respectively. Then*

$$\frac{\sin\alpha}{\sin L_\alpha}\cdot\frac{\sin\beta}{\sin L_\beta}\cdot\frac{\cos\gamma}{\cos L_\gamma}=-1.$$

Proof. From Theorem 4, we have

$$\tan L_\alpha=\frac{L}{T},\quad \tan L_\beta=\frac{M}{T},\quad \tan\ L_\gamma=\frac{N}{T}.$$

Hence

$$\sin^2 L_\alpha=\frac{L^2}{T^2+L^2},\quad \sin^2 L_\beta=\frac{M^2}{T^2+M^2},\quad \cos^2 L_\gamma=\frac{T^2}{T^2+N^2}$$

and by Lemma 2, we have

$$\frac{\sin^2\alpha}{\sin^2 L_\alpha}\cdot\frac{\sin^2\beta}{\sin^2 L_\beta}\cdot\frac{\cos^2\gamma}{\cos^2 L_\gamma} = \frac{(T^2+L^2)(T^2+M^2)(T^2+N^2)}{(1+L^2)(1+M^2)(1+N^2)T^2} = 1. \tag{2.3}$$

Since $\frac{\pi}{2} < \alpha, \beta, \gamma < \pi$ and the values $0 < L_\alpha, L_\beta, L_\gamma < \frac{\pi}{2}$, the theorem is proved. □

Remark 1. *Up to cyclic permutation of angles α, β and γ, Theorem 5 contains three independent equations which are sufficient to determine L_α, L_β and L_γ in terms of α, β and γ.*

3 The volume of spherical Lambert cube

From now on, for an arbitrary function $\phi : (\frac{\pi}{2}, \pi) \to \mathcal{R}$ we shall write $\phi(\frac{\pi}{2})$ and $\phi(\pi)$ instead of $\phi(\frac{\pi}{2}+0)$ and $\phi(\pi-0)$, respectively, if the corresponding values have a sense.

Let $V = V(\alpha, \beta, \gamma)$ be the volume of spherical Lambert cube $Q(\alpha, \beta, \gamma)$ with essential angles α, β and γ.

Notice first that $V(\alpha, \beta, \gamma) \to 0$, as $\alpha, \beta, \gamma \to \frac{\pi}{2}+0$.

It can be justified by the following arguments. Each face of $Q(\alpha, \beta, \gamma)$ is a Lambert quadrilateral with essential angles α, β and γ tending to $\frac{\pi}{2}$. Then there is a pair of opposite sides in the quadrilateral whose lengths tend to zero. We have decrease in the dimension in each face of $Q(\alpha, \beta, \gamma)$ and hence of $Q(\alpha, \beta, \gamma)$ itself. It means the volume tends to zero.

Thus, we shall consider the equality

$$V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0 \tag{3.1}$$

as a true one.

Let L_α, L_β and L_γ be the lengths of edges whose dihedral angles α, β and γ, respectively. By the Schläfli formula (see [Sh], [M]), we have

$$\frac{\partial V}{\partial \alpha} = \frac{1}{2}\, L_\alpha\,, \qquad \frac{\partial V}{\partial \beta} = \frac{1}{2}\, L_\beta\,, \qquad \frac{\partial V}{\partial \gamma} = \frac{1}{2}\, L_\gamma\,. \tag{3.2}$$

Proposition 1. *The volume of spherical Lambert cube $Q(\alpha, \beta, \gamma)$, $\frac{\pi}{2} < \alpha, \beta, \gamma < \pi$ is given by the formula*

$$V(\alpha, \beta, \gamma) = \frac{1}{4}\int\limits_{-\infty}^{T} \log\frac{(t^2+L^2)(t^2+M^2)(t^2+N^2)}{(1+L^2)(1+M^2)(1+N^2)t^2}\,\frac{dt}{t^2-1}, \tag{3.3}$$

where T is a negative root of the equation

$$T^4 + (L^2+M^2+N^2+1)T^2 - L^2M^2N^2 = 0,$$

$L = \tan\alpha,\ M = \tan\beta$ *and* $N = \tan\gamma$.

Proof. By the above arguments the volume function $V = V(\alpha, \beta, \gamma)$ satisfies (3.2) with initial data (3.1). We set

$$\widetilde{V} = \int\limits_{-\infty}^{T} F(t, L, M, N)\, dt,$$

where

$$F(t, L, M, N) = \frac{1}{4(t^2-1)} \log \frac{(t^2+L^2)(t^2+M^2)(t^2+N^2)}{(1+L^2)(1+M^2)(1+N^2)t^2}$$

and show that $\widetilde{V}$ satisfies conditions (3.1) and (3.2) . Then $V = \widetilde{V}$ and the proposition is proven.

By the Leibnitz formula we obtain

$$\frac{\partial \widetilde{V}}{\partial \alpha} = F(T, L, M, N) \frac{\partial T}{\partial \alpha} + \int\limits_{-\infty}^{T} \frac{\partial F(t, L, M, N)}{\partial L} \frac{\partial L}{\partial \alpha} dt \tag{3.4}$$

By (2.3) we have $F(T, L, M, N) = 0$. Moreover, since $\alpha = \arctan L$, we have

$$\frac{\partial L}{\partial \alpha} = 1 + L^2 \quad \text{and} \quad \frac{\partial F(t, L, M, N)}{\partial L} \frac{\partial L}{\partial \alpha} = -\frac{L}{2\,(t^2+L^2)}.$$

Hence, by Tangent Rule, we obtain from (3.4)

$$\frac{\partial \widetilde{V}}{\partial \alpha} = \int\limits_{-\infty}^{T} -\frac{L dt}{2\,(t^2+L^2)} = \frac{1}{2} \arctan \frac{L}{T} = \frac{1}{2}\, L_{\alpha}.$$

The equalities

$$\frac{\partial \widetilde{V}}{\partial \beta} = \frac{1}{2}\, L_{\beta} \quad \text{and} \quad \frac{\partial \widetilde{V}}{\partial \gamma} = \frac{1}{2}\, L_{\gamma}$$

can be obtained by the similar way.

To verify the initial condition (3.1) for function $\widetilde{V}$ we remark that $L, M, N \to +\infty$ as $\alpha, \beta, \gamma \to \frac{\pi}{2} + 0$, hence $T \to -\infty$. From the convergence of integral (3.3), we have $\widetilde{V} \to 0$ as $T \to -\infty$. □

4 The proof of the main theorem

Proof of Theorem 1. By Proposition 1, we have

$$\begin{aligned} V(\alpha, \beta, \gamma) &= \frac{1}{4} \int\limits_{-\infty}^{T} \log \left(\frac{t^2+L^2}{1+L^2} \frac{t^2+M^2}{1+M^2} \frac{t^2+N^2}{1+N^2} : \frac{t^2+0^2}{1+0^2} \right) \frac{dt}{t^2-1} \\ &= \frac{1}{4} \left(I(L,T) + I(M,T) + I(N,T) - I(0,T) \right) \end{aligned} \tag{4.1}$$

where

$$I(L,T) = \int\limits_{-\infty}^{T} \log\left(\frac{t^2+L^2}{1+L^2}\right)\frac{dt}{t^2-1}.$$

Let $T = \tan\theta$, $L = \tan\alpha$, $M = \tan\beta$ and $N = \tan\gamma$. Then under substitution $t = \tan\tau$ we obtain

$$\begin{aligned}
I(L,T) &= \int\limits_{\frac{\pi}{2}}^{\theta} \log\left(\frac{\tan^2\tau+\tan^2\alpha}{1+\tan^2\alpha}\right)\frac{d\tau}{\cos^2\tau\left(\tan^2\tau-1\right)} \\
&= \int\limits_{\theta}^{\frac{\pi}{2}} \frac{\log\left(1-\cos 2\tau\cos 2\alpha\right)d\tau}{\cos 2\tau} - \int\limits_{\theta}^{\frac{\pi}{2}} \frac{\log\left(1+\cos 2\tau\right)d\tau}{\cos 2\tau} \\
&= \delta(\alpha,\theta) - \delta\left(\frac{\pi}{2},\theta\right).
\end{aligned}$$

Hence, (4.1) yields

$$\begin{aligned}
V(\alpha,\beta,\gamma) &= \frac{1}{4}\left(I(L,T)+I(M,T)+I(N,T)-I(0,T)\right) \\
&= \frac{1}{4}\left(\delta(\alpha,\theta)+\delta(\beta,\theta)+\delta(\gamma,\theta)-2\delta\left(\frac{\pi}{2},\theta\right)-\delta(0,\theta)\right).
\end{aligned}$$

□

5 Explicit volume calculations

This section is devoted to the explicit volume calculations in some particular cases.

Proposition 2. *Let* $\alpha,\beta,\gamma,\ \frac{\pi}{2} < \alpha,\beta,\gamma < \pi$ *are related by the following equation*

$$\cos^2\alpha + \cos\beta^2 + \cos\gamma^2 = 1.$$

Then the volume of a spherical Lambert cube $Q(\alpha,\beta,\gamma)$ *is given by the formula*

$$V(\alpha,\beta,\gamma) = \frac{1}{4}\left(\frac{\pi^2}{2} - (\pi-\alpha)^2 - (\pi-\beta)^2 - (\pi-\gamma)^2\right).$$

Proof. Remind that

$$\tan\theta = -\sqrt{-p+\sqrt{p^2+L^2M^2N^2}},$$

where

$$p = \frac{L^2+M^2+N^2+1}{2}$$

and $L = \tan\alpha,\ M = \tan\beta,\ N = \tan\gamma$. Let

$$\cos^2\alpha + \cos\beta^2 + \cos\gamma^2 = 1.$$

By applying the elementary trigonometry we have

$$2p + 1 = \frac{1}{\cos^2\alpha} + \frac{1}{\cos^2\beta} + \frac{1}{\cos^2\gamma} - 1$$

and

$$\begin{aligned} L^2M^2N^2 &= \frac{1}{\cos^2\alpha} + \frac{1}{\cos^2\beta} + \frac{1}{\cos^2\gamma} - 1 \\ &\quad -\frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1}{\cos^2\alpha\cos^2\beta\cos^2\gamma} \\ &= 2p + 1. \end{aligned}$$

Hence $\tan\theta = -1$ and consequently, $\theta = \frac{3\pi}{4}$. By Theorem 1 and Corollary 2(iii) (see Appendix), we obtain

$$\begin{aligned} V(\alpha,\beta,\gamma) &= \frac{1}{4}\left(\delta(\alpha,\theta) + \delta(\beta,\theta) + \delta(\gamma,\theta) - 2\delta\left(\frac{\pi}{2},\theta\right) - \delta(0,\theta)\right) \\ &= \frac{1}{4}\left(\frac{\pi^2}{2} - (\pi-\alpha)^2 - (\pi-\beta)^2 - (\pi-\gamma)^2\right). \end{aligned}$$

□

In particular, we have the following.

Corollary 1.

$$V\left(\frac{2\pi}{3},\frac{2\pi}{3},\frac{3\pi}{4}\right) = \frac{31}{576}\pi^2.$$

Consider the process of degenerating of Lambert cube $Q(\alpha,\beta,\gamma)$ as $\gamma \to \pi$. Let L^*_γ be the length of the edge opposite to the edge with essential angle γ. The straightforward calculations based on Theorem 5 show that L^*_γ tends to zero as $\gamma \to \pi$. As a result the singular Lambert cube $Q(\alpha,\beta,\pi)$ can be defined as a cone under the plane spherical quadrilateral with the angles $\frac{\pi}{2},\alpha,\frac{\pi}{2},\beta$ (Fig 2) .

Applying Theorem 1 and Corollary 2(iv) in case $\gamma = \pi$, we obtain the following

Proposition 3. *The volume of a singular Lambert cube $Q(\alpha,\beta,\pi)$, $\frac{\pi}{2} < \alpha,\beta < \pi$ is given by the formula*

$$V(\alpha,\beta,\pi) = (\alpha+\beta-\pi)\,\pi.$$

6 Appendix

This section is devoted to elementary properties of the function $\delta(\alpha,\theta)$ and its relations to the Dilogarithm, Lobachevskij and Schläfli functions.

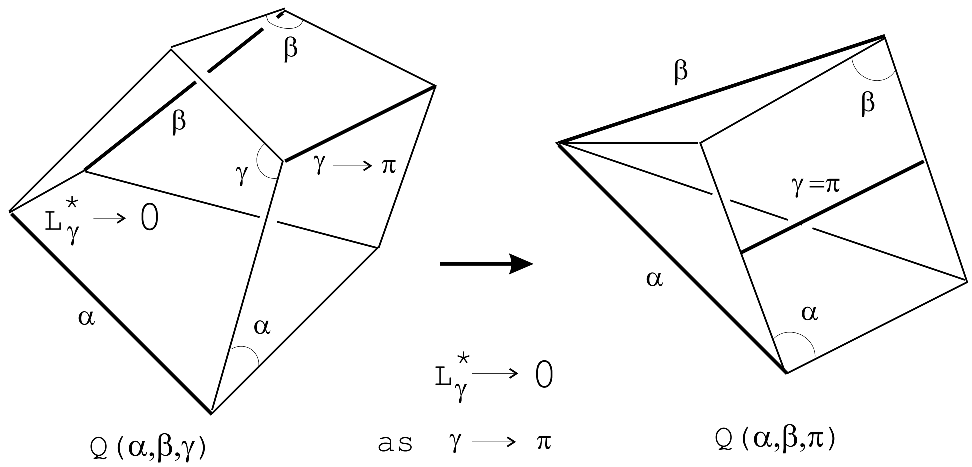

Figure 2:

6.1 Elementary properties of $\delta(\alpha,\theta)$

Now we list the following elementary properties of the function $\delta(\alpha,\theta)$.

Proposition 4. *The function*

$$\delta(\alpha,\theta)=\int_{\theta}^{\frac{\pi}{2}}\log\left(1-\cos 2\alpha\cos 2\tau\right)\frac{d\tau}{\cos 2\tau}$$

satisfies the following properties

(*i*) $\delta(\alpha,\theta)$ *is continuous for all* $(\alpha,\theta)\in R^2$ *and differentiable with respect to* (α,θ) *for* $\alpha\neq\pi/2+k\pi, k\in Z$;

(*ii*) $\delta(\alpha,\theta)$ *is even with respect to* α *and satisfies the relation* $\delta(\alpha,\theta)+\delta(\alpha,-\theta)=2\delta(\alpha,0)$ *for all* $(\alpha,\theta)\in R^2$;

(*iii*) $\delta(\alpha,\theta)=\delta(\pi-\alpha,\theta)$ *and* $\delta(\alpha,\theta)=-\,\delta(\alpha,\pi-\theta)$;

(*iv*) $\delta(\alpha,\theta)$ *is* π*-periodic with respect to* α ;

(*v*) $\delta(\alpha,\theta)$ *is linear periodic with respect to* θ *in the following sense*

$\delta(\alpha,\theta+k\pi)=\delta(\alpha,\theta)-2k\delta(\alpha,0)\,,k\in Z$

and

$\delta(\alpha,0)=\pi^2/4-\left|\pi^2/2-\alpha\pi\right|,0\leq\alpha\leq\pi$;

(*vi*) *Let* $\widetilde{\delta}(\alpha,\theta)=\delta(\alpha,\theta)+(2\theta/\pi-1)\,\delta(\alpha,0)$. *Then*

a) $\widetilde{\delta}(\alpha,\theta)$ *is even and* π*-periodic with respect to* α;

b) $\widetilde{\delta}(\alpha,\theta)$ *is odd and* π*-periodic with respect to* θ;

c) $\left|\widetilde{\delta}(\alpha,\theta)\right|\leq\pi^2/4$ *and* $\widetilde{\delta}(\pi/2,3\pi/4)=\pi^2/4$.

The properties $(i)-(iv)$ can be deduced directly from the definition of $\delta(\alpha,\theta)$. To prove the second statement of the property (ii) we observe that the equality is evidently true for $\theta=0$. By definition of $\delta(\alpha,\theta)$, the derivatives of the both sides of the equality with respect to θ are equal to zero. To see the equality

$$\delta(\alpha,\theta+k\pi)=\delta(\alpha,\theta)-2k\delta(\alpha,0)\,,k\in Z$$

we notice that the derivatives of the both sides with respect to θ are equal. To establish the property we have to check the equality for one fixed value of θ. Let $\theta=0$. The equality holds for $k=0$. For any $k\in Z$ from properties (iii) and (ii) we have

$$\delta(\alpha,(k+1)\pi)=-\delta(\alpha,-k\pi)$$

and

$$-\delta(\alpha,-k\pi)=\delta(\alpha,k\pi)-2\delta(\alpha,0)\,.$$

Hence

$$\delta(\alpha,(k+1)\pi)=\delta(\alpha,k\pi)-2\delta(\alpha,0)$$

and the first statement of (v) follows by induction. The value of $\delta(\alpha,0)$ can be taken from the corollary 2 bellow. The property (vi) follows from properties $(i)-(iv)$.

For explicit calculations it is more convenient to use the following alternative form of $\delta(\alpha,\theta)$.

Proposition 5. *Let* $\pi/2<\alpha,\theta<\pi$*, then the function* $\delta(\alpha,\theta)$ *can be represented in the form*

$$\delta(\alpha,\theta)=2\int_{\frac{3\pi}{4}}^{\alpha} arccot\,\left(\frac{\cot\nu}{\cot\theta}\right)d\nu,$$

where $0<arccot\ x<\pi$ *for all* x*.*

Proof. By definition, we have

$$\delta(\alpha,\theta)=\int_{\theta}^{\frac{\pi}{2}}\log(1-\cos 2\alpha\cos 2\tau)\frac{d\tau}{\cos 2\tau}.$$

Hence

$$\begin{aligned}\frac{\partial\delta(\alpha,\theta)}{\partial\alpha}&=\int_{\theta}^{\frac{\pi}{2}}\frac{2\sin 2\alpha\cos 2\tau}{1-\cos 2\alpha\cos 2\tau}\frac{d\tau}{\cos 2\tau}\\&=\pi-2\arctan(\cot\alpha\tan\theta)=2\,arccot\,(\cot\alpha\tan\theta)\,.\end{aligned}$$

Notice that

$$\delta\left(\frac{3\pi}{4},\theta\right)=0.$$

Hence

$$\delta(\alpha,\theta)=\int\limits_{\frac{3\pi}{4}}^{\alpha}\frac{\partial\delta(\nu,\theta)}{\partial\nu}d\nu=2\int\limits_{\frac{3\pi}{4}}^{\alpha}arccot\left(\frac{\cot\nu}{\cot\theta}\right)d\nu.$$

□

We use the proposition to calculate the value of $\delta(\alpha,\theta)$ in some cases. The results of our calculations are collected in the following corollary.

Corollary 2. *For any* α, $0\le\alpha\le\pi$ *we have*

(i) $\delta(\alpha,0)=\pi\left(\frac{\pi}{4}-\left|\frac{\pi}{2}-\alpha\right|\right)$,

(ii) $\delta\left(\alpha,\frac{\pi}{4}\right)=\left(\frac{\pi}{2}-\left|\frac{\pi}{2}-\alpha\right|\right)^2-\frac{\pi^2}{16}$,

(iii) $\delta\left(\alpha,\frac{3\pi}{4}\right)=\frac{\pi^2}{16}-\left(\frac{\pi}{2}-\left|\frac{\pi}{2}-\alpha\right|\right)^2$,

(iv) $\delta(\alpha,\pi)=\pi\left(\left|\alpha-\frac{\pi}{2}\right|-\frac{\pi}{4}\right)$.

The result easy follows from Proposition 5 and properties $(i)-(iii)$ of function δ.

6.2 The relation $\delta(\alpha,\theta)$ to the Schläfli function

Let $T(\alpha,\beta,\gamma)$ be a double-rectangular spherical, Euclidean or hyperbolic tetrahedron with dihedral angles $\pi/2-\alpha$, β and $\pi/2-\gamma$. Define the Schläfli function by the following formula

$$S(\alpha,\beta,\gamma)=\sum_{1}^{\infty}\frac{(-X)^n}{n^2}(\cos 2n\alpha-\cos 2n\beta+\cos 2n\gamma-1)-\alpha^2+\beta^2-\gamma^2,$$

where

$$X=\frac{\sin\alpha\sin\gamma-D}{\sin\alpha\sin\gamma+D},$$

$$D=\sqrt{\cos^2\alpha\cos^2\gamma-\cos^2\beta},$$

$$0\le\alpha\le\frac{\pi}{2},\qquad 0\le\beta\le\pi,\qquad 0\le\gamma\le\frac{\pi}{2}$$

(see Coxeter [C] for details). It was shown by Schläfli (1898) that in the spherical case ($\cos^2\alpha\cos^2\gamma>\cos^2\beta$) the volume of $T(\alpha,\beta,\gamma)$ and $S(\alpha,\beta,\gamma)$ are related by

$$4Vol(T(\alpha,\beta,\gamma))=S(\alpha,\beta,\gamma).$$

In Euclidean case ($\cos^2\alpha\cos^2\gamma=\cos^2\beta$, see [$C$], *p.* 16)

$$S(\alpha,\beta,\gamma)=0.$$

In hyperbolic case ($\cos^2\alpha\cos^2\gamma<\cos^2\beta$, $\alpha,\gamma<\beta$), Coxeter notice ([C] , *p.* 27), that

$$iS(\alpha,\beta,\gamma)=4Vol(T(\alpha,\beta,\gamma)).$$

From the other side, we know (see, for instance [C], p. 23 or [V], p. 125) that

$$\begin{aligned}4Vol(T(\alpha,\beta,\gamma)) &= \Lambda(\alpha+\theta)+\Lambda(-\alpha+\theta)-\Lambda(\beta+\theta)-\Lambda(-\beta+\theta)\\ &\quad +\Lambda(\gamma+\theta)+\Lambda(-\gamma+\theta)+2\Lambda(\theta),\end{aligned}$$

where θ is defined by

$$\tan\theta = \frac{i\sin\alpha\sin\gamma}{D} = \frac{\sin\alpha\sin\gamma}{\sqrt{\cos^2\beta-\cos^2\alpha\cos^2\gamma}}.$$

Hence, the function $S(\alpha,\beta,\gamma)$ is connected with Lobachevskij function by the relation

$$iS(\alpha,\beta,\gamma) = -\Delta(\alpha,\theta)+\Delta(\beta,\theta)-\Delta(\gamma,\theta)+\Delta(0,\theta),$$

where $\Delta(\alpha,\theta)=\Lambda(\alpha+\theta)-\Lambda(\alpha-\theta)$. The following proposition gives relation $S(\alpha,\beta,\gamma)$ and $\delta\,(\alpha,\theta)$ in spherical case.

Proposition 6. *Let* $0\le\alpha,\beta,\gamma\le\frac{\pi}{2}$ *and*

$$\tan\theta = \frac{\sin\alpha\sin\gamma}{\sqrt{\cos^2\alpha\cos^2\gamma-\cos^2\beta}},$$

where $\cos^2\alpha\cos^2\gamma>\cos^2\beta$. *Then*

$$S(\alpha,\beta,\gamma) = -\delta(\alpha,\theta)+\delta(\beta,\theta)-\delta(\gamma,\theta)+\delta(0,\theta).$$

Proof. Let $T(\alpha,\beta,\gamma)$ be a double-rectangular spherical tetrahedron with essential dihedral angles $\frac{\pi}{2}-\alpha$, β and $\frac{\pi}{2}-\gamma$. By the Schläfli theorem $S(\alpha,\beta,\gamma) = 4Vol\,(T(\alpha,\beta,\gamma))$. We find $Vol\,(T(\alpha,\beta,\gamma))$ in a few steps.

1 step. The Tangent Rule (see, for instance [V], p.125)

$$\frac{\tan\alpha}{\tan a} = \frac{\tan\beta}{\tan b} = \frac{\tan\gamma}{\tan c} = T,$$

where a,b and c are the spherical lengths of edges correspondent to dihedral angles $\frac{\pi}{2}-\alpha,\beta$ and $\frac{\pi}{2}-\gamma$ respectively,

$$T = \frac{\sin\alpha\sin\gamma}{D}$$

and $D=\sqrt{\cos^2\alpha\cos^2\gamma-\cos^2\beta}$.

2 step. We note that T is a root of the following biquadratic equation

$$\frac{1+A^2}{T^2+A^2}\,\frac{T^2+B^2}{1+B^2}\,\frac{1+C^2}{T^2+C^2}T^2 = 1, \tag{6.1}$$

where $A=\tan\alpha$, $B=\tan\beta$ and $C=\tan\gamma$. Indeed, the above equation has four roots $T_{1,2}=\pm1$ and $T_{3,4}=\pm\sin\alpha\sin\gamma/D$. The equation (6.1) has the following geometrical sense. By the Tangent Rule, we have $\cos^2 a = T^2/\left(T^2+A^2\right)$, $\cos^2 b = T^2/\left(T^2+B^2\right)$ and $\cos^2 c = T^2/\left(T^2+C^2\right)$. Hence (6.1) gives

$$\frac{\cos^2 a}{\cos^2\alpha}\cdot\frac{\cos^2\beta}{\cos^2 b}\cdot\frac{\cos^2 c}{\cos^2\gamma}=1$$

and after the suitable choosing the sign we obtain

$$\frac{\cos\beta}{\cos b}=\frac{\cos\alpha}{\cos a}\cdot\frac{\cos\gamma}{\cos c}. \tag{6.2}$$

This is the Cosine Rule for double-rectangular spherical tetrahedron $T(\alpha,\beta,\gamma)$.

3 step. Let $V=V(\alpha,\beta,\gamma)$ be the volume of $T(\alpha,\beta,\gamma)$. By the Schläfli formula we have

$$\frac{\partial V}{\partial\alpha}=-\frac{a}{2},\quad \frac{\partial V}{\partial\beta}=\frac{b}{2},\quad \frac{\partial V}{\partial\gamma}=-\frac{c}{2}. \tag{6.3}$$

It follows from the Tangent Rule that

$$V\to 0 \quad\text{as}\quad T\to+\infty. \tag{6.4}$$

4 step. It is easy to check (see the proof of Proposition 1) that the function

$$\widetilde{V}(\alpha,\beta,\gamma)=\frac{1}{4}\int\limits_T^{+\infty}\log\frac{(1+A^2)(t^2+B^2)(1+C^2)t^2}{(t^2+A^2)(1+B^2)(t^2+C^2)}\frac{dt}{t^2-1}$$

is a solution of (6.3) with the initial condition (6.4). Hence $Vol\,(T(\alpha,\beta,\gamma))=\widetilde{V}(\alpha,\beta,\gamma)$. Since $S(\alpha,\beta,\gamma)=4Vol\,(T(\alpha,\beta,\gamma))$ we have

$$\begin{aligned}S(\alpha,\beta,\gamma) &= \int\limits_T^{+\infty}\log\frac{(1+A^2)(t^2+B^2)(1+C^2)t^2}{(t^2+A^2)(1+B^2)(t^2+C^2)}\frac{dt}{t^2-1}\\ &= -I\,(T,A)+I\,(T,B)-I\,(T,C)+I\,(T,0)\,,\end{aligned}$$

where

$$I\,(T,A)=\int\limits_T^{+\infty}\log\frac{t^2+A^2}{1+A^2}\frac{dt}{t^2-1}$$

Let $T=\tan\theta$. Then under substitution $t=\tan\tau$ we obtain

$$\begin{aligned}I\,(T,A) &= \\ &\int\limits_\theta^{\frac{\pi}{2}}\frac{\log\left(1-\cos 2\tau\cos 2\alpha\right)d\tau}{\cos 2\tau}-\int\limits_\theta^{\frac{\pi}{2}}\frac{\log\left(1+\cos 2\tau\right)d\tau}{\cos 2\tau}\\ &= \delta\,(\alpha,\theta)-\delta\left(\frac{\pi}{2},\theta\right).\end{aligned}$$

Hence

$$S(\alpha,\beta,\gamma)=-\delta(\alpha,\theta)+\delta(\beta,\theta)-\delta(\gamma,\theta)+\delta(0,\theta).$$

□

6.3 The relation $\delta(\alpha,\theta)$ to the Lobachevskij function

Let

$$\Delta(\alpha,\theta) = -\int_{\alpha-\theta}^{\alpha+\theta} \log|2\sin t|\,dt = \Lambda(\alpha+\theta) - \Lambda(\alpha-\theta)$$

and

$$\delta(\alpha,\theta) = \int_{\theta}^{\frac{\pi}{2}} \log(1-\cos 2\alpha\cos 2\tau)\frac{d\tau}{\cos 2\tau}.$$

Proposition 7. *Let* $\alpha,\ \theta, \widetilde{\theta} \in R$, $|\widetilde{\theta}| < \pi/4$ *and* $\tan\widetilde{\theta} = \tanh\theta$. *Then*

$$i\,\delta\left(\alpha,\widetilde{\theta}\right) - i\,\delta(\alpha,0) = \Delta(\alpha,i\theta) - \Delta\left(\frac{\pi}{4},i\theta\right).$$

Proof. Let

$$F(\alpha,\theta) = i\,\delta\left(\alpha,\widetilde{\theta}\right) - i\,\delta(\alpha,0) - \Delta(\alpha,i\theta) + \Delta\left(\frac{\pi}{4},i\theta\right).$$

We have to show that $F(\alpha,\theta)$ satisfy the equation

$$\frac{\partial F(\alpha,\theta)}{\partial\theta} = 0, \tag{6.5}$$

with the initial condition $F(\alpha,0)=0$. It means $F(\alpha,\theta)=0$ and the proposition follows.

Note, that

$$\frac{\partial\widetilde{\theta}}{\partial\theta} = \frac{\cos^2\widetilde{\theta}}{\cosh^2\theta} = \frac{1-\tanh^2\theta}{1+\tan^2\widetilde{\theta}} = \frac{1-\tan^2\widetilde{\theta}}{1+\tan^2\widetilde{\theta}} = \cos 2\widetilde{\theta}. \tag{6.6}$$

From (6.6) and definition of $\delta(\alpha,\theta)$ we have

$$\begin{aligned} \frac{\partial\left(i\,\delta\left(\alpha,\widetilde{\theta}\right) - i\,\delta(\alpha,0)\right)}{\partial\theta} &= i\frac{\partial\delta\left(\alpha,\widetilde{\theta}\right)}{\partial\widetilde{\theta}}\frac{\partial\widetilde{\theta}}{\partial\theta} \\ &= -i\log\left(1-\cos 2\alpha\cos 2\widetilde{\theta}\right). \end{aligned} \tag{6.7}$$

From definition of $\Delta(\alpha,\theta)$ we deduce

$$\begin{aligned} \frac{\partial\left(-\Delta(\alpha,i\theta) + \Delta\left(\frac{\pi}{4},i\theta\right)\right)}{\partial\theta} &= i\log\left|\frac{\sin(\alpha+i\theta)\sin(\alpha-i\theta)}{\sin\left(\frac{\pi}{4}+i\theta\right)\sin\left(\frac{\pi}{4}-i\theta\right)}\right| \\ &= i\log\left|\frac{\cos 2i\theta - \cos 2\alpha}{\cos 2i\theta - \cos\frac{\pi}{2}}\right| \\ &= i\log\left|1-\frac{\cos 2\alpha}{\cos 2i\theta}\right| = i\log\left(1-\frac{\cos 2\alpha}{\cosh 2\theta}\right). \end{aligned}$$

Since

$$\cos 2\widetilde{\theta} = \frac{1-\tan^2\widetilde{\theta}}{1+\tan^2\widetilde{\theta}} = \frac{1-\tanh^2\theta}{1+\tanh^2\theta} = \frac{1}{\cosh 2\theta}$$

we obtain

$$\frac{\partial\left(-\Delta(\alpha,i\theta)+\Delta\left(\frac{\pi}{4},i\theta\right)\right)}{\partial\theta} = i\log\left(1-\cos 2\alpha\cos 2\widetilde{\theta}\right). \tag{6.8}$$

The equalities (6.7) and (6.8) give (6.5). The initial condition $F(\alpha,0)=0$ follows directly from definitions of $\delta(\alpha,\theta)$ and $\Delta(\alpha,\theta)$. □

6.4 The relation $\delta(\alpha,\theta)$ to the Dilogarithm

Here we consider the relation

$$\delta(\alpha,\theta) = \int\limits_{\theta}^{\frac{\pi}{2}} \log(1-\cos 2\alpha\cos 2\tau)\frac{d\tau}{\cos 2\tau}$$

to the Dilogarithm function ([Le], p. 292)

$$\mathrm{Li}_2\,(r,t) = -\frac{1}{2}\int\limits_0^r \log\left(1-2x\cos t+x^2\right)\frac{dx}{x}.$$

Recall that $\delta\left(\alpha,\frac{\pi}{4}\right)$ is a π-periodic function with respect to α and

$$\delta\left(\alpha,\frac{\pi}{4}\right) = \left(\frac{\pi}{2}-\left|\frac{\pi}{2}-\alpha\right|\right)^2 - \frac{\pi^2}{16}$$

for $0<\alpha<\pi$.

Proposition 8. *Let $\alpha\in R$ and $-\pi/4<\theta<3\pi/4$. Then*

$$\delta(\alpha,\theta)-\delta\left(\alpha,\frac{\pi}{4}\right) = \mathrm{Li}_2\left(\tan\left(\frac{\pi}{4}-\theta\right),\frac{\pi}{2}\right) - \mathrm{Li}_2\left(\tan\left(\frac{\pi}{4}-\theta\right),2\alpha\right).$$

Proof. Since

$$\mathrm{Li}_2\,(\tan\mu,\alpha) = -\int\limits_0^{\mu}\log\frac{1-\sin 2\mu\cos\alpha}{\cos^2\mu}\frac{d\mu}{\sin 2\mu}$$

for $-\pi/2<\mu<\pi/2$ we have

$$\mathrm{Li}_2\,(\tan\mu,\frac{\pi}{2}) - \mathrm{Li}_2\,(\tan\mu,2\alpha) = \tag{6.9}$$

$$\int\limits_0^{\mu}\log(1-\sin 2\mu\cos 2\alpha)\frac{d\mu}{\sin 2\mu} = -\delta\left(\alpha,\mu-\frac{\pi}{4}\right)+\delta\left(\alpha,-\frac{\pi}{4}\right).$$

The property (ii) of function $\delta(\alpha, \mu)$ gives

$$\delta\left(\alpha, \frac{\pi}{4} - \mu\right) - \delta\left(\alpha, \frac{\pi}{4}\right) = -\delta\left(\alpha, \mu - \frac{\pi}{4}\right) + \delta\left(\alpha, -\frac{\pi}{4}\right) \tag{6.10}$$

and under substitution $\theta = \pi/4 - \mu$ we obtain from (6.9) and (6.10)

$$\delta(\alpha, \theta) - \delta\left(\alpha, \frac{\pi}{4}\right) = \mathrm{Li}_2\left(\tan\left(\frac{\pi}{4} - \theta\right), \frac{\pi}{2}\right) - \mathrm{Li}_2\left(\tan\left(\frac{\pi}{4} - \theta\right), 2\alpha\right)$$

for $-\pi/4 < \theta < 3\pi/4$. □

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