UNIFORMIZATION OF SOME RIEMANN SURFACES WITH NODES

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The present article is devoted to studying the spaces of deformations of Kleinian groups representing Riemann surfaces with nodes and introduced by Bers in [1]. Nodes are the simplest case of degeneration of a Riemann surface when the surface is compressed along several simple closed loops.

We study these deformations on the example of Kleinian groups with a simple geometric structure, the extended Schottky groups of type (g, s, m).

In the article we construct the so-called augmented space $ST^*_{(g,s,m)}$ of extended Schottky groups and demonstrate that this space is a domain in $\overline{\mathbb{C}}^n$ such that to each point in this domain there corresponds some Riemann surface with nodes.

The augmented spaces for extended Schottky groups of types $(g, 0, 0)$ and $(g, 0, m)$ were considered in the articles [1–3].

§1. Definitions and Preliminaries

We let $\mathbb{M}$ stand for the group of all conformal automorphisms of the extended complex plane $\overline{\mathbb{C}}$.

A group $G \subset \mathbb{M}$ is called an *extended Schottky group of type* (g, s, m) with standard generators $T_1, \dots, T_g$, $W_1, \dots, W_s$, $U_1, V_1, \dots, U_m, V_m$ and defining curves $C_1, C_1', \dots, C_g, C_g'$, $B_1, B_1', \dots, B_s, B_s'$, $L_1, \dots, L_m$, where L_k is a topological quadrilateral with sides K_k, K_k', P_k, and P_k', $k = 1, \dots, m$, if the following conditions are satisfied:

(a) all defining curves are simple closed curves in $\overline{\mathbb{C}}$; the curves B_j and B_j' have one common point p_j, $j = 1, \dots, s$; all other curves are pairwise disjoint; and all curves jointly bound a $(2g+s+m)$-connected domain D such that

$$T_i(D) \cap D = W_j(D) \cap D = U_k(D) \cap D = V_k(D) \cap D = \varnothing;$$

(b) $T_i(C_i) = C_i'$, $i = 1, \dots, g$; $W_j(B_j) = B_j'$, $j = 1, \dots, s$; $U_k(K_k) = K_k'$, $V_k(P_k) = P_k'$, $k = 1, \dots, m$;

(c) U_k and V_k are commuting parabolic elements generating the Kleinian group $\langle U_k, V_k \rangle$, $k = 1, \dots, m$;

(d) W_j is a parabolic mapping with the fixed point p_j, $j = 1, \dots, s$.

An extended Schottky group G of type (g, s, m) with some ordered system of standard generators is referred to as a *marked extended Schottky group of type* (g, s, m).

We say that two marked extended Schottky groups of type (g, s, m)

$$G = \langle T_1, \dots, T_g,\ W_1, \dots, W_s,\ U_1, V_1, \dots, U_m, V_m \rangle,$$

$$\widetilde{G} = \langle \widetilde{T}_1, \dots, \widetilde{T}_g,\ \widetilde{W}_1, \dots, \widetilde{W}_s,\ \widetilde{U}_1, \widetilde{V}_1, \dots, \widetilde{U}_m, \widetilde{V}_m \rangle$$

are *equivalent* if there is a Möbius transformation B such that

$$BT_iB^{-1} = \widetilde{T}_i, \quad BW_jB^{-1} = \widetilde{W}_j, \quad BU_kB^{-1} = \widetilde{U}_k, \quad BV_kB^{-1} = \widetilde{V}_k,$$
$$i = 1, \dots, g, \quad j = 1, \dots, s, \quad k = 1, \dots, m.$$

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We denote the set of all equivalence classes of marked extended Schottky groups of type (g,s,m) by $S_{(g,s,m)}$. We endow $S_{(g,s,m)}$ with a topology as follows: a sequence $[G_n] \in S_{(g,s,m)}$ converges to $[G] \in S_{(g,s,m)}$ if and only if there are marked extended Schottky groups of type (g,s,m)

$$\langle T_1^{(n)},\dots,T_g^{(n)},\ W_1^{(n)},\dots,W_s^{(n)},\ U_1^{(n)},V_1^{(n)},\dots,U_m^{(n)},V_m^{(n)}\rangle \in [G_n]$$

and a marked extended Schottky group of type (g,s,m)

$$\langle T_1,\dots,T_g,\ W_1,\dots,W_s,\ U_1,V_1,\dots,U_m,V_m\rangle \in [G]$$

such that

$$T_i^{(n)} \to T_i,\quad W_j^{(n)} \to W_j,\quad U_k^{(n)} \to U_k,\quad V_k^{(n)} \to V_k,$$
$$i=1,\dots,g,\quad j=1,\dots,s,\quad k=1.\dots,m,\ \text{as } n\to\infty$$

in the topology of uniform convergence of mappings on the Riemann sphere $\overline{\mathbb{C}}$ ($[G]$ is the equivalence class of a group G).

We call the so-defined topological space $S_{(g,s,m)}$ the *space of extended Schottky groups of type* (g,s,m) or simply the *Schottky space of type* (g,s,m).

It was shown in [4] that we can endow the space $S_{(g,s,m)}$ with the structure of a complex manifold by embedding $S_{(g,s,m)}$ into $\overline{\mathbb{C}}^{3g+3m+2s-3}$. We denote the image of $S_{(g,s,m)}$ under this embedding by $ST_{(g,s,m)}$.

For definiteness, we shall assume that $\tau \in ST_{(g,s,m)}$ looks as follows (in the case when $g\ge 2$, $s\ge 0$, and $m \ge 0$):

$$\tau = (a_3,\dots,a_g,\ b_2,\dots,b_g,\ \lambda_1,\dots,\lambda_g,\ c_1,\dots,c_s, w_1,\dots,w_s,\ d_1,\dots,d_m,\ u_1,\dots,u_m,\ v_1,\dots,v_m),$$

where a_i and b_i are the fixed points and λ_i^{-1} is the factor of the loxodromic mapping T_i ($0<|\lambda_i|<1$); c_j and w_j are the fixed point and the radius of the isometric circle of the parabolic mapping W_j; and d_k and u_k, v_k are the fixed point and the radii of isometric circles of the parabolic mappings U_k and V_k.

We denote by $\partial ST_{(g,s,m)}$ the boundary of $ST_{(g,s,m)}$ in $\overline{\mathbb{C}}^{3g+3m+2s-3}$ and denote by $\delta ST_{(g,s,m)}$ the set of $\tau \in \partial ST_{(g,s,m)}$ satisfying at least one of the following conditions:

(1) one of the parameters w_j or u_k equals zero or infinity, $j\in\{1,\dots,s\}$, $k\in\{1,\dots,m\}$;

(2) one of the factors λ_i equals zero, $i\in\{1,\dots,g\}$;

(3) one of the parameters v_k is real or equals infinity, $k\in\{1,\dots,m\}$;

(4) two fixed points in the set $\{a_1,\dots,a_g,\ b_1,\dots,b_g,\ c_1,\dots,c_s,\ d_1,\dots,d_m\}$ coincide.

Observe that $\delta ST_{(g,s,m)}$ is the intersection of $\partial ST_{(g,s,m)}$ with finitely many analytic hypersurfaces and therefore has positive real codimension in $\partial ST_{(g,s,m)}$.

The group $G(\tau)$ is soundly defined for every point $\tau \in \partial ST_{(g,s,m)} \setminus \delta ST_{(g,s,m)}$. As it was demonstrated in [4], such a group is discrete and isomorphic to an extended Schottky group of type (g,s,m), and either is not Kleinian or contains random parabolic elements.

§ 2. Construction of the Augmented Schottky Space

In this section we construct the so-called augmented space of extended Schottky groups of type (g,s,m). We obtain this space by adjoining some points of $\overline{\mathbb{C}}^{3g+3m+2s-3}$ to $ST_{(g,s,m)}$. For definiteness, we suppose that $g\ge 2$, $s\ge 0$, and $m\ge 0$. We denote the coordinates of a point τ and the generators of the group $G(\tau)$ by $a_i(\tau)$, $b_i(\tau)$, $\lambda_i(\tau)$, $c_j(\tau)$, $w_j(\tau)$, $d_k(\tau)$, $u_k(\tau)$, $v_k(\tau)$ and $T_i(\tau,\hat{\ })$, $W_j(\tau,\hat{\ })$, $U_k(\tau,\hat{\ })$, $V_k(\tau,\hat{\ })$ respectively.

In particular, we shall consider those points $\tau \in \delta ST_{(g,s,m)}$ for which at least one of the parameters $w_j(\tau)$ and $u_k(\tau)$ vanishes, some of the factors $\lambda_i(\tau)$ are equal to zero, or two fixed points of some generator coincide.

The fulfillment of the above conditions for the elements $T_i(\tau,\hat{\ })$, $W_j(\tau,\hat{\ })$, $U_k(\tau,\hat{\ })$, and $V_k(\tau,\hat{\ })$ implies that the latter turn into constants. Thus, we consider those points on the boundary of the space of extended Schottky groups for which we obtain a constant for at least one generator in the limit for a sequence of marked Schottky groups of type (g,s,m).

We now proceed to constructing the augmented space. We define the set $\delta^{I,J,K}ST_{(g,s,m)}$, where $I \subset \{1,\dots,g\}$, $J \subset \{1,\dots,s\}$, and $K \subset \{1,\dots,m\}$.

For $I = J = K = \varnothing$ we put $\delta^{I,J,K}ST_{(g,s,m)} = ST_{g,s,m}$.

For $I \cup J \cup K \neq \varnothing$ we denote by $\delta^{I,J,K}ST_{(g,s,m)}$ the set of the points $\tau \in \overline{\mathbb{C}}^{3g+3m+2s-3}$ satisfying the following conditions:

(1a) the elements $T_i(\tau,\hat{\ })$, $W_j(\tau,\hat{\ })$, $U_k(\tau,\hat{\ })$, and $V_k(\tau,\hat{\ })$, $i \notin I$, $j \notin J$, $k \notin K$, are well defined and generate an extended Schottky group, say, $G_0(\tau)$;

(2a) $\lambda_i(\tau)(a_i(\tau) - b_i(\tau)) = 0$, $0 \leq |\lambda_i(\tau)| < 1$ for $i \in I$;

(3a) $w_j(\tau) = 0$ for $j \in J$ and $u_k(\tau) = 0$ for $k \in K$;

(4a) all points of the set $\{a_1(\tau),\dots,a_g(\tau),\ b_1(\tau),\dots,b_g(\tau), c_1(\tau),\dots,c_s(\tau),\ d_1(\tau),\dots,d_m(\tau)\}$ are different but possibly $a_i(\tau) = b_i(\tau)$, $i \in I$.

To introduce the last condition in the definition of $\delta^{I,J,K}ST_{g,s,m}$, given a point τ, we associate with it some collection of groups to be listed below.

Let $I_1 = \{i \in I \mid \lambda_i(\tau) = 0, a_i(\tau) \neq b_i(\tau)\}$, $I_2 = \{i \in I \mid \lambda_i(\tau) \neq 0, a_i(\tau) = b_i(\tau)\}$, and $I_3 = \{i \in I \mid \lambda_i(\tau) = 0, a_i(\tau) = b_i(\tau)\}$.

For $i \in [\{1,\dots,g\} \setminus I] \cup I_1$, put $G_i(\tau) = A_i G_0(\tau) A_i^{-1}$, where A_i is a Möbius transformation defined by the conditions: $A_i(a_i(\tau)) = \infty$, $A_i(b_i(\tau)) = 0$, and $A_i(\alpha) = 1$, where $\alpha = a_{i+1}(\tau)$ if $i < g$ and $\alpha = c_1(\tau)$ if $i = g$.

We agree that

$$G_i(\tau) = \left\langle z \to \frac{z}{\lambda_i(\tau)} \right\rangle \text{ for } i \in I_2, \quad G_i(\tau) = \langle \mathrm{id} \rangle \text{ for } i \in I_3.$$

If $j \in [\{1,\dots,s\} \setminus J]$ then we put $G_{j+g}(\tau) = R_j G_0(\tau) R_j^{-1}$, where R_j is a Möbius transformation such that $R_j(c_j(\tau)) = \infty$, $R_j W_j(\tau,\hat{\ }) R_j^{-1} = z + 1$, and $R_j(\alpha) = 0$, with $\alpha = c_{j+1}(\tau)$ if $j < s$ and $\alpha = d_1(\tau)$ if $j = s$.

For $j \in J$, we set $G_{j+g}(\tau) = \langle z+1 \rangle$.

Given $k \in [\{1,\dots,m\} \backslash K]$, we put $G_{k+g+s}(\tau) = Q_k G_0(\tau) Q_k^{-1}$. Here Q_k is a Möbius transformation such that $Q_k(d_k(\tau)) = \infty$, $Q_k U_k(\tau,\hat{\ }) Q_k^{-1} = z + 1$, and $Q_k(\alpha) = 0$, with $\alpha = d_{k+1}(\tau)$ if $k < m$ and $\alpha = a_1(\tau)$ if $k = m$.

If $k \in K$ then we set $G_{k+g+s}(\tau) = \langle z+1, z+v_k \rangle$.

Thus, a point τ is associated with the collection of groups

$$\{G_0(\tau), G_i(\tau), G_{j+g}(\tau), G_{k+g+s}(\tau),\ i = 1,\dots,g,\ j = 1,\dots,s,\ k = 1,\dots,m\}.$$

Now, we introduce the last condition in the definition of $\delta^{I,J,K}ST_{(g,s,m)}$:

(5a) the set $P_0 = \{a_i(\tau), b_i(\tau), c_j(\tau), d_k(\tau), i \in I, j \in J, k \in K\}$ lies in a suitable fundamental domain of $G_0(\tau)$ (we call this set the *set of distinguished points* for the group).

For $i \in I_2$, we choose a fundamental domain of the group $G_i(\tau)$ which contains the point 1. We call 1 the distinguished point for $G_i(\tau)$.

If $i \in I_3$ then we consider the set $P_i = \{0, 1, \infty\}$ to be distinguished for the group $G_i(\tau) = \langle \mathrm{id} \rangle$.

For $G_{j+g}(\tau)$ and $G_{k+g+s}(\tau)$, $j \in J$, $k \in K$, we can choose appropriate fundamental domains that contain the point 0. This point is said to be distinguished for $G_{j+g}(\tau)$ and $G_{k+g+s}(\tau)$.

It is the set $ST^*_{(g,s,m)} = \cup \delta^{I,J,K}ST_{(g,s,m)}$, with the union taken over all subsets $I \subset \{1,\dots,g\}$, $J \subset \{1,\dots,s\}$, and $K \subset \{1,\dots,m\}$, that we call the *augmented space* of extended Schottky groups of type (g,s,m), or simply the *augmented* Schottky space of type (g,s,m).

Theorem 1. *The augmented space $ST^*_{(g,s,m)}$ of extended Schottky groups is a subset of $ST_{(g,s,m)} \cup \partial ST_{(g,s,m)}$ and forms a domain in $\overline{\mathbb{C}}^{3g+3m+2s-3}$.*

PROOF. Let us demonstrate that the space $ST^*_{(g,s,m)}$ is a subset of $ST_{(g,s,m)} \cup \partial ST_{(g,s,m)}$.

Assume $\tau \in ST^*_{(g,s,m)}$, with $\tau \in \delta^{I,J,K} ST_{(g,s,m)}$ for some sets I, J, and K.

CASE 1: $I \neq \varnothing$, $J = \varnothing$, $K = \varnothing$.

Since $I = I_1 \cup I_2 \cup I_3$, we separately consider three subcases.

(1a) $I_1 \neq \varnothing$, $I_2 = I_3 = \varnothing$. The coordinates of the point τ satisfy the conditions $\lambda_i(\tau) = 0$, $i \in I_1$. Consider a sequence of numbers $\lambda_{in} \to 0$, $\lambda_{in} \in \mathbb{R}$, $0 < |\lambda_{in}| < 1$. Let T_{in} be a hyperbolic mapping with fixed points $a_i(\tau)$ and $b_i(\tau)$ and factor λ_{in}^{-1}. Denote by I_{in} the isometric circle of T_{in}. Put $I'_{in} = T_{in}(I_{in})$. Since $a_i(\tau)$ and $b_i(\tau)$ lie in the fundamental domain of the group, for n sufficiently large the curves I_{in} and I'_{in} also lie in the fundamental domain. By Maskit's combination theorem, the groups $G_n = \langle G_0(\tau), T_{in}, i \in I_1 \rangle$ are extended Schottky groups for n sufficiently large. Order the generators of the groups G_n so that the mapping T_{in}, $i \in I_1$, stand on the ith position. As in [4], associate the sequence $[G_n]$ with the sequence of the points $\tau_n \in ST_{(g,s,m)}$. As $n \to \infty$ we have $\tau_n \to \tau$. Thus, $\tau \in \partial ST_{(g,s,m)}$.

(1b) $I_2 \neq \varnothing$, $I_1 = I_3 = \varnothing$. The coordinates of the point τ satisfy the conditions $a_i(\tau) = b_i(\tau)$, $i \in I_2$. The point τ is associated with $|I_2|$ groups $G_i(\tau) = \langle z \to \lambda_i^{-1}(\tau) z \rangle$ ($|I_2|$ is the cardinality of the set I_2). Denote by C_i and C'_i the defining curves of the group $G_i(\tau)$. The point 1 is distinguished for $G_i(\tau)$ and lies in the fundamental domain of the group.

Consider the sequence of the points $a_{in} = a_i(\tau) + \varepsilon_n^2 e^{i\varphi}$, where $\varepsilon_n \in \mathbb{R}$, $\varepsilon_n \to 0$, $n \to \infty$.

Construct some mapping A_{in} for $i \in I_2$ and $n \in \mathbb{N}$ as follows: $A_{in}(0) = a_i(\tau)$, $A_{in}(\infty) = a_{in}$, and $A_{in}(1) = \infty$.

Let $T_{in} = A_{in} \frac{z}{\lambda_i(\tau)} A_{in}^{-1}$. The mapping T_{in} has fixed points a_{in} and $a_i(\tau)$ and factor $\lambda_i^{-1}(\tau)$. The curves $\Gamma_{in} = A_{in}(C_i)$ and $\Gamma'_{in} = A_{in}(C'_i)$ are defining for T_{in}; i.e., $T_{in}(\Gamma_{in}) = \Gamma'_{in}$, $i \in I_2$.

In the fundamental domain for $G_i(\tau)$, consider a circle c with center 1 and radius ε_n for n sufficiently large. Under the mapping A_{in}^{-1}, the circle c transforms into the circle $\tilde{c}$: $|w - a_{in}| = \varepsilon_n$. Moreover, the defining curves Γ_{in} and Γ'_{in} will lie inside $\tilde{c}$, whereas the defining curves for $G_0(\tau)$, outside $\tilde{c}$. By Maskit's combination theorem, the groups $G_n = \langle G_0(\tau), T_{in}, i \in I_2 \rangle$ are extended Schottky groups for n sufficiently large. Order the generators of the group G_n so that the mapping T_{in}, $i \in I_2$, stand on the ith position. Associate the canonical representatives of the classes $[G_n]$ with the sequence of the points τ_n in $ST_{(g,s,m)}$. As $n \to \infty$ we have $\tau_n \to \tau$. Therefore, $\tau \in \partial ST_{(g,s,m)}$.

(1c) $I_3 \neq \varnothing$, $I_1 = I_2 = \varnothing$. The proof is conducted by combining the methods of cases (1a) and (1b).

CASE 2: $I = \varnothing$, $J \neq \varnothing$, $K = \varnothing$.

A point $\tau \in \delta^{\varnothing,J,\varnothing} ST_{(g,s,m)}$ has coordinates $w_j(\tau) = 0$ for $j \in J$. Consider a sequence of points $w_{jn} \to 0$, $w_{jn} \neq 0$, $j \in J$. Let W_{jn} be a parabolic mapping with fixed point $c_j(\tau)$ and parameter w_{jn}. Since $c_j(\tau)$ lies in the fundamental domain for $G_0(\tau)$, the isometric circle I_{jn} of W_{jn} of radius $|w_{jn}|$ lies in the fundamental domain of the group $G_0(\tau)$ for a sufficiently large n. Let $I'_{jn} = W_{jn}(I_{jn})$. The curves I_{jn} and I'_{jn} are defining for W_{jn} and lie in the fundamental domain of the group $G_0(\tau)$. By the combination theorem, $G_n = \langle G_0(\tau), W_{jn}, j \in J \rangle$ is a Schottky group of type (g,s,m) for a sufficiently large n. As above, the corresponding sequence of the points $\tau_n \in ST_{(g,s,m)}$ converges to τ. Whence, $\tau \in \partial ST_{(g,s,m)}$.

CASE 3: $I = \varnothing$, $J = \varnothing$, $K \neq \varnothing$.

For such points τ, the coordinates $u_k(\tau)$ are equal to zero, $k \in K$. Let u_{kn} be a sequence of points vanishing as $n \to \infty$. Consider parabolic mappings U_{kn} and V_{kn} having the common fixed point $d_k(\tau)$ and parameters u_{kn} and $v_k(\tau)$. Let α_{kn} be the isometric circle of U_{kn} and let β_{kn} be the isometric circle of V_{kn}. Put $\alpha'_{kn} = U_{kn}(\alpha_{kn})$ and $\beta'_{kn} = V_{kn}(\beta_{kn})$. Then α_{kn}, α'_{kn} and β_{kn}, β'_{kn} are defining curves for U_{kn} and V_{kn} respectively. By the combination theorem, $G_n = \langle G_0(\tau), U_{kn}, V_{kn}, k \in K \rangle$ are extended Schottky groups for n sufficiently large. Associate the sequence $[G_n]$ with the sequence of the points $\tau_n \in ST_{(g,s,m)}$. As $n \to \infty$ we have $\tau_n \to \tau$. Therefore, $\tau \in \partial ST_{(g,s,m)}$. The first part of the theorem is proven.

Demonstrate that $ST^*_{(g,s,m)}$ is open.

Assume $\tau \in \delta^{I,J,K} ST_{(g,s,m)}$. If $I = J = K = \varnothing$ then $\delta^{I,J,K} ST_{(g,s,m)} = ST_{(g,s,m)}$ and $ST_{(g,s,m)}$ is open by a theorem proven in [4].

Suppose that $I \cup J \cup K \neq \varnothing$. Denote by $\{C_i, C_i', B_j, B_j', L_k, i \notin I, j \notin J, k \notin K\}$ the set of defining curves bounding the fundamental domain of the group $G_0(\tau)$. For $i \in I$, let C_i be a circle of a sufficiently small radius centered at $b_i(\tau)$. If $\hat{\tau} \in \mathbb{C}^{3g+3m+2s-3}$ is sufficiently close to τ then the set $L \subset \{1, \ldots, g\}$ of the indices l such that $\lambda_l(\hat{\tau}) = 0$ or $a_l(\hat{\tau}) = b_l(\hat{\tau})$ satisfies the condition $L \subset I$. Analogously, the sets $Q \subset \{q \in \{1, \ldots, s\} \mid w_q(\hat{\tau}) = 0\}$ and $P = \{p \in \{1, \ldots, m\} \mid u_p(\hat{\tau}) = 0\}$ are subsets of J and K respectively; i.e., $Q \subset J$ and $P \subset K$.

Given $i \in [\{1, \ldots, g\} \setminus L]$, put $\widetilde{C}_i' = T_i(\hat{\tau}, C_i)$. For $j \in [\{1, \ldots, s\} \setminus Q]$, denote by $\widetilde{B}_j$ the isometric circle of the mapping $W_j(\hat{\tau}, \hat{\ })$. Put $\widetilde{B}_j' = W_j(\hat{\tau}, \widetilde{B}_j)$. For $k \in [\{1, \ldots, m\} \setminus P]$, denote by $\tilde{\alpha}_k$ the isometric circle of the mapping $U_k(\hat{\tau}, \hat{\ })$ and denote by $\tilde{\beta}_k$ the isometric circle of the mapping $V_k(\hat{\tau}, \hat{\ })$. Let $\tilde{\alpha}_k' = U_k(\hat{\tau}, \tilde{\alpha}_k)$, $\tilde{\beta}_k' = V_k(\hat{\tau}, \tilde{\beta}_k)$, and let $\widetilde{L}_k$ be the topological quadrilateral formed by the curves $\tilde{\alpha}_k$, $\tilde{\alpha}_k'$, $\tilde{\beta}_k$, and $\tilde{\beta}_k'$. Then

$$\{C_i, \widetilde{C}_i', \widetilde{B}_j, \widetilde{B}_j', \widetilde{L}_k, i \in [\{1, \ldots, g\} \setminus L], j \in [\{1, \ldots, s\} \setminus Q], k \in [\{1, \ldots, m\} \setminus P]\}$$

is a set of pairwise disjoint Jordan curves bounding the standard fundamental domain of the extended Schottky group

$$G_0(\hat{\tau}) = \langle T_i(\hat{\tau}, \hat{\ }), W_j(\hat{\tau}, \hat{\ }), U_k(\hat{\tau}, \hat{\ }), V_k(\hat{\tau}, \hat{\ }), \quad i \notin L, j \notin Q, k \notin P \rangle.$$

Moreover, all points $a_l(\hat{\tau})$, $b_l(\hat{\tau})$, $c_q(\hat{\tau})$, $d_p(\hat{\tau})$, $l \in L$, $q \in Q$, $p \in P$, lie in the fundamental domain of $G_0(\hat{\tau})$. We call them distinguished for $G_0(\hat{\tau})$. Then $\tau \in \delta^{L,Q,P} ST_{(g,s,m)}$ and consequently $ST^*_{(g,s,m)}$ is open.

The connectedness of $ST^*_{(g,s,m)}$ is immediate from the relations

$$ST_{(g,s,m)} \subset ST^*_{(g,s,m)} \subset \overline{ST_{(g,s,m)}}$$

and the connectedness of $ST_{(g,s,m)}$ is shown in [4]. The theorem is proven.

§ 3. The Augmented Space and Riemann Surfaces with Nodes

In this section we shall interpret each point $\tau \in ST^*_{(g,s,m)}$ as a complex space, namely, as some Riemann surface with nodes.

A *Riemann surface with nodes* is a connected complex space S such that each point $X \in S$ has a neighborhood homeomorphic either to the disk $|z| < 1$ in $\mathbb{C}$ (X corresponds to $z = 0$) or to the set $\{|z| < 1, |w| < 1, zw = 0\}$ in $\mathbb{C}^2$ (X corresponds to $z = w = 0$).

In the last case, X is called a *node*.

We consider Riemann surfaces with nodes and punctures.

Every component of the complement to the nodes is called a *part* of S and represents a conventional Riemann surface.

The genus g of a Riemann surface with nodes is defined by the formula

$$g = \sum_{i=1}^{r} g_i + k + 1 - r,$$

where g_i is the genus of the ith part, k is the number of nodes, and r is the number of parts.

A node X is called *separating* if $S \setminus \{X\}$ is disconnected and *nonseparating* otherwise.

Let $\tau \in ST^*_{(g,s,m)}$. Then $\tau \in \delta^{I,J,K} ST_{(g,s,m)}$ for some sets $I \subset \{1,\dots,g\}$, $J \subset \{1,\dots,s\}$, and $K \subset \{1,\dots,m\}$.

The point τ is associated with the collection of the groups

$$\{G_0(\tau), G_i(\tau), G_{j+g}(\tau), G_{k+g+s}(\tau),\ i=1,\dots,g,\ j=1,\dots,s,\ k=1,\dots,m\}.$$

To this collection of groups there corresponds a collection of Riemann surfaces

$$S_0 = R(G_0(\tau))/G_0(\tau), \quad S_i = R(G_i(\tau))/G_i(\tau),$$
$$S_{j+g} = R(G_{j+g}(\tau))/G_{j+g}(\tau), \quad S_{k+g+s} = R(G_{k+g+s}(\tau))/G_{k+g+s}(\tau),$$

where $R(G)$ is the fundamental domain of the corresponding group, $i=1,\dots,g$, $j=1,\dots,s$, $k=1,\dots,m$.

Denote the corresponding natural projections by

$$\pi_0 : R(G_0(\tau)) \to S_0, \quad \pi_i : R(G_i(\tau)) \to S_i,$$
$$\pi_{j+g} : R(G_{j+g}(\tau)) \to S_{j+g}, \quad \pi_{k+g+s} : R(G_{k+g+s}(\tau)) \to S_{k+g+s},$$

where $i=1,\dots,g$, $j=1,\dots,s$, and $k=1,\dots,m$.

The Riemann surface S_0 represents a Riemann surface of genus $g+m-(|I_2|+|I_3|+K)$ with $2(s-|J|)$ punctures. On S_0 distinguished are $|I_1|$ pairs of the points $q_i=\pi_0(a_i(\tau))$ and $q_i'=\pi_0(b_i(\tau))$, $i\in I_1$ as well as $(|I_2|+|I_3|+|J|+|K|)$ points $r_i=\pi_0(a_i(\tau)$, $i\in I_2$, $u_i=\pi_0(a_i(\tau))$, $i\in I_3$, $v_k=\pi_0(d_k(\tau))$, $k\in K$, and $p_j=\pi_0(c_j(\tau))$, $j\in J$.

For $i\in[\{1,\dots,g\}\setminus I]\cup I_1$, $j\in[\{1,\dots,s\}\setminus J]$, and $k\in[\{1,\dots,m\}\setminus K]$ the Riemann surfaces S_i, S_{j+g}, and S_{k+g+s} are conformally equivalent to S_0 with the same distinguished points. Henceforth we identify these Riemann surfaces with S_0.

For $i\in I_2$, the surface S_i is a torus with distinguished point $r_i'=\pi_i(1)$. For $i\in I_3$, it is a sphere with distinguished points $w_i=0$, $w_i'=\infty$, and $u_i'=1$.

For $j\in J$, the surface S_{j+g} is a sphere with two punctures and distinguished point $p_j'=\pi_{j+g}(0)$.

For $k\in K$, the surface S_{k+g+s} is a torus with distinguished point $v_k'=\pi_{k+g+s}(0)$.

Denote by $S(\tau)$ the union of the Riemann surfaces S_0, S_i, $i\in I_2\cup I_3$, S_{j+g}, $j\in J$, S_{k+g+s}, $k\in K$, with identified pairs of corresponding points q_i and q_i', r_i and r_i', w_i and w_i', u_i and u_i', p_j and p_j', v_k and v_k'.

The so-obtained surface $S(\tau)$ represents a Riemann surface with nodes. More precisely, for $\tau\in\delta^{I,J,K}ST_{(g,s,m)}$ the corresponding Riemann surface $S(\tau)$ has $(|I_2|+|I_3|+|J|+K|)$ separating nodes and $(|I_1|+|I_3|)$ nonseparating nodes. We say that $S(\tau)$ is *associated with* τ.

Thereby, we have proven the following

Theorem 2. *For $\tau\in\delta^{I,J,K}ST_{(g,s,m)}$, there is a Riemann surface S associated with τ which is of genus $g+m$ and has $2s$ punctures, $(|I_2|+|I_3|+|J|+|K|)$ separating nodes, and $(|I_1|+|I_3|)$ nonseparating nodes.*

It is easy to show that the converse assertion is also valid; i.e., to a Riemann surface of genus $g+m$ with $2s$ punctures and a distinguished system of loops and nodes of the considered type, there corresponds some point $\tau\in ST^*_{(g,s,m,)}$. Observe that such point τ is not determined by a Riemann surface uniquely but depends on the choice of fundamental domains for corresponding groups.

References

1. L. Bers, "Automorphic forms for Schottky groups," Adv. in Math. (China), **16**, 332–361 (1975).
2. H. Sato, "On augmented Schottky spaces and automorphic forms. I," Nagoya Math. J., **75**, 151–175 (1979).
3. R. E. Rodriques, "On Schottky-type groups with applications to Riemann surfaces with nodes. II," Complex Variables Theory Appl., **1**, No. 2/3, 293–310 (1983).
4. N. A. Gusevskiĭ and N. S. Zindinova, "On the space of extended Schottky groups," Sibirsk. Mat. Zh., **2**, No. 6, 65–78 (1986).

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