

Unitary Solutions to the Yang-Baxter Equation in Dimension Four

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Abstract

In this paper, we determine all unitary solutions to the Yang-Baxter equation in dimension four. Quantum computation motivates this study. This set of solutions will assist in clarifying the relationship between quantum entanglement and topological entanglement. We present a variety of facts about the Yang-Baxter equation for the reader unfamiliar with the equation.

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Introduction

In this paper we classify all unitary solutions to the Yang-Baxter equation in dimension four. These solutions represent both braid operators and quantum operators which describe topological entanglement and quantum entanglement respectively. The motivating question for this work is found in [1] in which Kauffman and Lomonaco compare the concepts of entanglement in the areas of topology and quantum computing. They suggest that quantum entanglement may be described in relation to a set of entangled links.

This classification describes all 4×4 unitary matrices that are both braid operators and quantum operators. Determining all unitary, 4×4 matrix solutions to the Yang-Baxter equation will assist us in studying the relationship

between these two forms of entanglement.

We introduce quantum operators and braid operators in Section 1. We also discuss the connection between the Yang-Baxter equation and braids in this section. In Section 2, we examine the algebraic properties of the Yang-Baxter equation. We will apply these properties to determine unitary solutions to the Yang-Baxter equation.

In Section 3, we describe these properties in terms of matrices. We recall facts about matrices that are conjugate to unitary matrices. We state the equation used to determine the unitary matrices in Proposition 3.4 and prove a variety of facts chosen to simplify the proof of Theorem 4.1.

In Section 4, we present a complete list of unitary 4×4 matrix solutions to the algebraic Yang-Baxter equation that consists of five families of unitary matrices. These families are conjugacy classes determined by matrices that preserve unitarity and the property that the conjugate is a solution to the Yang-Baxter equation. However, the four families are conjugate under a larger class of matrices that do not necessarily preserve these properties. We obtain the unitary families by analyzing each solution given in [3] and applying facts from the earlier sections. In Section 5, we determine which unitary solutions are also solutions to the bracket skein equation given in [5].

1 Braid Operators and Quantum Gates

Topological entanglement and quantum entanglement are non-local structural features that occur in topological and quantum systems, respectively. Kauffman introduces this fact in [1] and asks the following question. What is the relationship between these two concepts? We describe these concepts as an introduction to the Yang-Baxter equation.

Topological entanglement is described in terms of link diagrams and via the Artin braid group. The *Artin braid group* on n -strands is denoted by B_n and is generated by $\{\sigma_i | 1 \leq i \leq n - 1\}$. The group B_n consists of all words of the form $\sigma_{j_1}^{\pm 1} \sigma_{j_2}^{\pm 1} \dots \sigma_{j_n}^{\pm 1}$ modulo the following relationships.

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n - 1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ such that } |i - j| > 1$$

This group has the following diagrammatic representation. A *n-strand braid* is the immersion of n arcs with over and under crossing information at each singularity. Furthermore, any transverse arc intersects the braid at n points, except at the levels at which singularities occur. Each generator σ_i of the Artin braid group is associated to a diagram as shown in Figure 1.

Multiplication of diagrams is performed by concatenation of the diagrams from upper to lower. We show an example in Figure 2.

The diagrammatic form of the relationships in the Artin braid group are shown in Figure 3.

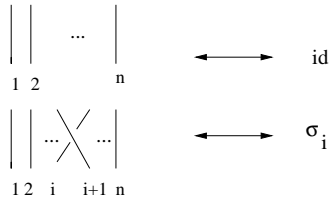


Figure 1: Generators of the Artin Braid Group

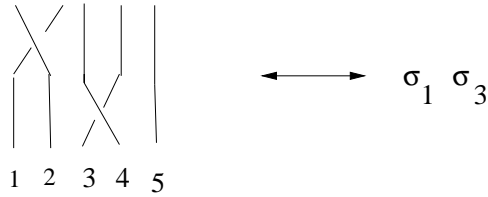


Figure 2: Multiplicaton

To obtain the braided Yang-Baxter equation, we associate a vector space V to each endpoint in the diagram. Each n -strand braid represents a linear map from $V^{\otimes n}$ to $V^{\otimes n}$. The generator σ_i is associated to a map $R : V \otimes V \rightarrow V \otimes V$ in the following manner: $\sigma_i \mapsto I \otimes I \cdots \otimes R \otimes \cdots I \otimes I$ with the R in the i th position and all the I 's representing the identity map. This representation will respect the relations of the Artin braid group.

In the three strand case, this corresponds to the braided Yang-Baxter equation.

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \quad (1)$$

Notice that R is a linear map and may be expressed as a matrix for some basis of V . For the remainder of this paper we will refer to the Yang-Baxter equation as the ybe.

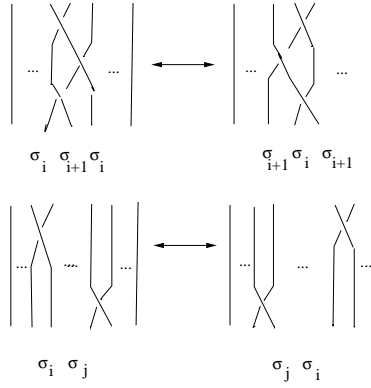


Figure 3: Relations

We algebraically describe quantum entanglement as the action of a unitary, linear map with certain properties. In a quantum system, quantum states are elements of $V^{\otimes n}$ where V is two dimensional vector space over \mathbf{C} . Using Dirac notation, we express the basis of V as $\{|0\rangle, |1\rangle\}$. The term *qubit* refers to vectors of the form $\alpha|0\rangle + \beta|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$. If α and β are both non-zero then the qubit is said to be in a state of *superposition*. This state collapses to $|0\rangle$ with probability $|\alpha|^2$ and the state $|1\rangle$ with probability $|\beta|^2$. The *n-qubit* is an element of $V^{\otimes n}$. If the n-qubit is in superposition, it may simultaneously represent 2^n states of classical information with associated probabilities. The n-qubit will collapse to a single outcome when measured, but until then will carry information about all 2^n states.

A *quantum gate* acts on a qubit so that the sum of the probabilities of all states is preserved. In quantum computing, we assemble a series of quantum

gates to perform a computation. Each quantum gate may be represented as a unitary matrix.

Consider the case of a two-qubit. This is a quantum state of the form: $\alpha|0\rangle \otimes |0\rangle + \beta|0\rangle \otimes |1\rangle + \gamma|1\rangle \otimes |0\rangle + \delta|1\rangle \otimes |1\rangle$. This state is said to be *entangled* if it can not be written in the form: $(x|0\rangle + y|1\rangle)(\hat{x}|0\rangle + \hat{y}|1\rangle)$. Note that this implies that $\alpha\delta - \beta\gamma \neq 0$. We may transform an unentangled state into a entangled state by application of a quantum gate. In general, a n-qubit state ψ is said to be *entangled* if ψ can not be written in the form $P(\psi_1 \otimes \psi_2)$ where $P : V^{\otimes n} \rightarrow V^{\otimes n}$ is a permutation of the tensor factors of $V^{\otimes n}$ and $\psi_1 \in V^{\otimes l}$, $\psi_2 \in V^{\otimes k}$ such that $l + k = n$.

The relationship between topological and quantum entanglement is not fully understood. We examine the properties of the Yang-Baxter equation in the next section.

2 The Yang-Baxter Equation

Let V be a vector space over a field F . Let R be a linear map:

$$R : V \otimes V \rightarrow V \otimes V$$

Let I be the identity map on V then R is said to be a solution the *braided Yang-Baxter equation* if the following holds:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

Now assume that V is finite dimensional and that $\{v_0, v_1, \dots, v_{n-1}\}$ is a basis for V over F . We may denote the basis of $V \otimes V$ as $\{v_i \otimes v_j | i, j \in \{0, 1, 2, \dots, n-1\}\}$. Using this basis we may describe R by its action on the generators of $V \otimes V$:

$$R(v_i \otimes v_j) = \sum_{k,l} R_{ij}^{kl} v_k \otimes v_l$$

Applying the original definition of the Yang-Baxter equation, we may describe R as a series of equations that are dependent on the choice of i, j, k and x, y, z .

$$\sum_{a,b,c} R_{ij}^{ab} R_{bk}^{cz} R_{ac}^{xy} = \sum_{m,n,p} R_{jk}^{np} R_{in}^{xm} R_{mp}^{yz}$$

We prove the following propositions about solutions to the braided Yang-Baxter Equations. These facts will be used to obtain unitary solutions to the braided Yang-Baxter equation. These propositions also indicate that a single solution produces a class of solutions by conjugation and scalar multiplication [3], [4].

Proposition 2.1. *If R is a solution to the braided Yang-Baxter equation then the following hold.*

- i) If $\alpha \in F$, then αR is a solution to the braided Yang-Baxter equation.*
- ii) If Q is an invertible map, $Q : V \rightarrow V$, then $(Q \otimes Q)R(Q \otimes Q)^{-1}$ is a solution to the braided Yang-Baxter equation.*
- iii) If R is invertible, then R^{-1} is a solution to the braided Yang-Baxter equation.*

Proof: See [4] page 168. ■

We now define the *algebraic Yang-Baxter equation*. Let $\tau : V \otimes V \rightarrow V \otimes V$ be the map such that $\tau(u \otimes v) = (v \otimes u)$. Let $R_{12}, R_{13}, R_{23} : V \otimes V \rightarrow V \otimes V$ be linear maps. Let $R_{12} = (R \otimes I)$, $R_{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$, and $R_{23} = (I \otimes R)$. Then R is a solution to the algebraic Yang-Baxter equation if:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2}$$

In this paper, we will refer to the equation as the algebraic Yang-Baxter equation and do not follow the notation of [3]. In [3], the equation is referred to as the quantum Yang-Baxter equation. We obtain a complete list of solutions to the algebraic Yang-Baxter equation from [3]. The algebraic Yang-Baxter equation is related to the braided ybe by the linear transformation τ of $V \otimes V$.

Proposition 2.2. *If R is a solution to the braided Yang Baxter equation, and $\tau : V \otimes V \rightarrow V \otimes V$ such that $\tau(u \otimes v) = v \otimes u$ then $R \circ \tau$ is a solution to the algebraic Yang-Baxter equation. Similarly, if ρ is a solution to the algebraic Yang-Baxter equation, then $\rho \circ \tau$ is a solution to the braided Yang-Baxter equation.*

Proof: See [4]. ■

Note that if we obtain a solution to the algebraic Yang-Baxter equation, we may apply τ to obtain a solution to the braided Yang-Baxter equation.

Using the basis for $V \otimes V$, we rewrite equation 2 as a sum of basis elements.

$$\begin{aligned}
R_{12}R_{13}R_{23}(v_i \otimes v_j \otimes v_k) &= R_{12}R_{13}(I \otimes R)(v_i \otimes v_j \otimes v_k) \\
&= R_{12}R_{13} \sum_{b,c} R_{j,k}^{b,c}(v_i \otimes v_b \otimes v_c) \\
&= R_{12}(I \otimes \tau)(R \otimes I) \sum_{b,c} R_{j,k}^{b,c}(v_i \otimes v_c \otimes v_b) \\
&= R_{12}(I \otimes \tau) \sum_{x,a,b,c} R_{j,k}^{b,c} R_{i,c}^{x,a}(v_x \otimes v_a \otimes v_b) \\
&= R_{12} \sum_{x,a,b,c} R_{j,k}^{b,c} R_{i,c}^{x,a}(v_x \otimes v_b \otimes v_a) \\
&= \sum_{y,z,x,a,b,c} R_{k,j}^{b,c} R_{i,c}^{x,a} R_{b,a}^{y,z}(v_x \otimes v_y \otimes v_z)
\end{aligned}$$

Consider the right hand side of the equation and obtain:

$$\begin{aligned}
R_{23}R_{13}R_{12}(v_i \otimes v_j \otimes v_k) &= R_{23}R_{13} \sum_{m,n} R_{i,j}^{m,n}(v_m \otimes v_n \otimes v_k) \\
&= R_{23}(I \otimes \tau)(R \otimes I) \sum_{m,n} R_{i,j}^{m,n}(v_m \otimes v_n \otimes v_k) \\
&= \sum_{x,y,z,m,n,p} R_{i,j}^{m,n} R_{m,k}^{p,x} R_{p,n}^{y,z}(v_x \otimes v_y \otimes v_z)
\end{aligned}$$

Therefore, a solution to the algebraic Yang-Baxter equation satisfies the following equations:

$$\sum_{a,b,c} R_{k,j}^{b,c} R_{i,c}^{x,a} R_{b,a}^{y,z}(v_x \otimes v_y \otimes v_z) = \sum_{m,n,p} R_{i,j}^{m,n} R_{m,k}^{p,x} R_{p,n}^{y,z}(v_x \otimes v_y \otimes v_z).$$

3 Matrix Representation

We may represent solutions to the braided Yang-Baxter equation and the algebraic Yang-Baxter equation as matrices. We study matrix notation for the case where the dimension of V is two. In this section, we present facts about matrix solutions to the Yang-Baxter equation and facts about unitary matrices. We present a method that determines families of unitary solutions to the Yang-Baxter equation. Recall that we may conjugate R by $Q \otimes Q$ or multiply by a scalar. The representation of $Q \otimes Q$ has a specific matrix form that will assist in determining the unitary solutions.

Let V be finite dimensional. If R is a solution to the quantum Yang-Baxter equation, we may rewrite R as a matrix \mathbf{R} .

Let R be a solution to the algebraic Yang-Baxter equation, and recall that:

$$R(v_i \otimes v_j) = \sum_{a,b} R_{ij}^{ab} v_a \otimes v_b$$

R may be written in matrix form \mathbf{R}_{ij}^{ab} where ij represents the column and ab the row of the matrix. In particular, if the dimension of V is two then V has a basis $\{v_0, v_1\}$. Hence $V \otimes V$ has a basis of the form $\{v_{00}, v_{01}, v_{10}, v_{11}\}$.

$$\mathbf{R} = \begin{bmatrix} R_{00}^{00} & R_{01}^{00} & R_{10}^{00} & R_{11}^{00} \\ R_{00}^{01} & R_{01}^{01} & R_{10}^{01} & R_{11}^{01} \\ R_{00}^{10} & R_{01}^{10} & R_{10}^{10} & R_{11}^{10} \\ R_{00}^{11} & R_{01}^{11} & R_{10}^{11} & R_{11}^{11} \end{bmatrix}$$

The matrix \mathbf{R} acts on a vector of the form

$$\begin{bmatrix} a \\ b \\ c \\ c \end{bmatrix}$$

that represents $a(v_0 \otimes v_0) + b(v_0 \otimes v_1) + c(v_1 \otimes v_0) + d(v_1 \otimes v_1)$.

Suppose $Q : V \rightarrow V$ is an invertible linear map. We may rewrite Q as the matrix \mathbf{Q} :

$$\begin{bmatrix} Q_0^0 & Q_1^0 \\ Q_0^1 & Q_1^1 \end{bmatrix} \text{ such that } Q_0^0 Q_1^1 - Q_1^0 Q_0^1 \neq 0$$

Let \mathbf{A} denote the linear map $Q \otimes Q : V \otimes V \rightarrow V \otimes V$. This map may be represented as a matrix \mathbf{A} . In the case where the dimension of V is two, \mathbf{A} is a 4×4 matrix of the form:

$$\begin{bmatrix} Q_0^0 Q_0^0 & Q_0^0 Q_1^0 & Q_1^0 Q_0^0 & Q_1^0 Q_1^0 \\ Q_0^0 Q_0^1 & Q_0^0 Q_1^1 & Q_1^0 Q_0^1 & Q_1^0 Q_1^1 \\ Q_1^0 Q_0^0 & Q_1^0 Q_1^0 & Q_1^1 Q_0^0 & Q_1^1 Q_1^0 \\ Q_1^0 Q_0^1 & Q_1^0 Q_1^1 & Q_1^1 Q_0^1 & Q_1^1 Q_1^1 \end{bmatrix}$$

For each matrix solution in [3], there is a family of solutions to the algebraic Yang-Baxter equations. To obtain another element of this family, we conjugate \mathbf{R} by \mathbf{A} . We wish to obtain all unitary families of solutions. Note that it is not sufficient to determine which representatives are unitary. Although \mathbf{R} may not be unitary, it is possible that $\mathbf{A}\mathbf{R}\mathbf{A}^{-1}$ is unitary for some \mathbf{A} .

Linear Algebra Facts

We prove facts about matrix solutions to the algebraic Yang-Baxter equation. These facts determine a preliminary approach to evaluating whether or not a family of solutions will produce any unitary solutions. Let M be a finite dimensional matrix. M^T denotes the transpose of the matrix M . Let \bar{M} denote the conjugate of M . Let M^* denote the conjugate transpose of the matrix M . Recall that M is a *unitary* matrix if $M^* = M^{-1}$.

Proposition 3.1. *Recall the following matrix facts.*

- i) Let A be an invertible matrix then $(A^*)^{-1} = (A^{-1})^*$.*
- ii) If A is a unitary matrix and let $\alpha \in \mathbb{C}$, then αA is a unitary matrix if and only if $|\alpha| = 1$.*
- iii) If A is a unitary matrix and λ is an eigenvalue of A , then $|\lambda| = 1$.*

Proposition 3.2. *If ARA^{-1} is unitary, then*

- i) $AR^{-1}A^{-1} = (A^*)^{-1}R^*A^*$ implying that $A^*AR^{-1} = R^*A^*A$, and*
- ii) R^{-1} and R^* have the same set of eigenvalues.*

Proof: Suppose ARA^{-1} is unitary. By the definition of unitary:

$$\begin{aligned}(ARA^{-1})^{-1} &= (ARA^{-1})^* \\ AR^{-1}A^{-1} &= (A^{-1})^*R^*A^* \\ AR^{-1}A^{-1} &= (A^*)^{-1}R^*A^* \\ A^*AR^{-1} &= R^*A^*A\end{aligned}$$

R^{-1} and R^* are conjugate by A^*A . Therefore, they have the same set of eigenvalues. ■

We make the following observations about an invertible matrix Q of the form

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Proposition 3.3. *If*

$$c = -\frac{a\bar{b}}{\bar{d}}$$

*then Q^*Q is diagonal. If $A = Q \otimes Q$ then A^*A is also diagonal.*

Proof: Recall that the matrix form of $Q \otimes Q$ is:

$$A = \begin{bmatrix} aa & ab & ba & bb \\ ac & ad & bc & bd \\ ca & cb & da & db \\ cc & cd & dc & dd \end{bmatrix}$$

Let $H = A^*A$. Let $x = (a\bar{a} + c\bar{c})$, $y = (b\bar{b} + d\bar{d})$, and $z = (a\bar{b} + c\bar{d})$.

$$H = \begin{bmatrix} x^2 & x\bar{z} & \bar{z}x & \bar{z}^2 \\ xz & xy & \bar{z}z & \bar{z}y \\ zx & z\bar{z} & yx & y\bar{z} \\ z^2 & zy & yz & y^2 \end{bmatrix}$$

If

$$c = -\frac{a\bar{b}}{\bar{d}}$$

then $z = 0$ and H is a diagonal matrix. ■

The relationship between a matrix Q of this form and a 4×4 matrix B may be simplified using the following facts.

Proposition 3.4. *Let $H = A^*A$ and let $D = HB^{-1} - B^*H$. Then*

$$D_{ij} = \sum_k H_{ik} B_{kj}^{-1} - \sum_m B_{im}^* H_{mj}$$

If H is a diagonal matrix, then

$$D_{ij} = H_{ii} B_{ij}^{-1} - B_{ij}^* H_{jj}$$

We will apply these facts to determine unitary solutions to the algebraic Yang Baxter equation.

Proposition 3.5. *If H is constructed as before then $H_{ii} \neq 0$ for all i .*

Proof: If $H_{i,i} = 0$ for some i , then either $x = 0$ or $y = 0$. This contradicts the assumption that Q is an invertible matrix. ■

Proposition 3.6. *Suppose H is constructed as before. If $H_{ij} = 0$ for some $i \neq j$ then $H_{ij} = 0$ for all $i \neq j$*

Proof: We observe that

$$H_{ij} = \sum_k A_{ik}^* A_{kj}.$$

If $H_{ij} = 0$ for some $i \neq j$ then we observe that $x = 0, y = 0$, or $z = 0$. If $x = 0$ then $|a|^2 + |c|^2 = 0$ implying that $a = c = 0$ and Q is not invertible. This contradicts our definition of Q . We have a similar contradiction if $y = 0$.

Note that z is a factor of each off diagonal entry. We have eliminated the the other possibilities. Therefore $x \neq 0, y \neq 0$ and $z = 0$ which forces H to be a diagonal matrix. ■

We apply these facts to the families of solutions in [3]. If the matrix \mathbf{R} produces unitary solutions under conjugation, then \mathbf{R} is of rank 4 and \mathbf{R}^* and \mathbf{R}^{-1} have equivalent sets of eigenvalues. Further, if \mathbf{ARA}^{-1} is a solution to the Yang-Baxter equation, then $\mathbf{A}^*\mathbf{AR} = \mathbf{R}^{-1}\mathbf{A}^*\mathbf{A}$. If \mathbf{R} is a unitary solution to the quantum Yang-Baxter, then $\alpha\mathbf{R}$ is a unitary solution if and only if $|\alpha| = 1$.

4 Solutions to the Yang-Baxter Equation

Theorem 4.1. *There are five families of 4×4 unitary matrix solutions to the braided Yang-Baxter equation. Each solution has the form:*

$$kARA^{-1}T$$

where k is a scalar of norm one, Q is an invertible matrix such that:

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A = Q \otimes Q = \begin{bmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & cd & d^2 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Additional information about the matrices R and A are specified for each family.

Family 1:

The matrix R has the form

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \text{ and} \quad (3)$$

$$1 = p\bar{p} = q\bar{q} = r\bar{r}.$$

The variable c in the matrix Q has the restriction that

$$c = -\frac{a\bar{b}}{d}.$$

Family 2:

The matrix R has the form:

$$R = \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix} \text{ and} \quad (4)$$

when $c \neq -\frac{\bar{a}b}{d}$ and

$$|pq| = 1,$$

$$p = \frac{(b\bar{b} + d\bar{d})(\bar{a}b + \bar{c}d)}{(a\bar{a} + c\bar{c})(\bar{a}b + \bar{c}d)},$$

$$q = \frac{(a\bar{a} + c\bar{c})(\bar{a}b + \bar{c}d)}{(b\bar{b} + d\bar{d})(\bar{a}b + \bar{c}d)}.$$

Family 3:

The third family consists of matrices with the form:

$$R = \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix} \text{ and} \quad (5)$$

$$|pq| = 1,$$

$$p\bar{p} = \frac{(d\bar{d})^2}{(a\bar{a})^2},$$

$$q\bar{q} = \frac{(a\bar{a})^2}{(d\bar{d})^2}.$$

The matrix Q has the restriction that

$$c = \frac{-a\bar{b}}{\bar{d}}.$$

Family 4:

The matrix R has the form

$$R = \begin{bmatrix} \frac{1}{\sqrt{(2)}} & 0 & 0 & \frac{1}{\sqrt{(2)}} \\ 0 & \frac{1}{\sqrt{(2)}} & \frac{1}{\sqrt{(2)}} & 0 \\ 0 & \frac{1}{\sqrt{(2)}} & -\frac{1}{\sqrt{(2)}} & 0 \\ -\frac{1}{\sqrt{(2)}} & 0 & 0 & \frac{1}{\sqrt{(2)}} \end{bmatrix}. \quad (6)$$

The matrix Q has the following restrictions:

$$c = -\frac{a\bar{b}}{d}$$

$$|a| = |d|.$$

Family 5:

The matrix R has the form

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

There are no additional restrictions on the matrix Q .

Remark 4.1. *We may transform the elements of families 2 and 3 into elements from family 1 by conjugation by unitary matrices. However, these matrices do not have the form $Q \otimes Q$.*

Remark 4.2. *The matrices of families 2 and 3 perform quantum entanglement. The matrices from families 2 and 3 do not detect knotting but do detect linking. Family 4 performs quantum entanglement and detects knotting and linking [2].*

Proof: If ϕ is a solution to the algebraic Yang-Baxter equation and $\tau : V \otimes V \rightarrow V \otimes V$ is a linear transformation such that $\tau(v_i \otimes v_j) = v_j \otimes v_i$, then $\phi \circ \tau$ is a solution to the braided Yang Baxter equation. We may express τ as the unitary matrix T since $V \otimes V$ has a basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$.

Note that R is a unitary solution to the algebraic Yang-Baxter equation if and only if RT is a unitary solution to the braided Yang-Baxter equation.

To find all unitary solutions to the braided Yang-Baxter equation, it suffices to determine all unitary solutions to the algebraic Yang-Baxter and multiply by T .

The families of solutions to the algebraic Yang-Baxter equation in [3] are indicated by a single element. This representative element is conjugated by the matrix $Q \otimes Q$ and multiplied by a scalar k to produce the entire family. If a family of solutions produces any unitary solutions, the representative of the family must be invertible. From [3] we obtain the following list of invertible

representatives:

$$R01 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R02 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$R03 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R11 = \begin{bmatrix} p^2 + 2pq - q^2 & 0 & 0 & p^2 - q^2 \\ 0 & p^2 + q^2 & p^2 - q^2 & 0 \\ 0 & p^2 - q^2 & p^2 + q^2 & 0 \\ p^2 - q^2 & 0 & 0 & p^2 - 2pq - q^2 \end{bmatrix}$$

$$R12 = \begin{bmatrix} p & 0 & 0 & k \\ 0 & p & p - q & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q \end{bmatrix}$$

$$R13 = \begin{bmatrix} k^2 & kp & -kp & pq \\ 0 & k^2 & 0 & kq \\ 0 & 0 & k^2 & -kq \\ 0 & 0 & 0 & k^2 \end{bmatrix}$$

$$R14 = \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 0 & k & 0 \\ 0 & k & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
R21 &= \begin{bmatrix} k^2 & 0 & 0 & 0 \\ 0 & kp & k^2 - pq & 0 \\ 0 & 0 & kq & 0 \\ 0 & 0 & 0 & k^2 \end{bmatrix} \\
R22 &= \begin{bmatrix} k^2 & 0 & 0 & 0 \\ 0 & kp & k^2 - pq & 0 \\ 0 & 0 & kq & 0 \\ 0 & 0 & 0 & -pq \end{bmatrix} \\
R23 &= \begin{bmatrix} k & p & q & s \\ 0 & k & 0 & q \\ 0 & 0 & k & p \\ 0 & 0 & 0 & k \end{bmatrix} \\
R31 &= \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & s \end{bmatrix}
\end{aligned}$$

If R produces unitary solutions, then R^* and R^{-1} are conjugate. We eliminate from this list all matrices that do not satisfy the condition that R^* and R^{-1} have the same set of eigenvalues. From this computation, we determine only the representatives: $R01, R02, R03, R12, R13, R14, R21, R22, R23,$ and $R31$ could produce unitary solutions to the Yang-Baxter equation. The eigenvalues in each of these matrices are non-zero, using this fact we have

removed one degree of freedom from each representative. Multiplying by a non-zero eigenvalue determines the new representatives used in the following cases. To determine if a representative produces unitary solutions, we refer to Proposition 3.4. If R and Q produce a unitary matrix, then $D_{ij} = 0$ for all i, j .

Case: R31

We consider the case of the diagonal matrix $R31$. Note that if $R31^*$ and $R31^{-1}$ have the same set of eigenvalues then $R31$ is unitary and each eigenvalue has norm one. Further, the eigenvalues are the diagonal elements.

$$R31 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & k \end{bmatrix}.$$

The inverse and conjugate transpose of the matrix are:

$$R31^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & 0 \\ 0 & 0 & \bar{q} & 0 \\ 0 & 0 & 0 & \bar{k} \end{bmatrix} \quad R31^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{p} & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ 0 & 0 & 0 & \frac{1}{k} \end{bmatrix}.$$

To determine the family of solutions given by $R31$ we examine the equation from Proposition 3.4.

$$D_{ij} = \sum_k k H_{ik} R31_{k,j}^{-1} - \sum_m R31_{im}^* H_{mj}$$

To obtain a unitary R matrix under conjugation by Q , each $D_{ij} = 0$. R_{31} is a diagonal matrix and by Proposition 3.3

$$\begin{aligned} D_{ij} &= H_{ij}R_{31}^{-1}_{jj} - R_{31}^*_{ii}H_{ij} \\ 0 &= (R_{31}^{-1}_{jj} - R_{31}^{-1}_{ii})H_{ij} \end{aligned}$$

From this equation, we obtain two cases. In case one, $R_{31}^{-1}_{jj} - R_{31}^{-1}_{ii} = 0$ for all i, j . This implies that R_{31} is the identity matrix, and we obtain solutions that are scalar multiples of the identity matrix.

In the second case, R_{31} is not a multiple of the identity. If R_{31} is not a multiple of the identity, then

$$(R_{31}^{-1}_{jj} - R_{31}^{-1}_{ii}) \neq 0 \text{ for some } i \neq j.$$

Therefore, $H_{ij} = 0$ for $i \neq j$. By Proposition 3.6, H is a diagonal matrix and

$$c = \frac{-a\bar{b}}{\bar{d}}$$

This produces the first family of solutions.

Case: R_{21}

The matrix R_{21} is not unitary. We observe that R_{21}^* and R_{21}^{-1} have the same set of eigenvalues if $|p| = |q| = 1$.

$$R21 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse and conjugate transpose of the matrix $R21$ are:

$$R21^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & 0 \\ 0 & 1 - \bar{p}\bar{q} & \bar{q} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R21^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{p} & \frac{-1}{pq} + 1 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To determine when $R21$ produces unitary solutions under conjugation, we examine:

$$D_{ij} = \sum_k H_{ik} R21_{kj}^{-1} - \sum_m R21_{im}^* H_{mj}$$

If H and $R21$ produce a unitary solution, then D is the zero matrix.

Examine the entry

$$\begin{aligned} D_{12} &= \sum_k H_{1k} R21_{k2}^{-1} - \sum_m R21_{1m}^* H_{m2} \\ 0 &= H_{12} \frac{1}{p} - H_{12} \\ 0 &= \left(\frac{1}{p} - 1\right) H_{12} \end{aligned}$$

From this equation, we obtain two cases: either H is a diagonal matrix by

Proposition 3.6 or $p = 1$. If $p = 1$

$$\begin{aligned}
D_{23} &= \sum_k H_{2k} R_{21}^{-1} - \sum_m R_{21}^* H_{m3} \\
0 &= H_{22} \left(1 - \frac{1}{q}\right) + H_{23} \bar{q} - H_{23} \bar{q} \\
0 &= H_{22} \left(1 - \frac{1}{q}\right)
\end{aligned}$$

We conclude that $q = 1$ since if $H_{22} = 0$ then Q is not invertible. Therefore if $p = 1$ then $q = 1$ which is an element of family 1.

If H is a diagonal matrix then

$$c = -\frac{a\bar{b}}{\bar{d}}$$

implying that

$$\begin{aligned}
D_{ij} &= H_{ii} R_{21}^{-1} - R_{21}^* H_{jj} \\
D_{23} &= H_{22} R_{21}^{-1} - R_{21}^* H_{33} \\
D_{23} &= H_{22} \left(-\frac{1}{pq} + 1\right) \\
D_{23} &= H_{22} \left(-\frac{1}{pq} + 1\right)
\end{aligned}$$

Hence $-\frac{1}{pq} + 1 = 0$ or $p = \frac{1}{q}$, which is a subcase of family 1. All unitary solutions produced by R_{21} are in family 1.

Case: R_{22}

The matrix R_{22} is of the following form after scaling. Note that $|p| = |q| = 1$.

$$R22 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -pq \end{bmatrix}.$$

The inverse and conjugate transpose of $R22$ have the form

$$R22^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{p} & -\frac{1}{pq} + 1 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ 0 & 0 & 0 & -\frac{1}{pq} \end{bmatrix} \quad \text{and} \quad R22^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & 0 \\ 0 & 1 - \bar{p}q & \bar{q} & 0 \\ 0 & 0 & 0 & -\bar{p}q \end{bmatrix}.$$

To determine if $R22$ produces unitary solutions under conjugation, we examine:

$$\begin{aligned} D_{ij} &= \sum_k H_{ik} R22_{kj}^{-1} - \sum_m R22_{im}^* H_{mj} \\ D_{12} &= \sum_k H_{1k} R22_{k2}^{-1} - \sum_m R22_{1m}^* H_{m2} \\ 0 &= \frac{1}{p} H_{12} - H_{12} \\ 0 &= \left(-\frac{1}{p} + 1\right) H_{12} \end{aligned}$$

This implies H is a diagonal matrix, by Proposition 3.6, or $p = 1$. Let $p = 1$,

and examine

$$\begin{aligned}
D_{23} &= \sum_k H_{2k} R_{22}^{-1} - \sum_m R_{22}^* H_{m3} \\
0 &= H_{22} \left(1 - \frac{1}{q}\right) + H_{23} \bar{q} - H_{23} \bar{q} \\
0 &= H_{22} \left(1 - \frac{1}{q}\right)
\end{aligned}$$

If $H_{22} = 0$ then Q is not invertible. The solution which consists of $p = q = 1$ is a subcase of family 1.

If H is a diagonal matrix, then

$$\begin{aligned}
c &= -\frac{a\bar{b}}{d} \text{ and} \\
D_{ij} &= H_{ii} R_{22}^{-1} - R_{22}^* H_{jj} \\
D_{23} &= H_{22} \left(-\frac{1}{pq} + 1\right) \\
0 &= H_{22} \left(-\frac{1}{pq} + 1\right)
\end{aligned}$$

Hence $p = \frac{1}{q}$ which is a subcase of family 1.

The matrix $R23$

We examine the matrix $R23$. We may multiply by a scalar and assume that

$$R23 = \begin{bmatrix} 1 & p & q & s \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case:

$$R23^{-1} = \begin{bmatrix} 1 & -p & -q & -s + 2pq \\ 0 & 1 & 0 & -q \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R23^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{p} & 1 & 0 & q \\ \bar{q} & 0 & 1 & p \\ \bar{s} & \bar{q} & \bar{p} & 1 \end{bmatrix}$$

We obtain

$$\begin{aligned} D_{12} &= \sum_k H_{1k} R23_{k2}^{-1} - \sum_m R23_{1m}^* H_{m2} \\ 0 &= H_{11}(-p) + H_{12} - H_{12} \\ 0 &= -pH_{1,1} \end{aligned}$$

Note that $p = 0$. If $H_{11} = 0$, the matrix Q is not invertible. We consider

the following two entries.

$$D_{13} = \sum_k H_{1k} R_{23_{k3}}^{-1} - \sum_m R_{23_{1m}^*} H_{m3}$$

$$0 = -H_{11}q + H_{13} - H_{13},$$

$$0 = -qH_{11}.$$

$$D_{14} = \sum_k H_{1k} R_{23_{k4}}^{-1} - \sum_m R_{23_{1m}^*} H_{m4}$$

$$0 = H_{11}(-2 + 2pq) - H_{12}p + -H_{13}q + H_{14} - H_{14},$$

$$0 = (-s + 2pq)H_{11} - H_{12}p + -H_{13}q.$$

Hence $p = q = s = 0$. R23 produces trivial solutions which are contained in family 1.

Case: $R12$

After scaling, the matrix $R12$ has the form:

$$R12 := \begin{bmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 1 - q & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q \end{bmatrix}$$

The inverse and the conjugate transpose of the matrix $R12$ are:

$$R12^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\frac{k}{q} \\ 0 & 1 & 1 - \frac{1}{q} & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ 0 & 0 & 0 & -\frac{1}{q} \end{bmatrix} \quad \text{and} \quad R12^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - \bar{q} & \bar{q} & 0 \\ \bar{k} & 0 & 0 & -\bar{q} \end{bmatrix}.$$

Observe that

$$\begin{aligned} D_{23} &= \sum_k H_{2k} R12_{k3}^{-1} - \sum_m R12_{2m}^* H_{m3} \\ &= H_{22} \left(1 - \frac{1}{q}\right) + H_{23} \frac{1}{q} - H_{23} \\ &= \left(1 - \frac{1}{q}\right) (H_{22} - H_{23}). \end{aligned}$$

Therefore, either $q = 1$ or $H_{22} = H_{23}$. If $q = 1$ then

$$\begin{aligned} D_{14} &= \sum_k H_{1k} R12_{k4}^{-1} - \sum_m R12_{1m}^* H_{m4} \\ &= -k H_{11} + H_{14} - H_{14} \\ &= -k H_{11}, \end{aligned}$$

and $k = 0$ since $H_{11} \neq 0$. This solution is part of family 1.

Suppose that $H_{22} = H_{23}$ then $\det(Q) = 0$ contradicting the assumption that Q was invertible.

Case: $R13$

The matrix $R13$ has the form

$$R13 := \begin{bmatrix} 1 & p & -p & pq \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The inverse and conjugate transpose of $R13$ are

$$R13^{-1} = \begin{bmatrix} 1 & -p & p & pq \\ 0 & 1 & 0 & -q \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R13^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{p} & 1 & 0 & 0 \\ -\bar{p} & 0 & 1 & 0 \\ -\bar{p}q & \bar{q} & -\bar{q} & 1 \end{bmatrix}.$$

We consider the equation:

$$D_{12} = \sum_k H_{1k} R13_{k2}^{-1} - \sum_m R13_{1,m}^* H_{m2}$$

$$0 = -pH_{11} + H_{12} - H_{12}$$

$$0 = -pH_{11}$$

$H_{11} \neq 0$ by Proposition 3.5 If $p = 0$ then

$$\begin{aligned}
D_{24} &= \sum_k H_{2k} R_{13_{k4}}^{-1} - \sum_m R_{13_{2m}^*} H_{m4} \\
0 &= -qH_{22} + qH_{23} + H_{24} - H_{24} \\
0 &= -qH_{22} + qH_{23}
\end{aligned}$$

If $H_{22} = H_{23}$ then $\det(Q) = 0$. Hence, $q = 0$ and this solution is a subcase of family 1.

Case: $R14$

The matrix has the following form after scaling:

$$R14 = \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix}$$

We assume that $|pq| = 1$ and that the inverse and conjugate transpose are

$$R14^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{q} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{p} & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R14^* = \begin{bmatrix} 0 & 0 & 0 & \bar{q} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{p} & 0 & 0 & 0 \end{bmatrix}.$$

Consider the individual entry

$$\begin{aligned}
D_{43} &= \sum_k H_{4k} R14_{k3}^{-1} - \sum_m R14_{4m}^* H_{m3} \\
0 &= H_{42} - \bar{p}H_{13} \\
0 &= (\bar{b}\bar{b} + d\bar{d})(\bar{a}\bar{b} + c\bar{d}) - \bar{p}(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + c\bar{d}).
\end{aligned}$$

As a result

$$p = \frac{(\bar{b}\bar{b} + d\bar{d})(\bar{a}\bar{b} + c\bar{d})}{(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + c\bar{d})}$$

or

$$c = -\frac{\bar{a}\bar{b}}{\bar{d}}.$$

Suppose that $c \neq -\frac{\bar{a}\bar{b}}{\bar{d}}$ and $p = \frac{(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + c\bar{d})}{(bb + dd)(ab + cd)}$ then

$$\begin{aligned}
D_{12} &= \sum_k H_{1k} R14_{k2}^{-1} - \sum_m R14_{1m}^* H_{m2} \\
0 &= H_{13} - \bar{q}H_{42} \\
0 &= (a\bar{a} + c\bar{c})(\bar{a}\bar{b} + c\bar{d}) - \bar{q}(\bar{b}\bar{b} + d\bar{d})(\bar{a}\bar{b} + c\bar{d})
\end{aligned}$$

Hence

$$q = \frac{(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + c\bar{d})}{(\bar{b}\bar{b} + d\bar{d})(\bar{a}\bar{b} + c\bar{d})}$$

Each entry is now zero in the matrix D. If

$$p = \frac{(b\bar{b} + d\bar{d})(\bar{a}b + \bar{c}d)}{(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + \bar{c}\bar{d})}$$

$$q = \frac{(a\bar{a} + c\bar{c})(\bar{a}\bar{b} + \bar{c}\bar{d})}{(b\bar{b} + d\bar{d})(\bar{a}b + \bar{c}d)}$$

we obtain the second family of solutions.

Now suppose that $c = -\frac{\bar{a}\bar{b}}{d}$. All entries in D except the following two entries are zero.

$$D_{14} = \sum_k H_{1k} R_{14_{k4}}^{-1} - \sum_m R_{14_{1m}}^* H_{m4}$$

$$0 = H_{11} \frac{1}{q} - \bar{q} H_{44}$$

$$0 = (a\bar{a} + c\bar{c})^2 \frac{1}{q} - \bar{q} (b\bar{b} + d\bar{d})^2$$

This implies that:

$$\frac{(d\bar{d} + b\bar{b})^2 (\bar{a}^2 a^2 - q\bar{q} d^2 \bar{d}^2)}{q d^2 \bar{d}^2} = 0$$

and $|a|^2 = |q||d|^2$.

$$D_{41} = \sum_k H_{4k} R_{41_{k1}}^{-1} - \sum_m R_{41_{4m}}^* H_{m1}$$

$$0 = H_{44} \frac{1}{p} - \bar{p} H_{11}$$

$$0 = (b\bar{b} + d\bar{d})^2 \frac{1}{p} - \bar{p} (a\bar{a} + c\bar{c})^2$$

This implies that:

$$\frac{(d\bar{d} + b\bar{b})^2(p\bar{p}\bar{a}^2a^2 - d^2\bar{d}^2)}{pd^2\bar{d}^2} = 0$$

and $|p||a|^2 = |d|^2$. This produces the third family of solutions.

Case: $R01$

Consider the matrix $R01$.

$$R01 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse and conjugate transpose of this matrix are

$$R01^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R01^* := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

We observe

$$D_{41} = \sum_k H_{4k} R01_{k1}^{-1} - \sum_m R01_{4,m}^* H_{m,1}$$

$$0 = H_{41} - H_{11} - H_{41}$$

$$0 = H_{11}.$$

If $H_{11} = 0$ then Q is not an invertible matrix. This matrix produces no unitary solutions.

Case: $R02$

Consider the matrix $R02$.

$$R02 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The inverse and conjugate transpose of this matrix are

$$R02^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad R02^* := \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $R02$ and $R02^*$ is not a unitary matrix. However, in [3], the solutions are families determined by constants and $Q \otimes Q$. We multiply the matrix $R02$ by the constant $2^{-\frac{1}{2}}$ and consider the modified matrix, $R02'$.

$$R02' := \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The inverse and conjugate transpose of this matrix are

$$R02'^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad R02'^* := \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $R02'$ is unitary, and we determine the family produced by $R02'$.

$$\begin{aligned} D_{11} &= \sum_k H_{1k} R02'_{k1}{}^{-1} - \sum_m R02'_{1m}{}^* H_{m1} \\ 0 &= H_{11} R02'_{11}{}^{-1} + H_{14} R02'_{41}{}^{-1} - R02'_{11}{}^* H_{11} - R02'_{14}{}^* H_{41} \\ 0 &= \frac{1}{\sqrt{2}}(H_{11} + H_{14} - H_{11} + H_{41}) \\ &= \frac{1}{\sqrt{2}}(H_{14} + H_{41}). \end{aligned}$$

Hence,

$$c = -\frac{a\bar{b}}{d}$$

Using this result, we determine that:

$$\begin{aligned} D_{14} &= \sum_k H_{1k} R02'_{k4}{}^{-1} - \sum_m R02'_{1m}{}^* H_{m4} \\ 0 &= H_{11} R02'_{14}{}^{-1} + H_{14} R02'_{44}{}^{-1} - R02'_{11}{}^* H_{14} - R02'_{14}{}^* H_{44} \\ 0 &= -\frac{1}{\sqrt{2}}(H_{11} - H_{44}). \end{aligned}$$

This reduces to: $|a| = |d|$. Hence, this forms the fourth family of solutions to the Yang-Baxter equation.

Case: $R03$

Consider the matrix $R03$. This is a unitary matrix and $R03 = R03^* = R03^{-1}$.

$$R03 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We examine the equation

$$\begin{aligned} D_{ij} &= \sum_k H_{ik} R03_{kj}^{-1} - \sum_m R03_{im}^* H_{mj} \\ 0 &= \sum_k H_{ik} R03_{kj} - R03_{ik} H_{kj} H_{22} \left(-\frac{1}{pq} + 1 \right) \end{aligned}$$

Computations will demonstrate that D is the zero matrix. This case gives rise to the solutions of type 5.

5 Solutions to the Bracket Equation

The Kauffman bracket skein relation determines solutions to the braided Yang-Baxter equation, [5]. We determine which 4×4 matrix solutions of the bracket skein relation are unitary. These solutions are a subcase of the family 3 indicated in Theorem 4. If \hat{R} is a 4×4 matrix solution to the bracket skein relation then it satisfies the following equation.

$$\hat{R} = \alpha I_4 + \alpha^{-1} U$$

where $U = N \odot K$ and N, K are 2×2 dimensional matrices. The operator \odot is defined for N and K

$$N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$K = \begin{bmatrix} g & h \\ k & l \end{bmatrix}$$

$$N \odot K = \begin{bmatrix} ag & ah & ak & al \\ bg & bh & bk & bl \\ cg & ch & ck & cl \\ dg & dh & dk & dl \end{bmatrix}$$

Referring to [5] the matrix U has the property that

$$U^2 = -(\alpha^2 + \alpha^{-2})U.$$

We will denote $-(\alpha^2 + \alpha^{-2})$ as δ . We will assume that α has norm one and that $\alpha = e^{i\theta}$. We obtain the following facts from [5].

$$\hat{R}^{-1} = \alpha^{-1}I_4 + \alpha U$$

$$K = N^{-1}$$

$$\delta = \sum_{a,b} N_{ab} N_{ab}^{-1}$$

If \hat{R} is unitary, we obtain the additional restriction that $\hat{R}^* = \hat{R}^{-1}$.

$$\begin{aligned}\hat{R}^* &= \hat{R}^{-1} \\ \bar{\alpha}I_4 + \bar{\alpha}^{-1}U^* &= \alpha^{-1}I_4 + \alpha U\end{aligned}$$

Recall that α has norm one implying that $\alpha = \bar{\alpha}^{-1}$. From this fact, we obtain $U = U^*$. We determine that $\bar{N} = N^{-1}$ from

$$\begin{aligned}U &= U^* \\ N \odot N^{-1} &= \bar{N}^{-1} \odot \bar{N}.\end{aligned}$$

We use the fact that $\bar{N} = N^{-1}$ to determine the value of α . Note that this argument did not refer to the dimensionality of N . If $|\alpha| = 1$, then \hat{R} is a 4×4 matrix. We demonstrate this for an $n \times n$ matrix N by calculating δ and the trace of $N\bar{N}$. Note that $N\bar{N} = I_n$.

$$\begin{aligned}\delta &= \sum_{i,j} N_{ij} \bar{N}_{ij} \\ \delta &= \sum_k N_{kk} \bar{N}_{kk} + \sum_{i \neq j} N_{ij} \bar{N}_{ij}\end{aligned}$$

We compute that value of the trace of $N\bar{N}$.

$$\begin{aligned}\text{trace}(N\bar{N}) &= n = \sum_{j,k} N_{kj} \bar{N}_{jk} \\ n &= \sum_k N_{kk} \bar{N}_{kk} + \sum_{i \neq j} N_{ij} \bar{N}_{ji}\end{aligned}$$

Combining these two calculations, we determine that

$$\begin{aligned}\delta - n &= \sum_{i \neq j} N_{ij} \bar{N}_{ij} - \sum_{i \neq j} N_{ij} \bar{N}_{ji} \\ \delta - n &= \sum_{i \leq j} |N_{ij} - N_{ji}|^2\end{aligned}$$

Hence, if N is a $n \times n$ matrix then $\delta \geq n$. This inequality contradicts the fact that $\delta = -(\alpha^2 + \alpha^{-2}) = -(e^{2i\theta} + e^{-2i\theta})$, unless $n \leq 2$. We consider the specific case when N is a 2×2 matrix.

$$\begin{aligned}N\bar{N} &= NN^{-1} \\ N\bar{N} &= \begin{bmatrix} a\bar{a} + b\bar{c} & \bar{b}a + b\bar{d} \\ \bar{a}c + \bar{c}d & \bar{b}c + d\bar{d} \end{bmatrix} \\ N\bar{N} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

We examine the case when N is a 2×2 matrix. Recall that $\delta = \sum_{a,b} N_{a,b} N_{a,b}^{-1}$. Note that $\sum_{a,b} N_{a,b} N_{a,b}^{-1} = \sum_{a,b} N_{a,b} \bar{N}_{a,b}$, since $\bar{N} = N^{-1}$. From these facts we obtain

$$\begin{aligned}\delta &= a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} \\ \delta &= 1 - b\bar{c} + b\bar{b} + c\bar{c} + 1 - \bar{b}c \\ \delta &= 2 + (b - c)(\bar{b} - \bar{c})\end{aligned}$$

so that $|\delta| \geq 2$. Recall that $\alpha = e^{i\theta}$ and $\delta = -(\alpha^2 + \alpha^{-2})$, we compute that

$$\delta = -2\cos(\theta).$$

Implying that $|\delta| \leq 2$. These inequalities indicate that $|\delta| = 2$. Using this result, we determine that $\alpha = i$ and $b = c$. With the restrictions that $b = c$ and $\bar{N} = N^{-1}$ then

$$N = \begin{bmatrix} re^{ig} & \sqrt{1-r^2}e^{i\frac{p}{2}} \\ \sqrt{1-r^2}e^{i\frac{p}{2}} & re^{i(p-g)} \end{bmatrix}$$

We determine the family from Theorem 4 of this solution. We determine a matrix Q such that

$$\hat{R}_M = (Q \otimes Q)\hat{R}(Q \otimes Q)^{-1}, \quad (8)$$

where \hat{R}_M is a representative of a family from Theorem 4.1 constructed from a matrix M . The family of the \hat{R} is also dependent on the value of Q .

Notice that

$$\begin{aligned} \hat{R}_M &= (Q \otimes Q)\hat{R}(Q \otimes Q)^{-1} \\ \alpha Id_4 + \alpha^{-1}M \odot \bar{M} &= (Q \otimes Q)\alpha I_4 + \alpha^{-1}N \odot \bar{N}(Q \otimes Q)^{-1} \\ \alpha Id_4 + \alpha^{-1}M \odot \bar{M} &= \alpha I_4 + \alpha^{-1}(Q \otimes Q)N \odot \bar{N}(Q \otimes Q)^{-1} \end{aligned}$$

Note that

$$(Q \otimes Q)N \odot \bar{N}(Q \otimes Q)^{-1} = (Q N Q^t) \odot (Q^{-1} N (Q^{-1})^t).$$

The conditions of the equation 8 are equivalent to determining a matrix Q such that $Q N Q^t = M$. If Q has the following form:

$$Q = \begin{bmatrix} x & 0 \\ z & r^{1/2}x \end{bmatrix}$$

and

$$z = \frac{-i\sqrt{1-r^2}e^{i(\frac{p}{2}-g)}}{\sqrt{r}}$$

then

$$M = \begin{bmatrix} x^2 r e^{ig} & 0 \\ 0 & x^2 e^{i(p-g)} \end{bmatrix}.$$

We determine the family of \hat{R}_M by considering a generic 2×2 diagonal matrix.

$$M = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

Then

$$M \odot M^{-1} = \begin{bmatrix} 1 & 0 & 0 & \frac{x}{y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{y}{x} & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{R}_M = \alpha I_4 + \alpha^{-1} M \odot M^{-1} = \begin{bmatrix} 0 & 0 & 0 & \alpha^{-1} \frac{x}{y} \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ \alpha^{-1} \frac{y}{x} & 0 & 0 & 0 \end{bmatrix}$$

The matrix \hat{R}_M is a solution to the braided Yang-Baxter equation. To obtain a solution to the algebraic Yang-Baxter equation, we multiply the matrix \hat{R}_M

by the matrix T given in Theorem 4.

$$\hat{R}_M T = R_M = \begin{bmatrix} 0 & 0 & 0 & \alpha^{-1} \frac{x}{y} \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ \alpha^{-1} \frac{y}{x} & 0 & 0 & 0 \end{bmatrix}$$

The matrix \hat{R}_M is an element of family 2 or family 3, dependent on the value of Q .

Finally, we observe that the R matrix and Q produced satisfy the conditions of family 3 from Theorem 4.1

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