DIFFERENCE AND DIFFERENTIAL EQUATIONS FOR THE COLORED JONES FUNCTION

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Abstract. A function of one variable is holonomic if it satisfies a linear differential equation with polynomial coefficients. This notion was extended by Zeilberger to discrete functions (that is, to sequences). Holonomicity is an important property studied from the point of view of algebraic geometry and D-module theory by I.N. Bernstein, and in combinatorics by D. Zeilberger. TTQ Le and the author recently proved that the colored Jones function of a knot is holonomic, i.e., that it satisfies a difference equation. In this paper, we convert the difference equation into a hierarchy of differential equations for the so-called Euler expansion of the colored Jones function. Moreover, we study holonomicity of the colored Jones function from the point of view of quantum field theory, and compare it with the skein theory approach of the colored Jones function initiated by Frohman and Gelca.

1. Introduction

1.1. **The goal.** It was recently shown by TTQ Le and the author that the colored Jones function of a knot is q-holonomic, [GL], in other words that it satisfies a nontrivial difference relation. In this paper, we convert the difference relation into a hierarchy of differential equations for the so-called Euler expansion of the colored Jones function.

Moreover, we study holonomicity of the colored Jones function from the point of view of quantum field theory, and compare it with the skein theory approach of the colored Jones function initiated by Frohman and Gelca.

1.2. **Holonomic functions.** A holonomic function $f(x)$ in one variable is one that satisfies a nontrivial linear differential equation with polynomial coefficients. Holonomicity was introduced by I.N. Bernstein [B1, B2] in relation to algebraic geometry, D-modules and differential Galois theory. In a stroke of brilliance, Zeilberger noticed that holonomicity can be applied to verify, in a systematic way, combinatorial identities among special functions, [Z]. This was later implemented on a computer, [WZ, PWZ].

A key idea is to study the recursion relations that a function satisfies, rather than the function itself. This idea leads in a natural way to noncommutative algebras of operators that act on a function, together with left ideals of annihilating operators.

To explain this idea concretely, consider the operators x and ∂ which act on a smooth function f defined on $\mathbb{R}$ (or a distribution, or whatever else can be differentiated) by

$$(xf)(x) = xf(x) \qquad\qquad (\partial f)(x) = \frac{\partial}{\partial x} f(x).$$

Leibnitz's rule $\partial(xf) = x\partial(f) + f$ written in operator form states that $\partial x = x\partial + 1$. The operators x and ∂ generate the *Weyl algebra* which is a free noncommutative algebra on x and ∂ modulo the two sided ideal $\partial x - x\partial - 1$:

$$\mathcal{A} = \frac{\mathbb{C}\langle x, \partial\rangle}{(\partial x - x\partial - 1)}.$$

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The Weyl algebra is nothing but the *algebra of differential operators in one variable with polynomial coefficients*. Given a function f of one variable, let us define the *recursion ideal* $\mathcal{I}_f$ by

$$\mathcal{I}_f = \{P \in \mathcal{A} | \, Pf = 0\}$$

It is easy to see that $\mathcal{I}_f$ is a left ideal of $\mathcal{A}$. Following Zeilberger and Bernstein, we say that f is *holonomic* iff $\mathcal{I}_f \neq 0$. In other words, a holonomic function is one that satisfy a linear differential equation with polynomial coefficients.

A key property of the Weyl algebra $\mathcal{A}$ (shared by its cousins, $\mathcal{B}$ and $\mathcal{C}$ defined below) is that it is *Noetherian*, which implies that every left ideal is *finitely generated*. In particular, a holonomic function is uniquely determined by a finitely list, namely the generators of its recursion ideal and a finite set of initial conditions.

The set of holonomic functions is closed under summation and product. Moreover, holonomicity can be extended to functions of several variables. For an excellent exposition of these results, see [Bj].

Zeilberger expanded the definition of holonomic functions of a continuous variable to *discrete functions* f (that is, functions with domain $\mathbb{Z}$) by replacing differential operators by *shift* operators. More precisely, consider the operators N and E which act on a discrete function $f(n)$ by

$$(Nf)(n) = nf(n) \qquad\qquad (Ef)(n) = f(n+1).$$

It is easy to see that $EN = NE + E$. The *discrete Weyl algebra* $\mathcal{B}$ is a noncommutative algebra with presentation

$$\mathcal{B} = \frac{\mathbb{Q}\langle N^{\pm}, E^{\pm}\rangle}{(EN - NE - E)}.$$

The field coefficients $\mathbb{Q}$ are not so important, and neither is the fact that we allow positive as well as negative powers of N and E. Given a discrete function f, one can define the *recursion ideal* $\mathcal{I}_f$ in $\mathcal{B}$ as before. We will call a discrete function f *holonomic* iff the ideal $\mathcal{I}_f \neq 0$.

In our paper we will consider a q-variant of the Weyl algebra. Consider the operators E and Q which act on a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R} := \mathbb{Z}[1/2, q^{\pm 1/2}]$ by:

$$(Qf)(n) = q^n f(n) \qquad\qquad (Ef)(n) = f(n+1)$$

It is easy to see that $EQ = qQE$. We define the *q-Weyl ring* $\mathcal{C}$ to be a noncommutative algebra with presentation

$$\mathcal{C} = \frac{\mathcal{R}\langle Q, E\rangle}{(EQ - qQE)}.$$

$\mathcal{C}$ is the famous *quantum torus* [Ka, Chapter IV], [M]. Given a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R}$, one can define the left ideal $\mathcal{I}_f$ in $\mathcal{C}$ as before, and call a discrete function f *q-holonomic* iff the ideal $\mathcal{I}_f \neq 0$. Concretely, a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R}$ is q-holonomic iff there exists a nonzero element $\sum_{a,b} c_{a,b} E^a Q^b \in \mathcal{C}$ such that

$$\sum_{a,b} c_{a,b} q^{(n+a)b} f(n+a) = 0. \tag{1}$$

The sequence $J : \mathbb{Z} \longrightarrow \mathbb{Z}[q^{\pm 1}]$ of Laurent polynomials that we have in mind is the celebrated *colored Jones function.*

1.3. **The colored Jones function.** In 1985 Jones introduced his polynomial of a knot in 3-space, and a few years later Witten interpreted the Jones polynomial in terms of a topological quantum field theory. From that point of view, it is natural to consider the Jones polynomial of a knot, together with all of its cables. This information can be described in terms of a sequence $\{J_n\}$ of polynomials, where J_n is the polynomial associated with the n-dimensional representation of $\mathfrak{sl}_2$. The sequence $\{J_n\}$, (normalized by $J_1 = 1$, $J_0 = 0$ and extended to integer indices by $J_n = -J_{-n}$), is often called the *colored Jones function* of the knot.

Intuitively, it is clear that a knot is a finite object. On the other hand, its colored Jones function is a sequence of polynomials, and thus an infinite object. Is there some repetition in the information encoded by the colored Jones function? An answer to this puzzle is the following:

Theorem 1. [GL] *The colored Jones function of every knot in S^3 is q-holonomic.*

The history of this theorem is interesting. The statement of the theorem can be motivated by TQFT (topological quantum field theory) arguments about the colored Jones function, encoded skein theory. As we will see shortly, Frohman and Gelca use Kauffman bracket skein theory to associate to a knot two further ideals, the peripheral and the orthogonal ideal. Frohman and Gelca claimed that the peripheral ideal is nonzero, [FGL, Prop.8] and [Ge1, Proof of Cor.1]. Combined with our Theorem 2 below, it would prove Theorem 1. Unfortunately, there is an error in the argument of Frohman and Gelca.

1.4. **The peripheral and orthogonal ideals of a knot.** The recursion relations (1) for the colored Jones function are motivated by the work of Frohman, Gelca, Przytycki, Sikora and others on the Kauffman bracket skein module, and its relation to the colored Jones function. Let us briefly recall the main ideas.

For a manifold N, of any dimension, possibly with nonempty boundary, let $\mathcal{S}_q(N)$ denote the *Kauffman bracket skein module*, which is an $\mathcal{R}$-module (where $\mathcal{R} = \mathbb{Z}[1/2, q^{\pm 1/2}]$) generated by the isotopy classes of *framed unoriented* links in N (including the empty one), modulo the relations of Figure 1

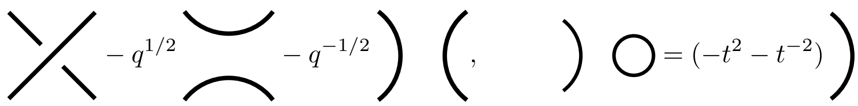

Figure 1. The relations of the Kauffman bracket skein module

Let us recall some elementary facts of skein theory, reminiscent of TQFT:

Fact 1: If $N = N' \times I$, where N' is a closed manifold, then $\mathcal{S}_q(N)$ is an algebra.
Fact 2: If N is a manifold with boundary ∂N, then $\mathcal{S}_q(N)$ is a module over the algebra $\mathcal{S}_q(\partial N \times I)$.
Fact 3: If $N = N_1 \cup_Y N_2$ is the union of N_1 and N_2 along their common boundary Y, then there is a map:

$$\langle \, , \rangle : \mathcal{S}_q(N_1) \otimes_{\mathcal{S}_q(\partial Y \times I)} \mathcal{S}_q(N_1) \longrightarrow \mathcal{S}_q(N)$$

We will apply the previous discussion in the following situation. Let K denote a knot in a homology sphere N, and let M denote the complement of a thinkening of K. Then, using the abbreviation $\mathcal{S}_q(\mathbb{T}) := \mathcal{S}_q(\mathbb{T}^2 \times I)$, Gelca and Frohman introduced in [FG, FGL] the peripheral and orthogonal ideals of K (the latter was called *formal* in [FGL, Sec.5]):

Definition 1.1. (a) We define the *peripheral ideal* $\mathcal{P}(K)$ of K to be the *annihilator* of the action of $\mathcal{S}_q(\mathbb{T})$ on $\mathcal{S}_q(M)$. In other words,

$$\mathcal{P}(K) = \{P \in \mathcal{S}_q(\mathbb{T}) | P.\emptyset = 0\}$$

(b) We define the *orthogonal ideal* of a knot K to be

$$\mathcal{O}(K) = \{v \in \mathcal{S}_q(\mathbb{T}) | \, \langle \mathcal{S}_q(S^1 \times D^2)v, \emptyset \rangle = 0\}.$$

Note that $\mathcal{O}(K)$ and $\mathcal{P}(K)$ are left ideals in $\mathcal{S}_q(\mathbb{T})$ and that $\mathcal{P}(K) \subset \mathcal{O}(K)$. Unfortunately, the peripheral and orthogonal ideals of a knot do not seem to be algorithmically computable objects.

To understand the peripheral and orthogonal ideals requires a better description of the ring $\mathcal{S}_q(\mathbb{T})$, and its module $\mathcal{S}_q(S^1 \times D^2)$.

The skein module $\mathcal{S}_q(F \times I)$ is well-studied for a closed surface F. It is a free $\mathcal{R}$-module on the set of free homotopy classes of finite (possibly empty) collections of disjoint unoriented curves in F without contractible components. In particular, for a 2-torus $\mathbb{T}$, $\mathcal{S}_q(\mathbb{T})$ is the quotient of the free $\mathcal{R}$-module on the set $\{(a,b) \, | a, b \in \mathbb{Z}\}$ modulo the relations $(a,b) = (-a,-b)$. The multiplicative structure of $\mathcal{S}_q(\mathbb{T})$ is well-known, and related to the *even trigonometric polynomials* of the *quantum torus*, [FG]. Let us recall this description due to Gelca and Frohman. Consider the ring involution given by

$$\tau : \mathcal{C} \longrightarrow \mathcal{C} \qquad E^a Q^b \longrightarrow E^{-a} Q^{-b} \tag{2}$$

and let $\mathcal{C}^{\mathbb{Z}_2}$ denote the invariant subring of $\mathcal{C}$. Frohman-Gelca [FG] prove that

Fact 4: The map

$$\Phi : \mathcal{S}_q(\mathbb{T}) \longrightarrow \mathcal{C}^{\mathbb{Z}_2} \tag{3}$$

given by

$$(a,b) \longrightarrow (-1)^{a+b} q^{-ab/2} (E^a Q^b + E^{-a} Q^{-b}).$$

is an isomorphism of rings.

Thus, to a knot one can associate three ideals: the recursion ideal in $\mathcal{C}$ and the peripheral and the orthogonal ideal in $\mathcal{S}_q(\mathbb{T})$. The next theorem explains the relation between the recursion and orthogonal ideals.

Theorem 2. (a) We have:

$$\Phi(\mathcal{O}) = \mathcal{C}^{\mathbb{Z}_2} \cap \mathcal{I}.$$

In particular, the colored Jones function of a knot determines its orthogonal ideal.
(b) $\mathcal{I}$ is invariant under the ring involution τ.

The proof of the above theorem proves the following corollary which compares the *orthogonality relations* of Gelca [Ge1, Sec.3] with the recursion relations given here:

Corollary 1.2. *Fix an element x of the orthogonal ideal of a knot. The orthogonality relation for the colored Jones function is $(E - E^{-1})xJ = 0$. On the other hand, Theorem 2 implies that $xJ = 0$. It follows that a $(d+2)$-term recursion relation for the colored Jones function given by Gelca is implied by a d-term recursion relation.*

1.5. **The Euler expansion of the colored Jones function.** The colored Jones function $\{J_n\}$ may be repackaged in a sequence into a sequence $\{Q_k\}$ of rational functions, via the *Euler expansion*, as follows:

$$J_n(q) \quad = \quad \frac{1}{q - q^{-1}} \sum_{k=0}^{\infty} Q_k(q^n)(q-1)^k, \tag{4}$$

where

$$Q_k(q) = \frac{P_k(q)}{\Delta(q)^{2k+1}},$$

$\Delta(q) \in \mathbb{Z}[q^{\pm 1}]$ is the *Alexander polynomial* and $P_k(q) \in \mathbb{Z}[q^{\pm 1}]$.

It is easy to see that the colored Jones function $\{J_n\}$ determines and is determined by the Euler function $\{Q_k\}$. The Euler expansion of the colored Jones function of a knot was established by Rozansky for $\mathfrak{sl}_2$ and in general by Kricker and the author, see [GK] and [Ga].

Our next result translates difference equations for $\{J_n\}$ to differential equations for $\{Q_k\}$.

Theorem 3. (a) Theorem 1 implies a hierarchy of ODEs for $\{Q_k\}$. More precisely, for every knot there exist a lower diagonal matrix of infinite size

$$D = \begin{pmatrix} D_0 & 0 & 0 & \dots \\ D_1 & D_0 & 0 & \ddots \\ D_2 & D_1 & D_0 & \ddots \\ \dots & \dots & \dots & \ddots \end{pmatrix}$$

such that $D_i \in A_1$, $0 \neq D_0$ and $DQ = 0$ where $Q = (Q_1, Q_2, \dots)^T$.
(b) The above hierarchy uniquely determines the sequence $\{Q_k\}$ up to a finite number of initial conditions $\{\frac{d^l}{ds^l}|_{s=1} Q_k(s)\}$ for $0 \leq l \leq \deg(D_0)$.

The above hierarchy is reminiscent of *matrix models* discussed in physics. See for example [DV] and Question 1 below.

1.6. **Questions.** Let us end with some questions and comments.

Question 1. Is there a *physical meaning* to the recursion relations of the colored Jones function, and in particular of the hierarchy of ODE which is satisfied by its Euler expansion? Differential equations often hint at a hidden *matrix model*, or an *M-theory* explanation.

Question 2. The hierarchy of ODEs that appear in Theorem 3 also appears, under the name of *semi-pfaffian chain*, in complexity questions of real algebraic geometry. We thank S. Basu for pointing this out to us. For a reference, see [GV]. Is this a coincidence?

Question 3. In [GK] Kricker and the author constructed a rational form Z^{rat} of the Kontsevich integral of a knot. As was explained in [Ga], this rational form becomes the Euler expansion of the colored Jones function, on the level of Lie algebras. In [GL] it is shown that the $\mathfrak{g}$-colored Jones function for any simple Lie algebra $\mathfrak{g}$ is q-holonomic. This raises the question: is there a notion of *differential hierarchy* for the Z^{rat} invariant? This would imply that the Z^{rat} invariant of a knot (and thus, also the Kontsevich integral of a knot) is determined uniquely from a finite list. Since knots are finite objects, this may be possible.

Question 4. Theorem 2 proves that $\mathcal{I}$ is an ideal in $\mathcal{C}^{\mathbb{Z}_2}$ invariant under the ring involution τ. Is it true that $\mathcal{I}$ is generated by its $\mathbb{Z}_2$-invariant part? In other words, is it true that $\mathcal{I} = \mathcal{C}(\mathcal{C}^{\mathbb{Z}_2} \cap \mathcal{I})$?

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2. Proof of Theorem 2

Let us begin by understanding the recursion relation for the colored Jones function which is obtained by a nonzero element of the orthogonal ideal of a knot. This uses work of Gelca, which we will quote here. For proofs, we refer the reader to [Ge1].

The problem is to understand the right action of the ring $\mathcal{S}_q(\mathbb{T})$ on the skein module $\mathcal{S}_q(S^1 \times D^2)$.

To begin with, the skein module $\mathcal{S}_q(S^1 \times D^2)$ can be identified with the polynomial ring $\mathcal{R}[\alpha]$, where α is a longitudonal curve in the solid torus, and $\mathcal{R} = \mathbb{Z}[1/2, q^{\pm 1/2}]$. Rather than using the $\mathcal{R}$-basis for $\mathcal{S}_q(S^1 \times D^2)$ given by $\{\alpha^n\}_n$, Gelca uses the basis given by $\{T_n(\alpha)\}_n$, where $\{T_n\}$ is a sequence of Chebytchev-like polynomials defined by $T_0(x) = 2$, $T_1(x) = x$ and $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$.

Recall that the ring $\mathcal{S}_q(\mathbb{T})$ is generated by symbols (a, b) for integers a and b and relations $(a, b) = (-a, -b)$. Gelca [Ge1, Lemma 1] describes the right action of $\mathcal{S}_q(\mathbb{T})$ on $\mathcal{S}_q(S^1 \times D^2)$ as follows:

$$
\begin{aligned}
(5) \qquad T_n(\alpha) \cdot (a,b) \quad &= \quad q^{ab/2}(-1)^b \left(q^{nb} \left[q^b S_{n+a}(\alpha) - q^{-b} S_{n+a-2}(\alpha) \right] \right. \\
&\quad \left. + \, q^{-nb} \left[-q^b S_{n-a-2}(\alpha) + q^{-b} S_{n-a}(\alpha) \right] \right),
\end{aligned}
$$

where $\{S_n\}$ is a sequence of Chebytchev polynomials defined by $S_0(x) = 1$, $S_1(x) = x$ and $S_{n+1}(x) = xS_n(x) - S_{n-1}(x)$.

Consider an element $x = \sum_{a,b} c_{a,b}(a, b)$ of the orthogonal ideal of a knot and recall the pairing $\langle \, , \rangle$ from Fact 3. Since the (shifted) nth colored Jones polynomial of a knot is given by $J_n = (-1)^{n-1}\langle S_{n-1}(\alpha), \emptyset \rangle$, Equation (5) implies the recursion relation

$$
\begin{aligned}
(6) \qquad 0 \quad = \quad & \sum_{a,b} c_{a,b} q^{ab/2}(-1)^{a+b} \left(q^{nb} \left[q^b J_{n+a+1}(K) - q^{-b} J_{n+a-1}(K) \right] \right. \\
& \left. + \, q^{-nb} \left[-q^b J_{n-a-1}(K) + q^{-b} J_{n-a+1}(K) \right] \right)
\end{aligned}
$$

for the colored Jones function corresponding to an element $\sum_{a,b} c_{a,b}(a, b)$ in the orthogonal ideal.

Remark 2.1. Our notation differs slightly from Gelca's. Gelca's t^2 equals to q and further, Gelca's $(-1)^n J_n$ is our J_{n+1}.

Let us write the above recursion relation in operator form. Recall that the operators E and Q act on the discrete function J by $(EJ)(n) = J(n + 1)$ and $QJ(n) = q^n J(n)$, and satisfy the commutation relation $EQ = qQE$. Then, Equation (6) becomes:

$$
0 = \sum_{a,b} c_{a,b} q^{ab/2}(-1)^{a+b} \left(q^b Q^b E^{a+1} - q^{-b} Q^b E^{a-1} - q^b Q^{-b} E^{-a-1} + q^{-b} Q^{-b} E^{-a+1} \right) J.
$$

Using the commutation relation $E^k Q^l = q^{kl} Q^l E^k$ for integers k, l and moving the E's on the left and the Q's on the right, we obtain that

$$0 = (E - E^{-1}) \sum_{a,b} c_{a,b} q^{-ab/2} (E^a Q^b + E^{-a} Q^{-b}) J.$$

Recall the isomorphism Φ of Equation (3). Using this isomorphism, our discussion so far implies that $x \in \mathcal{C}^{\mathbb{Z}_2}$ is an element of the orthogonal ideal of a knot iff $(E - E^{-1})x$ lies in the recursion ideal. It remains to show that for every $x \in \mathcal{C}^{\mathbb{Z}_2}$, $(E - E^{-1})x \in \mathcal{I}$ iff $x \in \mathcal{I}$. One direction is obvious since $\mathcal{I}$ is a left ideal. For the opposite direction, consider $x \in \mathcal{C}^{\mathbb{Z}_2}$, and let $y = (E - E^{-1})x$ and $f = xJ$. Assume that $y \in \mathcal{I}$. We need to show that $x \in \mathcal{I}$; in other words that $f = 0$.

We have $(E^2 - I)f = E(E - E^{-1})f = 0$, which implies that

$$f(n+2) = f(n) \tag{7}$$

for all $n \in \mathbb{Z}$.

Recall the symmetry relation $J_n + J_{-n} = 0$ for the colored Jones function. In order to write it in operator form, consider the operator S that acts on a discrete function f by $(Sf)(n) = f(-n)$. Then, $(S + I)J = 0$.

It is easy to see that

$$SE = E^{-1} \qquad SQ = Q^{-1} S. \tag{8}$$

Since $\mathcal{C}^{\mathbb{Z}_2}$ is generated by $E^a Q^b + E^{-a} Q^{-b}$, it follows that S commutes with every element of $\mathcal{C}^{\mathbb{Z}_2}$; in particular $Sx = xS$, and thus $(S + I)f = (S + I)xJ = x(S + I)J = 0$. In other words,

$$f(n) + f(-n) = 0 \tag{9}$$

for all n. Equations (7) and (9) imply that $f(2n) = f(0)$, $f(2n+1) = f(1)$, $f(0) = f(1) = 0$. Thus, $f = 0$. This completes part (a) of Theorem 2.

For part (b), consider $x \in \mathcal{I}$ and recall the involution τ of (2). Then, we have $x = x_+ + x_-$ where $x_\pm = 1/2(x \pm \tau(x)) \in \mathcal{C}_\pm$, where $\mathcal{C}_\pm$ is generated by $E^a Q^b \pm E^{-a} Q^{-b}$. (8) implies that $Sx_+ = x_+ S$ and $Sx_- = x_- S$. Now, we have

$$0 = xJ = x(-J) = xSJ = (x_+ + x_-)SJ = S(x_+ - x_-)J.$$

Since $S^2 = I$, it follows that $(x_+ - x_-)J = 0$. This, together with $0 = (x_+ + x_-)J$, implies that $x_\pm J = 0$. In other words, $x_\pm \in \mathcal{I}$. Since $\tau(x) = x_+ - x_-$, it follows that $\mathcal{I}$ is invariant under τ. □

The proof of Theorem 2 also proves Corollary 1.2.

Example 2.2. In [Ge2] Gelca computes that the following element

$$(1, -2k-3) - q^{-4}(1, -2k+1) + q^{(2k-5)/2}(0, 2k+3) - q^{(2k-1)/2}(0, 2k-1)$$

lies in the peripheral (and thus orthogonal) ideal of the left handed $(2, 2k+1)$ torus knot. Using the isomorphism Φ and Theorem 2 and a simple calculation, it follows that the following element

$$-q^2(EQ^{-2k-3} + E^{-1}Q^{2k+3}) + q^{-4}(EQ^{-2k+1} + E^{-1}Q^{2k-1}) + q^{-2}(Q^{2k+3} + Q^{-2k-3}) - (Q^{2k-1} + Q^{-2k+1})$$

lies in the recursion ideal of the left handed $(2, 2k+1)$ torus knot. This element gives rise to a 3-term recursion relation for the colored Jones function.

3. Proof of Theorem 3

Fix a discrete function $J : \mathbb{Z} \longrightarrow \mathbb{Z}[q^{\pm 1}]$ that satisfies $XJ = 0$ where $X = \sum_{a,b} c_{a,b} E^a Q^b \in \mathcal{C}$. In other words, we have:

$$0 = \sum_{a,b} c_{a,b} q^{(n+a)b} J_{n+a}.$$

Let us introduce a new variable $s = q^n$. Substituting for

$$J_{n+a}(q) = \frac{1}{q - q^{-1}} \sum_{k=0}^{\infty} Q_k(sq^a)(q-1)^k$$

(as follows from Equation (4)) into the recursion relation, and interchanging the order of summation, we obtain that

$$0 = \sum_{k=0}^{\infty} (q-1)^k X' Q_k \tag{10}$$

where $X' : \mathbb{Q}(s) \longrightarrow \mathbb{Q}(s)[[q-1]]$ is the operator defined by

$$f \mapsto X'f = \sum_{a,b} c_{a,b} s^b q^{ab} f(sq^a)$$

Let us denote by $\langle f \rangle_m$ the coefficient of $(q-1)^m$ in a power series f. Applying $\langle \cdot \rangle_m$ to (10), it follows that for all $m \geq 0$ we have

$$0 = \langle X' Q_0 \rangle_m + \langle X' Q_1 \rangle_{m-1} \cdots + \langle X' Q_m \rangle_0 \tag{11}$$

The chain rule implies that

$$\langle X' f \rangle_m = D'_m f \tag{12}$$

for some differential operator $D'_m \in A_1$ of degree at most m. Thus, we obtain that for all $m \geq 0$

$$0 = D'_m Q_0 + D'_{m-1} Q_1 \cdots + D'_0 Q_m. \tag{13}$$

We will show shortly that $D_m \neq 0$ for some m. Assuming this, let $l = \min\{m | D_m \neq 0\}$ and define $D_m = D'_{l+m}$. Equation (13) for $l+m$ implies that

$$0 = D_{m+l} Q_0 + D_{m+l-1} Q_1 + \dots D_0 Q_{m+l}.$$

In other words,

$$\begin{pmatrix} D_0 & 0 & 0 & \dots \\ D_1 & D_0 & 0 & \ddots \\ D_2 & D_1 & D_0 & \ddots \\ \dots & \dots & \dots & \ddots \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

as needed.

It remains to prove that $D'_m \neq 0$ for some m. Obviously we have

$$g = \sum_{m=0}^{\infty} (q-1)^m \langle g \rangle_m$$

for any function g which is a power series in $q-1$. Applying this to $X'f$ and using (12), it follows that

$$X' = \sum_{m=0}^{\infty} (q-1)^m D'_m$$

Assume that $D_m = 0$ for all m. Then, $X' = 0$. Thus, $X'f = 0$ for $f(x) = x^n$. Thus,

$$0 = \sum_{a,b} c_{a,b} s^b q^{ab} (sq^a)^n = s^n \sum_{a,b} c_{a,b} s^b q^{ab} (q^n)^a = s^n \sum_{a,b} c_{a,b} s^b q^{ab} u^a$$

where $u = q^n$. Recall that $X = \sum_{a,b} E^a Q^b \neq 0 \in \mathcal{C}$. Using $EQ = qQE$, it follows that $X = \sum_{a,b} q^{ab} Q^b E^a \neq 0 \in \mathcal{C}$. *Additively*, there is an $\mathcal{R}$-linear isomorphism $\mathcal{C} \leftrightarrow \mathcal{R}[s^{\pm 1}, u^{\pm 1}]$, given by $Q^b E^a \mapsto s^b u^a$, where $\mathcal{R} = \mathbb{Z}[1/2, q^{\pm 1/2}]$. Thus, $\sum_{a,b} c_{a,b} s^b q^{ab} u^a \neq 0$; a contradiction to the hypothesis that $D_m = 0$ for all m. This concludes part (a) of Theorem 3. Part (b) follows from Lemma 3.1 below.

Although we will not need it, let us discuss a bit further the degree of the operator D_0, which equals to the number of initial conditions needed to determine the $\{Q_k\}$.

Let us order the pairs (n, m) of integers with $0 \leq n \leq m$ as follows:

$$(0,0) < (1,1) < (1,0) < (2,2) < (2,1) < (2,0) < (3,3) < (3,2) < (3,1) < (3,0) \dots$$

In other words, $(n,m) \leq (n',m')$ iff $n < n'$ or $n = n'$ and $m > m'$. For a pair (n,m) where $0 \leq n \leq m$, let us define

$$b_{n,m} = s^n \sum_{a,b} \langle c_{a,b} s^b q^{ab} \rangle_{m-n} a^n$$

It is easy to see by induction on m that the following are equivalent:

- $D'_k = 0$ for all $k \leq m$.
- $b_{n,k} = 0$ for all $0 \leq n \leq k \leq m$.

Consider the operator $D_0 = D'_l$ as above, and let d denote its degree. Then,

$$(d,l) = \min\{(n,m) |\, 0 \leq n \leq m,\, b_{n,m} \neq 0\}$$

and the *principal symbol* $\sigma_d(D_0)$ of D_0 is given by

$$\sigma_d(D_0) = b_{d,l}.$$

For an ODE hierarchy as in Theorem 3, we define its *height* to be the degree of D_0. □

Lemma 3.1. *If $a_i, c \in \mathbb{C}[s^{\pm 1}]$, $a_n \neq 0$, the ODE*

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \dots a_0 f = c$$

has at most one solution which is a rational function with fixed initial condition for $f^{(k)}(x_0)$ for $k = 0, \dots, n-1$, where $a_n(x_0) \neq 0$.

Proof. Consider the set of real numbers s such that $a_0(s) \neq 0$. It is a finite set of open intervals. Uniqueness of the solution (modulo initial conditions) is well-known. Since f is a rational function, it is uniquely determined by its restriction on an open interval. The result follows. □

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