ASYMPTOTICS OF THE COLORED JONES POLYNOMIAL AND THE A-POLYNOMIAL

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A . We reveal a relationship between the colored Jones polynomial and the A-polynomial for twist knots. We demonstrate that an asymptotics of the N-colored Jones polynomial in large N gives the potential function, and that the A-polynomial can be computed. We also discuss on a case of torus knots.

1. Introduction

The N-colored Jones polynomial $J_{\mathcal{K}}(N)$ is a quantum invariant which is defined based on the N-dimensional irreducible representation of the quantum group $U_q(sl(2))$. Motivated by *Volume Conjecture* raised by Kashaev [16], it was pointed out that the colored Jones polynomial at a specific value should be related to the hyperbolic volume of knot complement [21].

As another example of the knot invariant related to $SL(2;\mathbb{C})$, we have the A-polynomial [2, 3]. This is defined as an algebraic curve of eigenvalues of the $SL(2;\mathbb{C})$ representation of the boundary torus of knot, and contrary to the quantum invariants such as the colored Jones polynomial it includes many geometrical informations such as the boundary slopes of the knot.

Those two knot invariants are superficially independent. Though, it is recently conjectured [4] that the homogeneous difference equation of the N-colored Jones polynomial for knot $\mathcal{K}$ with respect to N gives the A-polynomial for $\mathcal{K}$ (*AJ conjecture*). This fact was originally verified for both the trefoil and the figure-eight knot with a help of computer algebraic system [4], and was later proved for the torus knots [10]. We should note that in Ref. 26 a recursion relation of the summand of the colored Jones polynomial for the twist knots was shown to give the A-polynomial. See also Refs. 6, 7.

Recently pointed out is still another connection between the colored Jones polynomial and the A-polynomial. It was demonstrated [8, 22] that the A-polynomial has a relationship with an asymptotic limit of the colored Jones polynomial for a case of the figure-eight knot. Our purpose in this article is to show that this correspondence is also supported for a case of the twist knots and the torus knots.

We recall a fact [5] that the N-colored Jones function $J_{\mathcal{K}}(N)$ for knot $\mathcal{K}$ can be written in a form of the q-hypergeometric function. Once we obtain an invariant in the form of the q-hypergeometric series, we may define the H-function [20] for knot $\mathcal{K}$ based on the integrand of an asymptotics of

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the N-colored Jones polynomial for knot $\mathcal{K}$ as

$$J_{\mathcal{K}}(N) \sim \iiint \mathrm{d}\boldsymbol{x} \, \exp\left(\frac{N}{2\pi\mathrm{i}\, r} H_{\mathcal{K}}(\boldsymbol{x}, m^2)\right) \tag{1.1}$$

in a limit

$$N \to \infty \qquad\qquad r = \text{fixed} \tag{1.2}$$

Here we set a parameter q of the N-colored Jones polynomial as

$$q = \exp\left(\frac{2\pi\mathrm{i}\, r}{N}\right) \tag{1.3}$$

and define a parameter m by

$$\begin{aligned} m^2 &= q^N \\ &= \mathrm{e}^{2\pi \mathrm{i} r} \end{aligned} \tag{1.4}$$

In this article we demonstrate for twist knots $\mathcal{K}_p$ (Fig. 1) and torus knots $\mathcal{T}_{2,2p+1}$ (Fig. 4) that the H-function is regarded as the potential function [23, 28] under a constraint

$$x_i \frac{\partial H_{\mathcal{K}}(\boldsymbol{x}, m^2)}{\partial x_i} = 0 \tag{1.5a}$$

and that we have

$$m^2 \frac{\partial H_{\mathcal{K}}(\boldsymbol{x}, m^2)}{\partial (m^2)} = \log \ell \tag{1.5b}$$

Note that a constraint (1.5a) denotes a saddle point equation for the integral (1.1) in large N limit. Eliminating $\boldsymbol{x}$ from a set of eqs. (1.5), we obtain an algebraic equation of ℓ and m^2 which coincides with the A-polynomial of knot $\mathcal{K}$; the $SL(2;\mathbb{C})$ representation of the meridian μ and the longitude λ of the boundary torus of knot $\mathcal{K}$ is given by the upper triangular matrices,

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix} \qquad\qquad \rho(\lambda) = \begin{pmatrix} \ell & * \\ 0 & \ell^{-1} \end{pmatrix}$$

up to conjugation. This shows [8] an intriguing correspondence between the color N of the quantum knot invariant and the eigenvalue of the $SL(2;\mathbb{C})$ representation of the meridian.

This paper is organized as follows. In Section 2 we study the twist knots $\mathcal{K}_p$. Using the q-hypergeometric expression of the colored Jones polynomial derived in Ref. 18, we show that the H-function with constraints (1.5) gives the A-polynomial for the twist knots which was computed in Ref. 15. We also discuss on a relationship with the volume conjecture, and study a limit $p \to \pm\infty$. In Section 3 we show that this correspondence also works for the torus knot.

Throughout this paper we use a standard notation [1]; the q-product and the q-binomial coefficient are respectively defined as follows;

$$(x)_n = (x;q)_n = \prod_{i=1}^{n} \left(1 - x\, q^{i-1}\right)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_{n-k}\,(q)_k}$$

2. Twist Knot

We study the N-colored Jones polynomial for the twist knot $\mathcal{K}_p$. A case of $p = -1$ is the figure-eight knot (see Figure 1).

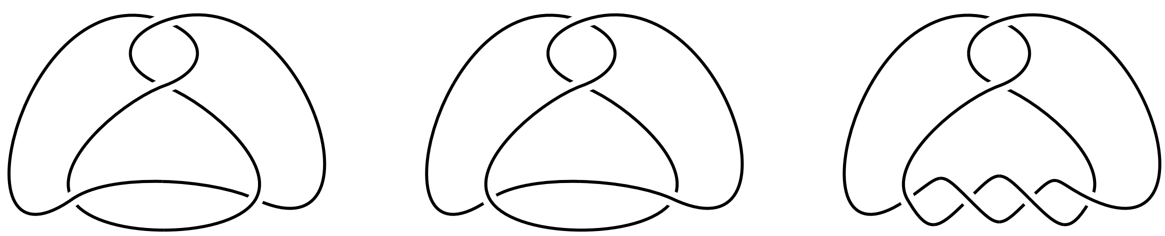

Figure 1: Twist knots $\mathcal{K}_p$ are depicted. There is a p-full twist at the bottom of each figure. From left to right, figure-eight knot ($p = -1$), left-hand trefoil ($p = 1$), and Stevedore's ribbon knot ($p = 2$).

The N-colored Jones polynomial for the p-twist knot $\mathcal{K}_p$ was computed skein-theoretically in Ref. 18 as follows.

Proposition 1 ([18]). *The N-colored Jones polynomial $J_{\mathcal{K}}(N)$ for the twist knot $\mathcal{K} = \mathcal{K}_p$ is given as follows (we set $p > 0$);*

$$J_{\mathcal{K}_{p>0}}(N) = \sum_{s_p \geq \cdots \geq s_2 \geq s_1 \geq 0}^{\infty} q^{(p-1)\, s_p\, (s_p+1) + s_p}\, (q^{1-N})_{s_p}\, (q^{1+N})_{s_p} \times \left(\prod_{i=1}^{p-1} q^{{s_i}^2 - (2 s_p + 1) s_i} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q \right) \tag{2.1}$$

and

$$J_{\mathcal{K}_{-p<0}}(N) = \sum_{s_p \geq \cdots \geq s_2 \geq s_1 \geq 0}^{\infty} (-1)^{s_p}\, q^{-(p-\frac{1}{2})\, s_p\, (s_p+1)}\, (q^{1-N})_{s_p}\, (q^{1+N})_{s_p} \times \left(\prod_{i=1}^{p-1} q^{-{s_i}^2 + (2 s_p + 1) s_i} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_{q^{-1}} \right) \tag{2.1′}$$

Here the colored Jones polynomial is normalized such that $J_{unknot}(N) = 1$.

We set a quantum parameter q as in eq. (1.3), and study an asymptotic behavior of the quantum invariant in a limit $N \to \infty$ (1.2). A limit in a case of $r = 1$, *i.e.*, $q = \exp(2\pi \mathrm{i}/N)$, corresponds to the "Volume Conjecture" [16, 21].

Proposition 2. *In a limit* (1.2) *we have*

$$J_{\mathcal{K}_p}(N) \sim \iiint \mathrm{d}x_0 \cdots \mathrm{d}x_{p-1} \exp\left(\frac{N}{2\,\pi\,\mathrm{i}\,r} H_{\mathcal{K}_p}(x_0, \dots, x_{p-1}, m^2)\right) \tag{2.2}$$

Here m is defined by eq. (1.4)*, and we have*

$$\begin{aligned} H_{\mathcal{K}_{p>0}}(x_0, \dots, x_{p-1}, m^2) = \sum_{i=1}^{p-1} \left(\log\left(x_i/x_0\right)\right)^2 + \mathrm{Li}_2(m^2) + \mathrm{Li}_2(1/m^2) \\ - \mathrm{Li}_2(x_0/m^2) - \mathrm{Li}_2(m^2\, x_0) - \mathrm{Li}_2(x_0) + \sum_{i=0}^{p-1} \mathrm{Li}_2(x_i/x_{i+1}) - (p-1)\frac{\pi^2}{6} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} H_{\mathcal{K}_{-p<0}}(x_0, \dots, x_{p-1}, m^2) = -\left(\log x_0\right)^2 - \sum_{i=1}^{p-1} \left(\log\left(x_i/x_0\right)\right)^2 + \mathrm{Li}_2(m^2) + \mathrm{Li}_2(1/m^2) \\ - \mathrm{Li}_2(x_0/m^2) - \mathrm{Li}_2(m^2\, x_0) - \mathrm{Li}_2(x_0) - \sum_{i=0}^{p-1} \mathrm{Li}_2(x_{i+1}/x_i) + (p-1)\frac{\pi^2}{6} \end{aligned} \tag{2.3$'$}$$

where we have set $x_p = 1$.

Proof. From a definition of the q-product, we have in a limit $N \to \infty$ (see *e.g.* Ref. 1, 24)

$$\log(x\,q)_n \sim \exp \frac{N}{2\,\pi\,\mathrm{i}\,r} \left(\mathrm{Li}_2(x) - \mathrm{Li}_2(x\,q^n)\right) \tag{2.4}$$

Setting

$$q^N = m^2 \qquad\qquad q^{s_i} = x_{p-i}$$

we obtain the H-function (2.3) and (2.3′). □

Our main theorem is as follows.

Theorem 3. *The function* $H_{\mathcal{K}_p}(x_0, \dots, x_{p-1}, m^2)$ *defined in eqs.* (2.3) *and* (2.3′) *is the potential function [23] for the p-twist knot* $\mathcal{K}_p$ *under a constraint* (1.5a). *Eliminating* $\boldsymbol{x}$ *with a condition* (1.5b) *gives the A-polynomial of the twist knot* $\mathcal{K}_p$.

To prove this theorem we rewrite a set of equations (1.5) for $\mathcal{K}_{p>0}$ as

$$\frac{1-m^2\,x_0}{x_0-m^2}=\ell, \tag{2.5a}$$

$$\left(1-\frac{x_0}{x_1}\right)\left(\prod_{i=1}^{p-1}{x_i}^2\right)=x_0^{2(p-1)}\,(1-x_0)\left(1-m^2\,x_0\right)\left(1-\frac{x_0}{m^2}\right) \tag{2.5b}$$

$$x_0^{\,2}\left(1-\frac{x_i}{x_{i+1}}\right)={x_i}^2\left(1-\frac{x_{i-1}}{x_i}\right)\qquad \text{for } i=1,\ldots,p-1 \tag{2.5c}$$

The first equation is solved as

$$x_0=\frac{1+\ell\,m^2}{\ell+m^2}, \tag{2.6}$$

When we set

$$x_{p-k}=x_0\,C_k(x_0) \tag{2.7}$$

for $k=1,2,\ldots,p-1$, we see that a rational function $C_k(x)$ is recursively defined by

$$C_{k+2}(x)=C_{k+1}(x)-\frac{1}{C_{k+1}(x)}+\frac{1}{C_k(x)} \tag{2.8}$$

where the first two rational functions are given by

$$C_0(x)=\frac{1}{x}$$

$$C_1(x)=\frac{1-(1-x)\left(1-m^2\,x\right)\left(1-x/m^2\right)}{x}$$

In the same way, a set of equations (1.5) for a negative case, $\mathcal{K}_{-p<0}$, is explicitly written as

$$\frac{1-m^2\,x_0}{x_0-m^2}=\ell, \tag{2.5a}$$

$$\left(\prod_{i=1}^{p-1}{x_i}^2\right)\left(1-\frac{x_0}{m^2}\right)\left(1-m^2\,x_0\right)(1-x_0)=x_0^{2p}\left(1-\frac{x_1}{x_0}\right), \tag{2.5b$'$}$$

$$x_0^{\,2}\left(1-\frac{x_i}{x_{i-1}}\right)={x_i}^2\left(1-\frac{x_{i+1}}{x_i}\right),\qquad \text{for } i=1,\ldots,p-1 \tag{2.5c$'$}$$

In this case x_0 is also fixed by eq. (2.6). When we also set

$$x_{p-k}=x_0\,C_{-k}(x_0) \tag{2.7$'$}$$

for $k=1,2,\ldots,p-1$, a rational function $C_{-k}(x)$ is recursively defined from

$$\frac{C_{-k-2}(x)-C_{-k-1}(x)}{C_{-k-1}(x)-C_{-k}(x)}=C_{-k-1}(x)\,C_{-k-2}(x) \tag{2.8$'$}$$

where the initial conditions are

$$C_0(x) = \frac{1}{x}$$

$$C_{-1}(x) = \frac{m^2\, x}{m^2\, x^2 - (1-x)(1-m^2\, x)(m^2 - x)}$$

As a result, a set of equations (1.5) reduces to an algebraic equation of ℓ and m;

$$C_p(x_0) = 1 \tag{2.10}$$

where the rational function $C_p(x)$ is recursively computed as above, and x_0 is defined in eq. (2.6). Then to prove Theorem 3, we must show that eq. (2.10) gives the A-polynomial of the twist knot $\mathcal{K}_p$.

We introduce rational function $\widetilde{C}_p(\ell, m)$ by

$$1 - C_p(x_0) = (1-\ell)\,(1-m^2)\,\widetilde{C}_p(\ell, m) \tag{2.11}$$

Eqs. (2.8) and (2.8′) are transformed into the recursion relation for $\widetilde{C}_k(\ell, m)$; in a case of positive twist knot $\mathcal{K}_{p>0}$, we have

$$\frac{\widetilde{C}_k - \widetilde{C}_{k+1}}{\widetilde{C}_{k+1} - \widetilde{C}_{k+2}} = \Big(1 - (1-\ell)\,(1-m^2)\,\widetilde{C}_k(\ell, m)\Big)\Big(1 - (1-\ell)\,(1-m^2)\,\widetilde{C}_{k+1}(\ell, m)\Big) \tag{2.12}$$

with initial conditions

$$\widetilde{C}_0(\ell, m) = \frac{1}{1 + \ell\, m^2}$$

$$\widetilde{C}_1(\ell, m) = \frac{\ell + m^6}{m^2\,(\ell + m^2)^2}$$

and for a negative case

$$\frac{\widetilde{C}_{-k-2} - \widetilde{C}_{-k-1}}{\widetilde{C}_{-k-1} - \widetilde{C}_{-k}} = \Big(1 - (1-\ell)\,(1-m^2)\,\widetilde{C}_{-k-1}\Big)\,\Big(1 - (1-\ell)\,(1-m^2)\,\widetilde{C}_{-k-2}\Big) \tag{2.12′}$$

with

$$\widetilde{C}_0(\ell, m) = \frac{1}{1 + \ell\, m^2}$$

$$\widetilde{C}_{-1}(\ell, m)$$
$$= \frac{-\ell + \ell\, m^2 + m^4 + 2\,\ell\, m^4 + \ell^2\, m^4 + \ell\, m^6 - \ell\, m^8}{-\ell + \ell^2 + 2\,\ell\, m^2 - \ell^2\, m^2 + m^4 + 2\,\ell\, m^4 + 2\,\ell^2\, m^6 + \ell^3\, m^6 - \ell\, m^8 + 2\,\ell^2\, m^8 + \ell\, m^{10} - \ell^2\, m^{10}}$$

Proposition 4. *Rational function satisfying eq.* (2.12) *is solved as*

$$\widetilde{C}_p(\ell, m) = \frac{A_p(\ell, m)}{B_p(\ell, m)} \tag{2.13}$$

Here polynomials $A_p(\ell, m)$ *and* $B_p(\ell, m)$ *are recursively defined as follows by use of a polynomial* $Z(\ell, m)$ *defined by*

$$Z(\ell, m) = -\ell + \ell^2 + 2\,\ell\, m^2 - \ell^2\, m^2 + m^4 + \ell^2\, m^4 - m^6 + 2\,\ell\, m^6 + m^8 - \ell\, m^8 \tag{2.14}$$

- *a positive case,* i.e. *twist knots* $\mathcal{K}_{p>0}$

$$\begin{aligned}\begin{pmatrix} A_{p+1}(\ell,m) \\ B_{p+1}(\ell,m) \end{pmatrix} &= \begin{pmatrix} -Z(\ell,m) & -(1-m^2)\,(\ell-m^4) \\ m^2\,(\ell+m^2)^2\,(1-\ell)\,(1-m^2) & -m^2\,(\ell+m^2)^2 \end{pmatrix} \begin{pmatrix} A_p(\ell,m) \\ B_p(\ell,m) \end{pmatrix} \\ &\equiv \mathbf{M}_+ \begin{pmatrix} A_p(\ell,m) \\ B_p(\ell,m) \end{pmatrix} \end{aligned} \tag{2.15}$$

with

$$\begin{pmatrix} A_1(\ell,m) \\ B_1(\ell,m) \end{pmatrix} = \begin{pmatrix} \ell+m^6 \\ m^2\,(\ell+m^2)^2 \end{pmatrix}$$

- *a negative case*, i.e., *twist knots* $\mathcal{K}_{-p<0}$

$$\begin{aligned}\begin{pmatrix} A_{-p-1}(\ell,m) \\ B_{-p-1}(\ell,m) \end{pmatrix} &= \begin{pmatrix} m^2\,(\ell+m^2)^2 & -(1-m^2)\,(\ell-m^4) \\ (1-\ell)\,(1-m^2)\,m^2\,(\ell+m^2)^2 & Z(\ell,m) \end{pmatrix} \begin{pmatrix} A_{-p}(\ell,m) \\ B_{-p}(\ell,m) \end{pmatrix} \\ &\equiv \mathbf{M}_- \begin{pmatrix} A_{-p}(\ell,m) \\ B_{-p}(\ell,m) \end{pmatrix} \end{aligned} \tag{2.15$'$}$$

with an initial condition

$$\begin{pmatrix} A_0(\ell,m) \\ B_0(\ell,m) \end{pmatrix} = \begin{pmatrix} 1 \\ 1+\ell\,m^2 \end{pmatrix}$$

We note that we have

$$(\mathbf{M}_\pm)^t \cdot \boldsymbol{\sigma}_y \cdot \mathbf{M}_\pm \cdot \boldsymbol{\sigma}_y = m^4\,(\ell+m^2)^4 \tag{2.16}$$

$$\mathbf{M}_+ \cdot \mathbf{M}_- = -m^4\,(\ell+m^2)^4 \tag{2.17}$$

with the Pauli spin matrix $\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, and that the characteristic polynomial of $\mathbf{M}_\pm$ is given by

$$\begin{aligned} &F_\pm(x) \\ &\quad = x^2 \pm x\Big((\ell^2+m^4)\,(1+m^4) + \ell\,(-1+2\,m^2+2\,m^4+2\,m^6-m^8)\Big) + m^4\,(\ell+m^2)^4 \end{aligned} \tag{2.18}$$

Proof of Prop. 4. We assume that $\widetilde{C}_p(\ell,m)$ is defined by eq. (2.13), and that the polynomials A_p and B_p satisfy eqs. (2.15) and (2.15′). Hereafter we use $A_p = A_p(\ell,m)$ and $B_p = B_p(\ell,m)$ for brevity.

We first prove a positive case $p > 0$. We see from eq. (2.15) that

$$(1-\ell)\,(1-m^2)\,A_p - B_p = m^{-2}\,(\ell+m^2)^{-2}\,B_{p+1}$$

$$A_p\,B_{p+1} - B_p\,A_{p+1} = m^4\,(\ell+m^2)^4\,\Big(A_{p-1}\,B_p - B_{p-1}\,A_p\Big)$$

With these identities we find that

$$\begin{aligned} &\left(\frac{A_{p+1}}{B_{p+1}} - \frac{A_{p+2}}{B_{p+2}}\right)\Big(B_p - (1-\ell)\,(1-m^2)\,A_p\Big)\Big(B_{p+1} - (1-\ell)\,(1-m^2)\,A_{p+1}\Big) \\ &\qquad = A_p\,B_{p+1} - B_p\,A_{p+1} \end{aligned}$$

which proves eq. (2.12).

For the negative case $-p < 0$, we easily see from a recursion relation (2.15′) that

$$B_{-p-1} - (1-\ell)(1-m^2)A_{-p-1} = m^2(\ell+m^2)^2 B_{-p}$$

$$A_{-p-1}B_{-p-2} - B_{-p-1}A_{-p-2} = m^4(\ell+m^2)^4\left(A_{-p}B_{-p-1} - B_{-p}A_{-p-1}\right)$$

which leads

$$\left(\frac{A_{-p}}{B_{-p}} - \frac{A_{-p-1}}{B_{-p-1}}\right)\left(B_{-p-1} - (1-\ell)(1-m^2)A_{-p-1}\right)\left(B_{-p-2} - (1-\ell)(1-m^2)A_{-p-2}\right) = A_{-p-1}B_{-p-2} - B_{-p-1}A_{-p-2}$$

This is nothing but eq. (2.12′) □

Proposition 5. *Polynomial $A_p(\ell,m)$ defined by eq.* (2.13) *coincides with the A-polynomial for the twist knot $\mathcal{K}_p$.*

Proof. We define a polynomial

$$X(\ell,m) = -\ell + \ell^2 + 2\,\ell\,m^2 + m^4 + 2\,\ell\,m^4 + \ell^2 m^4 + 2\,\ell\,m^6 + m^8 - \ell\,m^8 \tag{2.19}$$

The recursion relation (2.15) gives ($p > 0$)

$$A_{p+1}(\ell,m) = -X(\ell,m)\,A_p(\ell,m) - m^4\left(\ell+m^2\right)^4 A_{p-1}(\ell,m) \tag{2.20}$$

with

$$\begin{aligned} A_1(\ell,m) &= \ell + m^6 \\ A_2(\ell,m) &= \ell^2 - \ell^3 - 2\,\ell^2 m^2 - \ell\,m^4 - 2\,\ell^2 m^4 + \ell\,m^6 + \ell^2 m^8 \\ &\quad - 2\,\ell\,m^{10} - \ell^2 m^{10} - 2\,\ell\,m^{12} - m^{14} + \ell\,m^{14} \end{aligned}$$

For a negative case ($-p < 0$) we also see that the recursion relation (2.15′) reduces to

$$A_{-p-1}(\ell,m) = X(\ell,m)\,A_{-p}(\ell,m) - m^4\left(\ell+m^2\right)^4 A_{-p+1}(\ell,m) \tag{2.21}$$

with

$$\begin{aligned} A_0(\ell,m) &= 1, \\ A_{-1}(\ell,m) &= -\ell + \ell\,m^2 + m^4 + 2\,\ell\,m^4 + \ell^2 m^4 + \ell\,m^6 - \ell\,m^8 \end{aligned}$$

These recursion relations, eqs. (2.20) and (2.21), coincide with those for the twist knot $\mathcal{K}_p$ derived in Ref. 15, and we can conclude that the polynomial $A_p(\ell,m)$ defined by eq. (2.13) is the A-polynomial for the twist knot $\mathcal{K}_p$. □

These propositions indicate that an algebraic equation (2.10), which is a consequence of eqs. (1.5) for the twist knot $\mathcal{K}_p$, is nothing but an algebraic equation of the A-polynomial for the twist knot $\mathcal{K}_p$;

$$A_p(\ell,m) = 0 \tag{2.22}$$

if we suppose $\ell \neq 1$ and $m^2 \neq 1$. As we define the parameter ℓ, which now represents the eigenvalue of the longitude of the boundary torus of knot, by a derivative of the function $H_{\mathcal{K}}(\boldsymbol{x}, m^2)$ with respect to m, we can identify the H-function with a constraint (1.5a) as the potential function of the twist knot. This proves the statement of Theorem 3, *i.e.*, the H-function defined from an asymptotics of the colored Jones polynomial (1.1) is the potential function under a constraint (1.5a), and it gives the A-polynomial with a condition (1.5b). This fact may support the *Volume Conjecture* [16, 21] that the hyperbolic volume of the knot complements dominates an asymptotics of the colored Jones polynomial, as eq. (1.5a) denotes the saddle point equation of the integral (1.1). Indeed we see that under a constraint (1.5a) the H-function defined by eqs. (2.3) and (2.3′) becomes

$$\operatorname{Im} H_{\mathcal{K}_p}(x_0, \dots, x_{|p|-1}, m^2 = 1) = 3\, D(1/x_0) + \sum_{i=0}^{|p|-1} D(x_i/x_{i+1}) \tag{2.23}$$

where $(x_0, \dots, x_{|p|-1})$ is a solution of eqs. (2.5b)–(2.5c), or eqs. (2.5b′)–(2.5c′), under a constraint $m^2 = 1$. Here we have used the Bloch–Wigner function $D(z)$ defined by

$$D(z) = \operatorname{Im} \operatorname{Li}_2(z) + \arg(1-z) \cdot \log |z| \tag{2.24}$$

which denotes the hyperbolic volume of the ideal tetrahedron with modulus z.

In the case of $m^2 = 1$ we can simplify those equations as follows.

Proposition 6. *We consider the saddle point equations* (1.5a), i.e. *eqs.* (2.5b)–(2.5c) *or eqs.* (2.5b′)–(2.5c′), *of the H-function in a case of $m^2 = 1$. Let the polynomial $V_k(z)$ be defined by ($k > 0$)*

$$\begin{aligned} V_k(z) &= \sum_{j=0}^{2k} \binom{k + \lfloor \frac{j}{2} \rfloor}{j} z^j \\ V_{-k}(z) &= 1 + \sum_{j=1}^{2k-1} \binom{k + \lfloor \frac{j-1}{2} \rfloor}{j} z^j \end{aligned} \tag{2.25}$$

- *Positive case ($p > 0$);*

  Eqs. (2.5b) *and* (2.5c) *with $m^2 = 1$ are solved as*

  $$x_{p-k} = x_0 \, \frac{V_k(1-x_0)}{V_{k-1}(1-x_0)}, \tag{2.26}$$

  for $k = 1, \dots, p$, and x_0 is a solution of $V_p(1-x_0) = V_{p-1}(1-x_0)$, i.e.,

  $$V_{-p}(1-x_0) = 0 \tag{2.27}$$

- *Negative case ($-p < 0$);*

  Eqs. (2.5b′) *and* (2.5c′) *with $m^2 = 1$ are solved as*

  $$x_{p-k} = x_0 \, \frac{V_{-k}(x_0 - 1)}{V_{-k-1}(x_0 - 1)}, \tag{2.26′}$$

  for $k = 1, \dots, p$, and x_0 is a solution of $V_{-p}(x_0 - 1) = V_{-p-1}(x_0 - 1)$, i.e.,

  $$V_p(x_0 - 1) = 0 \tag{2.27′}$$

Proof. We see that the polynomials $V_p(z)$ satisfy

$$\begin{aligned} V_p(z) - V_{p-1}(z) &= z\, V_{-p}(z) \\ V_{-p-1}(z) - V_{-p}(z) &= z\, V_p(z) \end{aligned} \tag{2.28}$$

which gives the 3-term relations

$$\begin{aligned} V_{p+1}(z) - \left(z^2+2\right) V_p(z) + V_{p-1}(z) = 0 \\ V_{-p-1}(z) - \left(z^2+2\right) V_{-p}(z) + V_{-p+1}(z) = 0 \end{aligned} \tag{2.29}$$

To complete the proof, we need to show that the solution of eqs. (2.8) and (2.8′) with $m^2 = 1$ is given by $\frac{V_k(1-x_0)}{V_{k-1}(1-x_0)}$ and $\frac{V_{-k}(x_0-1)}{V_{-k-1}(x_0-1)}$ respectively, *i.e.*, the polynomial satisfies the bilinear equation

$$\begin{aligned} V_{k+2}(z)\cdot V_k(z) - (V_{k+1}(z))^2 &= V_{k+1}(z)\cdot V_{k-1}(z) - (V_k(z))^2 \\ &= -z^3 \\ V_{-k-2}(z)\cdot V_{-k}(z) - (V_{-k-1}(z))^2 &= V_{-k-1}(z)\cdot V_{-k+1}(z) - (V_{-k}(z))^2 \\ &= z^3 \end{aligned}$$

This can be done easily by induction using eq. (2.29). □

We note that the polynomials $V_k(z)$ are written as a sum of the hypergeometric functions,

$$\begin{aligned} V_k(z) &= {}_2F_1\left({-k, k+1 \atop \frac{1}{2}} ; -\frac{z^2}{4}\right) + k\,z\cdot {}_2F_1\left({1-k, k+1 \atop \frac{3}{2}} ; -\frac{z^2}{4}\right) \\ V_{-k}(z) &= {}_2F_1\left({1-k, k \atop \frac{1}{2}} ; -\frac{z^2}{4}\right) + k\,z\cdot {}_2F_1\left({1-k, k+1 \atop \frac{3}{2}} ; -\frac{z^2}{4}\right) \end{aligned} \tag{2.30}$$

With these results we conclude that when x_0 is a solution of eq. (2.27) or (2.27′) we have ($p > 0$)

$$\begin{aligned} &\operatorname{Im} H_{\mathcal{K}_p}(x_0, \dots, x_{p-1}, m^2 = 1) \\ &\quad = 3\,D(1/x_0) + \sum_{j=1}^{p-1} D\left(\frac{V_{j+1}(1-x_0)\,V_{j-1}(1-x_0)}{\left(V_j(1-x_0)\right)^2}\right) + D\left(x_0\,\frac{V_1(1-x_0)}{V_0(1-x_0)}\right) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} &\operatorname{Im} H_{\mathcal{K}_{-p}}(x_0, \dots, x_{p-1}, m^2 = 1) \\ &\quad = 3\,D(1/x_0) + \sum_{j=1}^{p} D\left(\frac{\left(V_{-j}(x_0-1)\right)^2}{V_{-j+1}(x_0-1)\,V_{-j-1}(x_0-1)}\right) \end{aligned} \tag{2.31′}$$

See Table 1 for numerical computation. We have checked that the largest value of $\operatorname{Im} H_{\mathcal{K}_p}(x_0, \dots, x_{|p|-1}, m^2 = 1)$ among solutions of eq. (1.5a) coincides with the hyperbolic volume of the complement of $\mathcal{K}_p$ [27] as was proposed as *Volume Conjecture* (see Ref. 25 for an ideal triangulation of the complement of the twist knots). We have plotted zeros of the polynomial $V_k(z)$ in Fig. 2 for convention.

p	$\mathrm{Im}\, H_{\mathcal{K}_p}\big\|_{m^2=1} = \mathrm{Vol}(S^3 \setminus \mathcal{K}_p)$	x_0
-5	3.57388	$0.99151 - 1.91177\,\mathrm{i}$
-4	3.52620	$0.98405 - 1.86641\,\mathrm{i}$
-3	3.42721	$0.96453 - 1.77530\,\mathrm{i}$
-2	3.16396	$0.89512 - 1.55249\,\mathrm{i}$
-1	2.02988	$0.50000 - 0.86603\,\mathrm{i}$
2	2.82812	$1.21508 - 1.30714\,\mathrm{i}$
3	3.33174	$1.05818 - 1.69128\,\mathrm{i}$
4	3.48666	$1.02317 - 1.82953\,\mathrm{i}$
5	3.55382	$1.01144 - 1.89257\,\mathrm{i}$

Table 1: Hyperbolic volume of the complement of the twist knot $\mathcal{K}_p$ coincides with the largest value of $\mathrm{Im}\, H_{\mathcal{K}_p}$. Given are values of x_0 which give the hyperbolic volume $\mathrm{Vol}(S^3 \setminus \mathcal{K})$ by eq. (2.31) or (2.31′). Knot $\mathcal{K}_{p=1}$ is the left-hand trefoil, which is not hyperbolic.

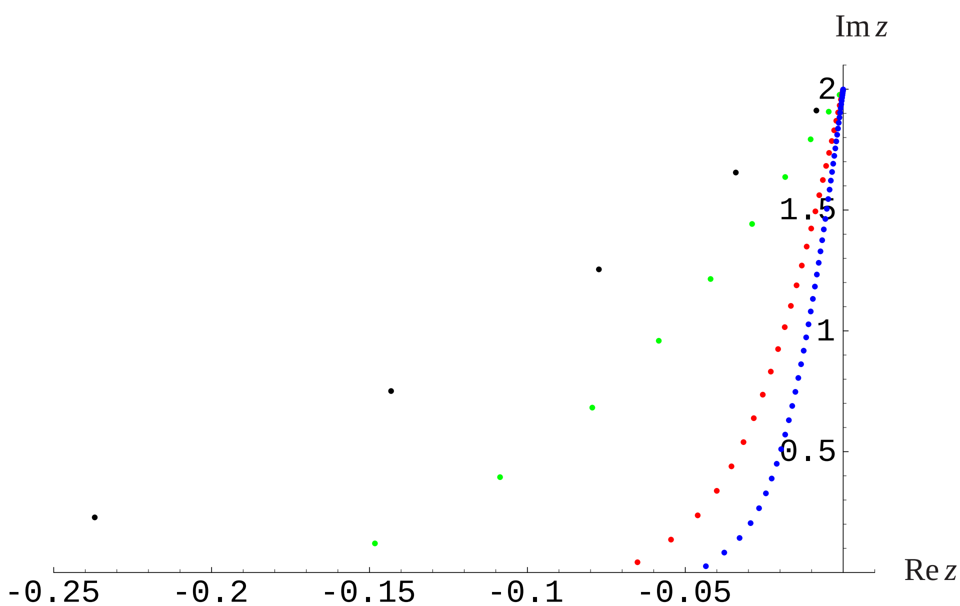

Figure 2: Zeros of the polynomials $V_k(z)$ for $k = 5(\bullet), 10(\bullet), 30(\bullet), 50(\bullet)$. We only plot zero points in the upper half plane.

In a limit $p \to \infty$, we may read off from both the above table and a numerical computation that $x_0 \to 1 - 2\,\mathrm{i}$ (see Fig. 2). It is known geometrically that in the limit of $|p| \to \infty$ the hyperbolic volume of the twist knot $\mathcal{K}_p$ is that of the Whitehead link [19] (Fig. 3), which coincides with the hyperbolic volume of the regular ideal octahedron $4\,D(\mathrm{i}) = 3.66386237670887606\ldots$. Applying

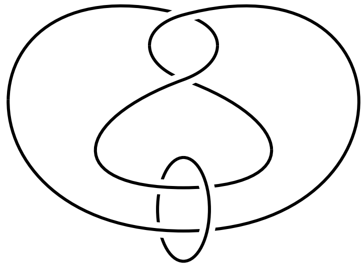

Figure 3: Whitehead link

identities

$$V_p(2\,\mathrm{i}) = (-1)^p \;(2\,p + 1 - 2\,p\,\mathrm{i})$$
$$V_{-p}(-2\,\mathrm{i}) = (-1)^p \;(-2\,p + 1 + 2\,p\,\mathrm{i}) \tag{2.32}$$

for $p > 0$, which result from the Chu–Vandermonde identity, to eqs. (2.31) and (2.31′), we may obtain formulae for the Bloch–Wigner function;

$$4\,D(\mathrm{i}) = 3\,D(2\,\mathrm{i}) + \sum_{k=0}^{\infty} D\left(\left(k + \frac{1}{4} + \frac{\mathrm{i}}{4}\right)^2\right) \tag{2.33}$$

$$= 3\,D(2\,\mathrm{i}) - \sum_{k=0}^{\infty} D\left(\left(-k - \frac{3}{4} + \frac{\mathrm{i}}{4}\right)^2\right) \tag{2.33′}$$

In fact these identities follow from the pentagon identity, especially an identity $D(z^2) = 2\,(D(z) - D(z+1))$ [*].

3. Torus Knot

We apply above story to the colored Jones polynomial for the torus knot $\mathcal{T}_{2,2p+1}$ (we study a case of $p > 0$. See Figure 4); we reveal a relationship between the A-polynomial and the H-function.

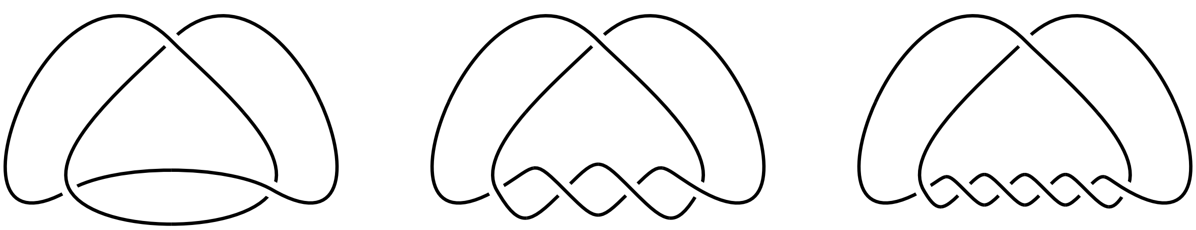

Figure 4: Torus knots $\mathcal{T}_{2,2p+1}$ with $p > 0$ are depicted. From left to right, $p = 1$ (right-hand trefoil), $p = 2$ (Solomon's seal knot), and $p = 3$.

We first recall the q-hypergeometric expression of the colored Jones polynomial for the torus knot.

[*] Anatol N. Kirillov kindly pointed out this fact.

Proposition 7 ([10])**.** *The N-colored Jones polynomial $J_{\mathcal{K}}(N)$ for the torus knot $\mathcal{K} = \mathcal{T}_{2,2p+1}$ is written as*

$$J_{\mathcal{T}_{2,2p+1}}(N) = q^{p\,(1-N^2)} \sum_{s_p \geq \cdots \geq s_2 \geq s_1 \geq 0}^{\infty} q^{(p-1)\,s_p\,(s_p+1)} \frac{(q^{1-N})_{s_p}\,(q^{1+N})_{s_p}}{(q)_{s_p}} \times \left(\prod_{i=1}^{p-1} q^{{s_i}^2 - (2s_p+1)s_i} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q \right) \tag{3.1}$$

Here the colored Jones polynomial is normalized to be $J_{unknot}(N) = 1$.

Applying eq. (2.4) to above expression, we easily obtain the H-function for the torus knot.

Proposition 8. *The asymptotic behavior of the N-colored Jones polynomial for the torus knot $\mathcal{T}_{2,2p+1}$ is written in an integral form in a limit $N \to \infty$ as*

$$J_{\mathcal{T}_{2,2p+1}}(N) \sim \iiint \mathrm{d}x_0 \cdots \mathrm{d}x_{p-1} \exp\left(\frac{N}{2\pi\,\mathrm{i}\,r} H_{\mathcal{T}_{2,2p+1}}(x_0, \ldots, x_{p-1}, m^2) \right) \tag{3.2}$$

where m is defined by eq. (1.4)*, and*

$$H_{\mathcal{T}_{2,2p+1}}(x_0, \ldots, x_{p-1}, m^2) = -p\,\left(\log(m^2)\right)^2 + \sum_{i=1}^{p-1} (\log(x_i/x_0))^2 + \mathrm{Li}_2(m^2) + \mathrm{Li}_2(1/m^2) - \mathrm{Li}_2(x_0/m^2) - \mathrm{Li}_2(m^2\,x_0) + \sum_{i=0}^{p-1} \mathrm{Li}_2(x_i/x_{i+1}) - p\,\frac{\pi^2}{6} \tag{3.3}$$

The theorem for the torus knots is as follows.

Theorem 9. *The H-function* (3.3) *is the potential function for the torus knot $\mathcal{T}_{2,2p+1}$ under a constraint* (1.5a)*, and it gives the A-polynomial for the torus knot $\mathcal{T}_{2,2p+1}$ by eliminating $\boldsymbol{x}$ with a help of a condition* (1.5b).

Proof. A set of equation (1.5) gives

$$\frac{1 - m^2\,x_0}{m^{4p}\,(x_0 - m^2)} = \ell, \tag{3.4a}$$

$${x_0}^{2p-2}\,\left(1 - m^2\,x_0\right)\left(1 - \frac{x_0}{m^2}\right) = \left(1 - \frac{x_0}{x_1}\right)\left(\prod_{i=1}^{p-1} {x_i}^2\right) \tag{3.4b}$$

$${x_i}^2\,\left(1 - \frac{x_{i-1}}{x_i}\right) = {x_0}^2\,\left(1 - \frac{x_i}{x_{i+1}}\right) \qquad \text{for } i = 1, 2, \ldots, p-1 \tag{3.4c}$$

In this case we have

$$x_0 = \frac{1 + \ell\,m^{4p+2}}{m^2\,(1 + \ell\,m^{4p-2})} \tag{3.5}$$

and we see that

$$x_k = x_0\,C_{p-k}(x_0) \tag{3.6}$$

where C_k is recursively solved as

$$C_{k+1}(x) = \left(1 - \frac{(1-m^2\,x)\,(m^2-x)}{m^2\,\left(C_1(x)\cdots C_k(x)\right)^2}\right) C_k(x) \tag{3.7}$$

$$C_0(x) = \frac{1}{x}$$

Then the algebraic equation for ℓ and m may reduce to

$$C_p(x_0) = 1 \tag{3.8}$$

where x_0 is solved in eq. (3.5). In this case we have

$$1 - C_p(x_0) = \frac{(1-\ell)\,(1-m^2)}{1+\ell\,m^2}$$

and we obtain an unwanted solution $\ell = 1$ or $m^2 = 1$. This suggests that a solution of eqs. (3.4) is rather given by

$$x_0 = 0, \qquad\qquad x_i = \pm 1, \quad \text{for } i > 0$$

This gives an algebraic equation as

$$A_{\mathcal{T}_{2,2p+1}}(\ell, m) = 1 + \ell\, m^{4p+2} \tag{3.9}$$

which is the A-polynomial for the torus knot $\mathcal{T}_{2,2p+1}$ [2, 15]. □

We note that an exact asymptotic expansion in $N \to \infty$ of the N-colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ with q being the N-th root of unity, $q = \exp(2\pi\,\mathrm{i}/N)$, was studied in Refs. 12, 14, 29 (see also Refs. 11, 13), and the invariant was identified with the Eichler integral of the modular form with half-integral weight 1/2 which is related to the character of the Virasoro minimal model $\mathcal{M}(s, t)$.

We shall restate our theorems for the trefoil $\mathcal{T}_{2,3}$ in more detail. The N-colored Jones polynomial for the right-hand trefoil was computed explicitly also in Refs. 9, 17, and collecting these results we have

$$J_{\mathcal{T}_{2,3}}(N) = q^{1-N} \sum_{n=0}^{\infty} q^{-n\,N}\,(q^{1-N})_n \tag{3.10a}$$

$$= \sum_{n=0}^{\infty} q^{-n(n+2)}\,(q^{1-N})_n\,(q^{1+N})_n \tag{3.10b}$$

$$= q^{1-N^2} \sum_{n=0}^{\infty} \frac{(q^{1-N})_n\,(q^{1+N})_n}{(q)_n} \tag{3.10c}$$

All these infinite sums reduce into finite sums due to $(q^{1-N})_k = 0$ for $k \geq N > 0$. Note that those q-hypergeometric type expressions are respectively from Refs. 10, 17, eq. (2.1) with $p = 1$ replacing q by q^{-1}, and eq. (3.1) with $p = 1$.

By use of eq. (2.4) we obtain the H-functions from three expressions (3.10) as follows;

$$H_a(x,m^2) = -(\log x)\,\Big(\log(m^2)\Big) + \mathrm{Li}_2\Big(\frac{1}{m^2}\Big) - \mathrm{Li}_2\Big(\frac{x}{m^2}\Big) \tag{3.11a}$$

$$H_b(x,m^2) = -\,(\log x)^2 + \mathrm{Li}_2\Big(\frac{1}{m^2}\Big) + \mathrm{Li}_2(m^2) - \mathrm{Li}_2\Big(\frac{x}{m^2}\Big) - \mathrm{Li}_2(m^2\,x) \tag{3.11b}$$

$$H_c(x,m^2) = -\Big(\log(m^2)\Big)^2 + \mathrm{Li}_2\Big(\frac{1}{m^2}\Big) + \mathrm{Li}_2(m^2) - \mathrm{Li}_2\Big(\frac{x}{m^2}\Big) - \mathrm{Li}_2(m^2\,x) + \mathrm{Li}_2(x) - \frac{\pi^2}{6} \tag{3.11c}$$

A set of equations (1.5) is solved as follows;

(a) We have from eq. (3.11a)

$$\frac{-1+m^2}{(m^2-x)\,x} = \ell \qquad\qquad \frac{m^2-x}{m^4} = 1$$

from which we have $x = (1-m^2)\,m^2$. We thus obtain an algebraic equation, $A(\ell,m)=0$, with

$$A(\ell,m) = 1 + \ell\,m^6 \tag{3.12}$$

This is the A-polynomial for the (right-hand) trefoil [2].

(b) Substituting eq. (3.11b) for eqs. (1.5), we get

$$\frac{-1+m^2\,x}{m^2-x} = \ell \qquad\qquad \frac{(m^2-x)\,(1-m^2\,x)}{m^2\,x^2} = 1$$

This gives $x = \frac{1+\ell\,m^2}{m^2+\ell}$, and an equation of ℓ and m is written as

$$\frac{(\ell+m^2)\,(1+\ell\,m^6)}{m^2\,(1+\ell\,m^2)^3} = 0$$

which suggests eq. (3.12).

(c) We have

$$\frac{1-m^2\,x}{m^4\,(m^2-x)} = \ell \qquad\qquad \frac{(m^2-x)\,(1-m^2\,x)}{m^2\,(1-x)} = 1$$

which gives $x = \frac{1+\ell\,m^6}{m^2\,(1+\ell\,m^2)}$ and

$$\frac{(1-\ell)\,(1+\ell\,m^6)}{(1+\ell\,m^2)\,(1-\ell\,m^4)} = 0$$

We may assume $\ell \neq 1$, and we obtain the A-polynomial (3.12).

To conclude, all three H-functions given from an asymptotics of three expressions (3.10), give the A-polynomial for the trefoil with constraints (1.5).

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