Hecke Algebra Representations of Braid Groups and Link Polynomials

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Hecke algebra representations of braid groups and link polynomials

By V. F. R. Jones

Abstract

By studying representations of the braid group satisfying a certain quadratic relation we obtain a polynomial invariant in two variables for oriented links. It is expressed using a trace, discovered by Ocneanu, on the Hecke algebras of type A. A certain specialization of the polynomial, whose discovery predated and inspired the two-variable one, is seen to come in two inequivalent ways, from a Hecke algebra quotient and a linear functional on it which has already been used in statistical mechanics. The two-variable polynomial was first discovered by Freyd-Yetter, Lickorish-Millet, Ocneanu, Hoste, and Przytycki-Traczyk.

0. Introduction

This paper initiates the detailed study of representations of Artin's braid groups B_n which arise from the Hecke algebras of type A_{n-1}. There appears to be no direct understanding of these representations so our approach will be via generators and relations. The braid group B_n has a presentation $\langle \sigma_1, \dots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, 2, \dots, n-2, \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \rangle$ and the Hecke algebra $H(q, n)$ of type A_{n-1} has a presentation $\langle g_1, \dots, g_{n-1} | g_0^2 = (q-1) g_i + q,\ i = 1, \dots, n-1,\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, i = 1, 2, \dots, n-2,\ g_i g_j = g_j g_i,\ |i-j| \geq 2 \rangle$, where q is a parameter. Our attitude will be that q is a complex number which may take any value. Thus for each $q \neq 0$, B_n has a representation inside $H(q, n)$ obtained by sending σ_i to g_i.

There seems to be no a priori reason why these representations should be of interest but we shall show that in fact they are. For generic q it is possible that they are faithful.

The geometric picture of braids gives relations with links in 3-space. We shall see how two such relations are connected with linear functionals on the Hecke algebras. The first is a trace defined, by Ocneanu, inductively from the relations $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, $\mathrm{tr}(1) = 1$, and $\mathrm{tr}(x g_n) = z\,\mathrm{tr}(x)$ where $x \in H(q, n)$

and $H(q, n)$ is embedded in $H(q, n+1)$ by identifying the g_i's. The parameter z is another complex number independent of q. The trace will be a two-variable polynomial invariant of oriented links discovered independently by Lickorish and Millet, Freyd and Yetter, Ocneanu, and Hoste ([14]). One of its specializations is the classical Alexander polynomial of [2]. We refer to [22] for a treatment of it from a different point of view.

The other linear functional is more difficult to define and lives on a quotient of the Hecke algebra in which the relation

$$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0$$

is satisfied. It corresponds to unoriented links via the theory of plats ([7]). Remarkably, the invariant so defined is a specialization of the one coming from the trace and was discovered first as an invariant of oriented links in [16]. This second linear functional occurs in statistical mechanics as the partition function in the Potts and "ice-type" models (see [4], [42]).

A topological interpretation of these invariants is lacking at present. In this direction it would seem very important to understand the representations of the braid groups in $H(q, n)$ in a more intrinsic manner, in particular the meaning of the parameters q and z. This might also show how to use the other Hecke algebras (not of type A_{n-1}), and their rich representation theory, in some field related to knots.

While very few of our results will use the theory of von Neumann algebras, it should be pointed out that they were the starting point of this work and continue to motivate many of the results. A deeper understanding of subfactors of finite index (see [17], [18]) will almost certainly clarify many of the topological questions.

The author would like to single out Joan Birman among the many recipients of his thanks. Her contribution to this new topic has been of inestimable importance.

1. Braids and links

Braids are formed when n points on a horizontal plane are connected by n strings to n points on another horizontal plane directly below the first n points. The strings are not allowed to go back upwards at any point in their travel. The braid group B_n on n strings is the group formed by appropriate isotopy classes of braids with the obvious concatenation operation. The n points may be supposed to lie on a single straight line which gives rise to an obvious preferred embedding of B_n in B_{n+1} and to a preferred set of generators called $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ given by the following picture:

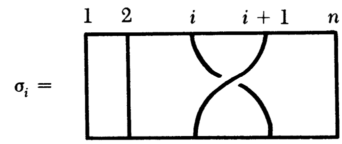


FIGURE 1.1

One may easily convince oneself that the σ_i's satisfy

(1.2) $$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

(1.3) $$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i - j| \geq 2.$$

That (1.2) and (1.3) give a presentation of B_n was proved by E. Artin. The importance of Artin's result for us is that, to construct a representation of B_n it suffices to find matrices satisfying (1.2) and (1.3). As a general reference on braids, see [6].

Given a braid $\alpha \in B_n$ one may form the oriented link $\hat{\alpha}$, called the closure of α, in a manner adequately described by the following diagram:

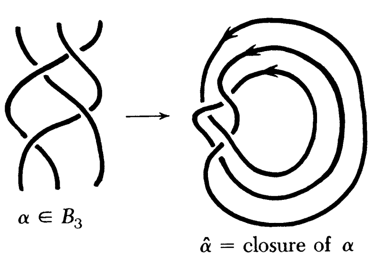


FIGURE 1.4

A result of J. Alexander asserts that any (tame) oriented link is isotopic to the closure of some braid, but attempts to exploit braids to study knots run into the following serious problem: The representation of a link L as a closed braid is highly non-unique. Fortunately, A. A. Markov found purely algebraic necessary and sufficient conditions for braids $\alpha \in B_n$ and $\beta \in B_m$ to have isotopic closures. This is stated in terms of "Markov moves" as follows. A Markov move of *type* I is changing $\alpha \in B_n$ to $\beta\alpha\beta^{-1} \in B_n$ for any $\beta \in B_n$, and a Markov move of *type* II is changing $\alpha \in B_n$ to $\alpha\sigma_n^{\pm 1} \in B_{n+1}$, or the inverse of this operation. Markov's theorem, whose first published proof appears in [6], is that if $\alpha \in B_n$ and $\beta \in B_m$ have isotopic closures then there is a finite sequence of Markov moves of types I and II which takes α to β. Unfortunately, Markov's theorem is not easy to apply directly as the sequence of moves may be long and go through several different braid groups.

There are more ways to obtain links from braids than by closing them as above. The plat method is discussed in detail in Section 14 and recently more general closures are being considered (see [11]).

2. The Burau representation

If t is a non-zero complex number let β_i, for $1 \leq i \leq n-1$, be the $n \times n$ matrix

$$\begin{pmatrix} \ddots & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1-t & t & \\ & & & 1 & 0 & \\ & & & & & 1 \\ & 0 & & & & & \ddots \end{pmatrix}$$

where $1-t$ is the $i-i$ entry. One may easily check that $\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$ and $\beta_i\beta_j = \beta_j\beta_i$ if $|i-j| \geq 2$. Sending σ_i to β_i defines the (non-reduced) Burau representation of B_n. It clearly leaves invariant the subspace of C^n of all vectors whose entries add up to 0 and on the quotient by this subspace a convenient $(n-1) \times (n-1)$ expression is

$$b_1 = \begin{pmatrix} -t & 0 & & & \\ & & & & 0 \\ -1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{pmatrix}, \quad b_{n-1} = \begin{pmatrix} 1 & & & & \\ & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & -t \\ 0 & & & 0 & -t \end{pmatrix},$$

$$b_i = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & -t & 0 & & \\ & & 0 & -t & 0 & & \\ & & 0 & -1 & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

$2 \leq i \leq n-2$ where the diagonal $-t$ is in the $i-i$ position.

These are the most simple non-trivial representations of the braid groups and are of fundamental importance. Except when $n = 2$ or 3, it is unknown at the time of writing if these representations are faithful, even for $t = 2$.

Burau recognized that his represention was related to closed braids. He knew that if $\alpha \in B_n$ and ψ is the reduced Burau representation then

$\det(1 - \psi(\alpha))$ is $(1 + t + \cdots + t^{n-1})$ times the Alexander polynomial of the link $\hat{\alpha}$. He showed this by connecting the matrix $\psi(\alpha)$ with a known way of calculating the Alexander polynomial from a presentation of the fundamental group of the complement of the link. In Section 7 we give a new proof of this result from an entirely different definition of the Alexander polynomial.

For positive braids there is also a mechanical interpretation of the Burau matrix: Lay the braid out flat and make it into a bowling alley with n lanes, the lanes going over each other according to the braid. If a ball travelling along a lane has probability t of falling off the top lane (and continuing in the lane below) at every crossing then the (i, j) entry of the (non-reduced) Burau matrix is the probability that a ball bowled in the ith lane will end up in the jth. Thus in particular every polynomial entry of the Burau matrix of a positive braid will be nonnegative for $0 \leq t \leq 1$.

One should think of the (reduced) Burau representation of B_n as deforming the fundamental representation of the Coxeter group of type A_{n-1} to a pseudo-reflection, since the Burau matrices of the generators σ_i have eigenvalues $-t$ with multiplicity 1 and 1 with multiplicity $n - 2$.

3. Representations of symmetric groups

A braid α defines in an obvious way a permutation on its end points. This homomorphism from B_n to S_n simply corresponds to adding the relation $\sigma_i^2 = 1$ to (1.2) and (1.3). Thus it is to be expected that some family of representations of B_n will be related to representations of symmetric groups. (Note in particular that putting $t = 1$ in the (non-reduced) Burau representation gives the representation of S_n as permutations of basis vectors of an n-dimensional space.) For this reason we devote this section to a discussion of some features of the representation theory of the symmetric groups.

As always for a finite group, irreducible representations (called irreps) are in one-to-one correspondence with conjugacy classes of the group, though not in a natural way. Conjugacy classes of S_n are indexed by *partitions* of n corresponding to the periods of the disjoint cycles of the permutation. These can be represented diagrammatically as follows.

partition: $3 + 2 + 1 + 1 = 7$
diagram:

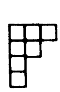
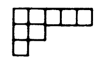
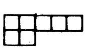
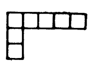
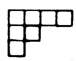


FIGURE 3.1

The length of the rows in the diagram is made to be non-decreasing. We will call such a diagram a *Young diagram*.

Thus irreps of S_n are indexed by Young diagrams. An explicit expression of the representation coming from a diagram is possible but will not be useful in this paper. What we shall describe is how a given irrep of S_n decomposes when it is restricted to S_{n-1}. In fact with some thought this rule suffices to construct the representation inductively. It is very simple: Given an irrep π of S_n with Young diagram Y, its restriction to S_{n-1} is the direct sum of (one copy of) each representation of S_{n-1} obtained from Y by removing a node so as to obtain a *Young diagram*.

Example 3.2. The representation  of S_8 restricts to the direct sum of  and  and  of S_7.

We can now build up by induction all the irreps of all the symmetric groups. As this will be extremely important let us record the picture up to S_5.

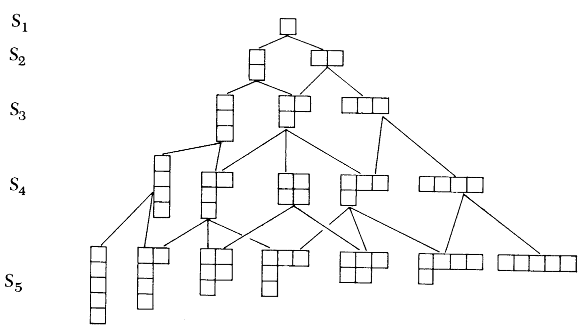
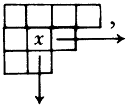
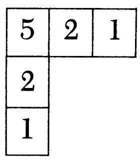


FIGURE 3.3

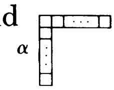
One is to imagine this diagram continuing indefinitely downwards. The lines connecting different rows represent the restrictions of representations. Several remarks are in order:

Remark 3.4. There is an ambiguity in assigning diagrams to representations created by the obvious row-column symmetry of Figure 3.3. To fix this problem it suffices to do it for S_2. We use  to represent the trivial representation of S_2 and  the parity. On the representation level the row-column symmetry corresponds to tensoring by the one-dimensional parity irrep of S_n.

Remark 3.5. The dimension of the representation associated with a given diagram is, by induction, given by the number of descending paths from the initial □ to the diagram in question. But there is a closed formula for the dimension known as the "hook length" formula. We will need this procedure, so here is how it works: record in each box its hook length, i.e. the number of boxes horizontally to the right and vertically below the box in question, e.g. for , the box marked x has hook length 3. The dimension of the representation is then $n!$ divided by the product of the hook lengths, e.g.



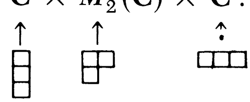
$$\text{dimension} = \frac{5!}{2 \cdot 2 \cdot 5} = 6$$

Remark 3.7. The characters of non-trivial conjugacy classes may also be calculated from the Young diagram. We record here the following fact which was known to Frobenius: in an irrep of S_n the trace of an n-cycle is non-zero if and only if the diagram has at most one corner, i.e. is of the form  and then it is $(-1)^{\alpha}$.

Remark 3.8. An irrep of S_n corresponding to a rectangular tableau has the property that, when restricted to S_{n-1} it remains irreducible.

Remark 3.9. Every representation of S_n extends uniquely to a representation of the complex group algebra $\mathbf{C}S_n$ which is, like any complex finite group algebra, semisimple. Thus Figure 3.3 can equivalently be considered as a description of the group algebras $\mathbf{C}S_n$ and the inclusions between them induced by the inclusions of the symmetric groups. For instance,

$$\mathbf{C}S_3 \cong \mathbf{C} \times M_2(\mathbf{C}) \times \mathbf{C}\,.$$



The irreps of S_n define the simple components of $\mathbf{C}S_n$ and the lines of Figure 3.3 show how the simple components of $\mathbf{C}S_n$ fit inside those of $\mathbf{C}S_{n+1}$.

4. Hecke algebras

The various generators σ_i of the braid groups are all conjugate, so that all one-dimensional representations of a braid group are classified by non-zero scalars. One could obtain all two-dimensional representations but we will see that a much richer structure emerges if we try to describe all representations of B_n in which the σ_i's have *at most two eigenvalues*. Writing g_i for the image of σ_i, under such a representation we must have an equation of the form $g_i^2 + ag_i + b = 0$ where a and b are scalars. It seems that two constants are involved but by modifying g_i by a fixed constant we may eliminate one. It is convenient to express the relation as $g_i^2 = (q-1)g_i + q$ (q a scalar). Let us call such a representation *quadratic*.

Thus knowledge of representations of B_n in which the g_i's have $\leq$ two eigenvalues is the same as knowledge of the algebra $H(q,n)$ with presentation on generators $g_1, g_2, \ldots, g_{n-1}$, and relations

$$g_i^2 = (q-1)g_i + q, \tag{4.1}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \tag{4.2}$$

$$g_i g_j = g_j g_i \qquad \text{if } |i-j| \geq 2. \tag{4.3}$$

Now recall that the symmetric group S_n has a presentation on $s_1, \ldots, s_{n-1}$, $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ if $|i-j| \geq 2$. Imagine trying to reduce words on the g_i's and s_i's to words of minimal length. The point is that relation (4.1) is as good as $s_i^2 = 1$ for this purpose. Thus a system of reduced words on the s_i's for S_n will furnish a basis for $H(q,n)$, by simply writing g_i for s_i. A convenient such basis is

$$\Big\{ (g_{i_1} g_{i_1 - 1} \cdots g_{i_1 - k_1})(g_{i_2} \cdots g_{i_2 - k_2}) \cdots (g_{i_p} g_{i_p - 1} \cdots g_{i_p - k_p}) \Big| \; 1 \leq i_1 < i_2 < \cdots < i_p \leq n-1 \Big\}. \tag{4.4}$$

We see that the dimension of $H(q,n)$ is $n!$. (In fact one must also show that the algebra does not collapse, but this is well established (see [12]). It would suffice to find an expression for the multiplication law in this basis and prove associativity.)

But we can do much better than this. For if we put $q = 1$ in (4.1) we see that $H(1,n)$ is the group algebra $\mathbf{C}S_n$ which, as we noted before, is semisimple. Now semisimplicity is an open condition; so if we change q slightly from 1, $H(q,n)$ will remain semisimple. Not only this, but the whole structure of $H(q,n)$ and the inclusions $H(q,n) \subseteq H(q,n+1)$ (defined from (4.4) in the obvious way) must remain the same under this deformation, at least for n not too large, depending on how far q is from 1. We conclude:

THEOREM 4.5. *For q close to* 1, *the simple $H(q,n)$ modules* (*or quadratic irreps of B_n*) *are in one-to-one correspondence with Young diagrams. Their decomposition rule and hence their dimensions are the same as for S_n.*

The above arguments are somewhat imprecise but can be made good without too much trouble. See [12], pages 54, 55 and 56.

In the same way that Figure 3.3 determines the representations of S_n, so it determines the representations of $H(q,n)$. Wenzl in [46] has written down explicit and demonstrably irreducible representations of $H(q,n)$ for each Young diagram. Once equipped with these formulae one may dispense with all the subtleties of the proof of Theorem 4.5, as there are sufficient representations to show that $\dim_{\mathbf{C}}(H(q,n)) = n!$. One is also able to make precise "sufficiently close to 1", for the formulae of [46] only require one to be able to invert q and $1 - q^p$ for the appropriate p. Thus the conclusion of Theorem 4.5 is true provided q is not a root of unity or zero. We do not wish to imply that the roots of unity are uninteresting. In fact they are probably of more importance than other values and will occur often later.

The algebra defined by (4.1), (4.2) and (4.3) is called the Hecke algebra of type A_{n-1} since the defining relations fit into the Coxeter-Dynkin picture. They arise in the study of representations of $GL(n, q)$ where they define the centralizer of the natural representation on the set of flags. Other Hecke algebras exist for other Coxeter-Dynkin diagrams and it would be nice to know if any of the ideas of this paper can be suitably modified for them.

We will always consider $H(q, n)$ as embedded in $H(q, n+1)$ via (4.4) and the representation of B_n inside $H(q, n)$ will be denoted π, so that $\pi(\sigma_i) = g_i$. We will often abbreviate $H(q, n)$ to H_n.

Note 4.6. The symmetry of Figure 3.3 corresponds to the automorphism $g_i \mapsto -qg_i^{-1}$ of the Hecke algebra in the obvious way.

Note 4.7. We decided in Remark 3.4 to let a Young diagram with one row correspond to the trivial irrep of S_n and a diagram with one column to the parity irrep. When this convention is extended to $q \neq 1$, we find that the "trivial" irrep of $H(q, n)$ must be defined by $g_i \mapsto q$ and the "parity" irrep by $g_i \mapsto -1$. This agrees with note 4.6.

5. Ocneanu's trace on $H(q, n)$

The following theorem is inspired directly from von Neumann algebras where normalized traces are building blocks of the theory.

THEOREM 5.1 (Ocneanu [14]). *For every $z \in \mathbf{C}$ there is a linear trace* tr *on $\bigcup_{n=1}^{\infty} H(q, n)$ uniquely defined by*

1) $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.
2) $\operatorname{tr}(1) = 1$.
3) $\operatorname{tr}(xg) = z \operatorname{tr}(x)$ for $x \in H(q, n)$.

Proof. The first thing to observe is that the map C: $H_n \oplus H_n \otimes_{H_{n-1}} H_n \Rightarrow H_{n+1}$ given by $C(x \oplus y_1 \otimes y_2) = x \oplus y_1 g_n y_2$ is an isomorphism of $H_n - H_n$ bimodules. This follows by considering the set (4.4) and noting that any word for $H(q, n+1)$ contains g_n at most once. Surjectivity is immediate and injectivity follows from a dimension count.

Now we are free to define a linear functional inductively from the formulae $\operatorname{tr}(1) = 1$ and $\operatorname{tr}(xg_n y) = z \operatorname{tr}(xy)$ for $x, y \in H(q, n)$. The problem is then to show property 1). By induction we may suppose it for $a, b \in H_n$. Now if S is a subset of an algebra A which generates it as an algebra, then to show that a linear functional is a trace it suffices to show $f(xs) = f(sx)$ for all $s \in S$ and $x \in A$. Applying this where $S = H_n \cup \{g_n\}$, we see that the only case which does not follow trivially from the definition of tr is $\operatorname{tr}(g_n x g_n y) = \operatorname{tr}(x g_n y g_n)$ for $x, y \in H_n$. But by the remark at the beginning of the proof it suffices to consider the four cases:

a) $x \in H_{n-1}$, $y \in H_{n-1}$,

b) $x = ag_{n-1}b$ where $a, b \in H_{n-1}$ and $y \in H_{n-1}$,

c) same as b) with the roles of x and y reversed,

d) $x = ag_{n-1}b$, $y = cg_{n-1}d$ where $a, b, c, d \in H_{n-1}$.

Case a) is trivial since g_n commutes with H_{n-1} so that we need only consider cases b) and d).

b)
$$\begin{aligned}\operatorname{tr}(g_n a g_{n-1} b g_n y) &= \operatorname{tr}(a g_n g_{n-1} g_n b y)\\ &= \operatorname{tr}(a g_{n-1} g_n g_{n-1} b y)\\ &= z \operatorname{tr}(a g_{n-1}^2 b y) \qquad \text{(definition of tr)}\\ &= (q-1) z \operatorname{tr}(a g_{n-1} b y) + qz \operatorname{tr}(aby)\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(a g_{n-1} b g_n y g_n) &= \operatorname{tr}(a g_{n-1} b g_n^2 y)\\ &= (q-1)\operatorname{tr}(a g_{n-1} b g_n y) + q \operatorname{tr}(a g_{n-1} b y)\\ &= z(q-1)\operatorname{tr}(a g_{n-1} b y) + qz \operatorname{tr}(aby).\end{aligned}$$

d)
$$\begin{aligned}\operatorname{tr}(g_n a g_{n-1} b g_n c g_{n-1} d) &= z \operatorname{tr}(a g_{n-1}^2 b c g_{n-1} d) \qquad \text{(as above)}\\ &= z(q-1)\operatorname{tr}(a g_{n-1} b c g_{n-1} d) + z^2 q \operatorname{tr}(abcd)\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(a g_{n-1} b g_n c g_{n-1} d g_n) &= z \operatorname{tr}(a g_{n-1} b g_{n-1}^2 d)\\ &= z(q-1)\operatorname{tr}(a g_{n-1} b c g_{n-1} d) + z^2 \operatorname{tr}(abcd).\end{aligned}$$ Q.E.D.

Remark 5.2. This proof differs from Ocneanu's original proof. He used a different basis for $H(q, n)$ corresponding to a different section for the natural homomorphism from B_n to S_n, which he called "layered" braids. The trace of a layered braid is a very simple expression but it is more difficult to prove property 1). On the other hand, we have used the basis (4.4) and the expression for the trace of one of these basis elements is quite complicated. It would be interesting to develop the theory using the basis $\{C_w\}$ of [20].

It should be apparent from the proof that properties 1), 2), and 3) suffice to calculate the trace of any element of $H(q, n)$. To emphasize this let us do a sample calculation, that of the trace of $g_2g_1g_3g_2$ which is of minimal length in the braid group:

$$\begin{aligned} \operatorname{tr}(g_2g_1g_3g_2) &= \operatorname{tr}(g_2^2g_1g_3) && \text{(property 1)} \\ &= z\operatorname{tr}(g_2^2g_1) && \text{(property 3)} \\ &= z(q-1)\operatorname{tr}(g_2g_1) + zq\operatorname{tr}(g_1) \\ &= (z^2(q-1)+zq)\operatorname{tr}(g_1) \\ &= z^3(q-1)+z^2q && \text{(properties 3 and 2).} \end{aligned}$$

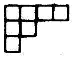
But another method of calculating tr is available. We know that the Hecke algebra is a direct product of matrix algebras, indexed by Young diagrams. The trace of Theorem 5.1 will be determined by its restrictions to these matrix algebras and a trace on a matrix algebra is necessarily a multiple of the usual trace. Thus if we know the scaling factors, called "weights", associated with each diagram we may calculate the trace of any element of the Hecke algebra by decomposing it according to the matrix algebras and taking the weighted sum of the traces.

The weights have been calculated by Ocneanu. They are Schur functions as we shall now describe (see [27]). Note that, given the solution, one only has to verify two things: first that the weights do indeed give a well-defined trace, i.e. the restriction to $H(q,n)$ of the trace defined on $H(q,n+1)$ by the weights is correct, and second, that it satisfies the Markov property $\operatorname{tr}(wg_n) = z\operatorname{tr}(w)$ for $w \in H(q,n)$. For this see [46]. We shall content ourselves simply to give the answer.

If Y is a Young diagram let π_Y denote the ensuing irreducible braid group representation. We must start with a Young diagram Y and specify a function W_Y of q and z which gives the trace of a minimal idempotent in the simple component of $H(q,n)$ specified by that Young diagram. First of all, define $S(q,z)$ as follows (where $w = 1-q+z$): superimpose the Young diagram on the following diagram

$w-z$	$w-qz$	$w-q^2z$	$\cdots$	$\cdots$
$qw-z$	$qw-qz$	$qw-q^2z$	$\cdots$	$\cdots$
q^2w-z	q^2w-qz	$\cdots$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		

FIGURE 5.3

and let $S(q, z)$ be the product of the terms covered by Y. For instance, if $Y =$  then

$$S(q, z) = (w - z)(w - qz)(w - q^2 z)(w - q^3 z)$$
$$\times (qw - z)(qw - qz)(q^2 w - z).$$

Now define $Q(q)$ as follows: First fill in the hook lengths as if calculating the dimension of the irrep of S_n corresponding to Y. Now replace each integer appearing, say m, by $1 - q^m$. Then $Q(q)$ is the product of these terms. For instance if Y is as before we form

6	4	2	1
3	1		
1			

and $Q(q) = (1 - q)(1 - q^2)(1 - q^4)(1 - q^6)(1 - q)(1 - q^3)(1 - q)$. The promised formula is then

$$W_Y(q, z) = S(q, z)/Q(q). \tag{5.4}$$

If Y is a Young diagram let tr_Y be the trace on the Hecke algebra obtained by evaluating the usual trace (sum of the diagonal entries) on the image of a Hecke algebra element in the representation π_Y. Then the significance of (5.4) is the "Fourier transform" formula:

$$\mathrm{tr}(x) = \sum_Y W_Y(q, z)\mathrm{tr}_Y(x). \tag{5.5}$$

Thus to calculate Ocneanu's trace of a word on the g_i's in this fashion we need only explicit matrices representing the g_i's for each π_Y. The explicit formulae of [46] are not well adapted for calculations as they involve square roots of certain polynomials. Much better from this point of view is the paper of Kazhdan and Lusztig ([20]) which gives a method of calculating, in principle, an extremely simple form of the g_i's in terms of what they call W-graphs. But a simple way to go from a Young diagram to a W-graph seems to be lacking.

There should be analogues of Ocneanu's trace for Hecke algebras other than those of type A_n. Some work has been done in this direction by B. Seifert in [39].

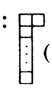
For knot theory it will be more convenient to change variables. We will use $\lambda = (1 - q + z)/qz = w/qz$, so that $z = -(1 - q)/(1 - \lambda q)$, $w = -\lambda q(1 - q)/(1 - \lambda q)$. The only change to the formula is that $S(q, z)$ becomes

$S(q, -(1-q)/(1-\lambda q))$ and is obtained by superimposing the Young diagram on the following version of Figure 5.3:

$$\begin{array}{cccc}
1-\lambda q & q-\lambda q & q^2-\lambda q & \cdots \\
1-\lambda q^2 & q-\lambda q^2 & q^2-\lambda q^2 & \cdots \\
1-\lambda q^3 & q-\lambda q^3 & q^2-\lambda q^3 & \cdots \\
1-\lambda q^4 & \vdots & \vdots & \\
\vdots & & &
\end{array}$$

FIGURE 5.6

and then multiplying by $((1-q)/(1-\lambda q))^n$, there being n nodes in the Young diagram. Let $R_Y(q, \lambda)$ be the product of the relevant terms in Figure 5.6, before multiplying by $((1-q)/(1-\lambda q))^n$.

Note 5.7. A final note is in order. Let us relate this section to Section 2. If $\psi(\sigma_i)$ are the reduced Burau matrices it is easy to check that $(-\psi(\sigma_i))^2 = (t-1)(-\psi(\sigma_i)) + t$ so that sending g_i to $-\psi(\sigma_i)$ gives a representation of the Hecke algebra with $q = t$. Since the reduced Burau representation is irreducible for generic t, this Hecke algebra representation must correspond to one of those already defined. Either by looking at $q = 1$ or by counting eigenvalues (see Lemma 9.1) we deduce that, for the n-string braid group, the representation has the following Young diagram:  (n nodes)

6. A two-variable knot polynomial

The key observation here is the similarity between condition 3) of Theorem 5.1 and the Markov moves of type II of Section 2. We call such traces Markov traces. The most natural way to obtain the invariant is to normalize the g_i's so that both type II Markov moves affect the trace in the same way; so let θ satisfy $\mathrm{tr}(\theta g_i) = \mathrm{tr}((\theta g_i)^{-1})$. Then

$$\theta^2 = \mathrm{tr}(g_i^{-1})/\mathrm{tr}(g_i) = \mathrm{tr}(g_i/q - (1-1/q))/z = (1-q+z)/qz.$$

This is the quantity we have called λ. Thus $\mathrm{tr}(\sqrt{\lambda}\, g_i) = \mathrm{tr}((\sqrt{\lambda}\, g_i)^{-1})$ and $\mathrm{tr}(\sqrt{\lambda}\, g_i) = z\sqrt{\lambda} = -\sqrt{\lambda}(1-q)/(1-\lambda q)$.

It is now immediate that if we represent B_n by π_λ, $\pi_\lambda(\sigma_i) = \sqrt{\lambda}\, g_i \in H(q, n)$, then the function of q and λ given by

$$\left(-(1-\lambda q)/\sqrt{\lambda}(1-q)\right)^{n-1}\mathrm{tr}\left(\pi_\lambda(\alpha)\right), \text{ for } \alpha \in B_n,$$

depends only on the link $\hat{\alpha}$. The representation π ($\pi(\sigma_i) = g_i$) has the advantage of only involving the variable q; so we define:

Definition 6.1. The two-variable invariant $X_L(q, \lambda)$ of the oriented link L is the function

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e \operatorname{tr}(\pi(\alpha))$$

where $\alpha \in B_n$ is any braid with $\hat{\alpha} = L$, e being the exponent sum of α as a word on the σ_i's and π the representation of B_n in $H(q, n)$, $\sigma_i \mapsto g_i$.

PROPOSITION 6.2. *To each oriented link L (up to isotopy) there is a Laurent polynomial $P_L(t, x)$ in the two variables t and x such that, if λ and q satisfy $t = \sqrt{\lambda}\sqrt{q}$, $x = (\sqrt{q} - 1/\sqrt{q})$ then $P_L(t, x) = X_L(q, \lambda)$. Moreover, $P_L(t, x)$ is uniquely defined by the "Skein rule": If L_+, L_- and L_0 are links that have projections identical, except in one crossing where they are as in Figure 6.3:*

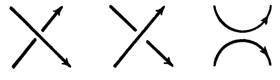


FIGURE 6.3

then $t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$.

(Note: This two-variable knot polynomial was discovered by Lickorish and Millett, Freyd and Yetter, Ocneanu, and Hoste in [14]. Another common choice of variables is $l = it^{-1}$, $m = ix$ where $i^2 = -1$.)

Proof. We begin by investigating the behavior of X under skein moves (see Figure 6.3). Note that if one chooses a single crossing of an oriented link projection one may turn it into braid form so that the crossing becomes a term σ_i (or σ_i^{-1} depending on whether it is a positive or negative crossing) in an expression of the braid as a word on the σ_i's. Thus after a Markov move of type I we may assume $L_+ = \widehat{\alpha\sigma_i^2}$, $L_- = \hat{\alpha}$ and $L_0 = \widehat{\alpha\sigma_i}$ for some $\alpha \in B_n$. By (4.1) we see $\operatorname{tr}(\pi(\alpha\sigma_i^2)) - q \operatorname{tr}(\pi(\alpha)) = (q-1)\operatorname{tr}(\pi(\alpha\sigma_i))$. Let the exponent sum of α be

e and multiply this equation by $T(\sqrt{\lambda})^{e+1}/\sqrt{q}$ with

$$T = \left(-(1-\lambda q)/\sqrt{\lambda}(1-q)\right)^{n-1}.$$

Then

$$\frac{T}{\sqrt{q}\sqrt{\lambda}}(\sqrt{\lambda})^{e+2}\operatorname{tr}\left(\pi\left(\alpha\sigma_i^2\right)\right) - \sqrt{q}\sqrt{\lambda}\,T(\sqrt{\lambda})^{e}\operatorname{tr}(\pi(\alpha))$$

$$= \left(\sqrt{q} - 1/\sqrt{q}\right)T(\sqrt{\lambda})^{e+1}\operatorname{tr}\left(\pi(\alpha\sigma_i)\right);$$

so by the definition of X, $t^{-1}X_{L_+} - tX_{L_-} = xX_{L_0}$.

That the skein relation suffices to calculate $P_L(t, x)$ uniquely follows from an induction implicit in [13] (see also [22]). This induction associates a Laurent polynomial in t and x with any skein decomposition of L. But provided we can find $\sqrt{\lambda}$ and $\sqrt{q}$ with $t = \sqrt{\lambda}\sqrt{q}$ and $x = \sqrt{q} - 1/\sqrt{q}$, the value of $P_L(t, x)$ can only depend on L since, as we have seen, $X_L(q, \lambda)$ is a link invariant and by induction they are equal. There is an open set of values of x and t for which $\sqrt{\lambda}$ and $\sqrt{q}$ exist; so since $P_L(x, t)$ is a Laurent polynomial, it can only depend on L and not on the skein decomposition.

Notes. 1) It is interesting that the values of t and x which do not admit corresponding q and λ values define precisely the specialization of P_L which is the Alexander-Conway polynomial. See Section 7.

2) We have deliberately chosen to use the notation $\sqrt{\lambda}$ rather than defining things in terms of another variable whose square would replace λ. This is because X_L is more than a Laurent polynomial in $\sqrt{\lambda}$—it is a power of $\sqrt{\lambda}$ times a Laurent polynomial in λ. So if L has an odd number of components the square root disappears.

Example 6.4. The (right-handed) trefoil is given by the closure of the braid $\sigma_1^3 \in B_2$. Thus

$$X_L(q, \lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)(\sqrt{\lambda})^3\operatorname{tr}\left(g_1^3\right).$$

But by (4.1), $g_1^3 = (q^2 - q + 1)g_1 + q(q-1)$ so that

$$\begin{aligned} X_L(q, \lambda) &= \left(\frac{\lambda(1-\lambda q)}{1-q}\right)\left((q^2 - q + 1)\frac{(1-q)}{1-\lambda q}q(q-1)\right) \\ &= \lambda\left(1 + q^2 - \lambda q^2\right) \\ &= (\lambda q)\left(\left(\sqrt{q} - 1/\sqrt{q}\right)^2 + 2 - \lambda q\right) \\ &= (2t^2 - t^4) + t^2x^2. \end{aligned}$$

Example 6.5. The figure-8 knot is given by the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in B_3$. So

$$\begin{aligned}
X_L(q,\lambda) &= \frac{(1-\lambda q)}{\lambda(1-q)^2}\operatorname{tr}\left(g_1 g_2^{-1} g_1 g_2^{-1}\right) \\
&= \frac{(1-\lambda q)^2}{\lambda(1-q)^2}\operatorname{tr}\left(\frac{1}{q^2} g_1 g_2 g_1 g_2 + \frac{1}{q}\left(\frac{1}{q}-1\right) g_1^2 g_2 \right. \\
&\qquad \left. + \frac{1}{q}\left(\frac{1}{q}-1\right) g_1 g_2 g_1 + \left(\frac{1}{q}-1\right)^2 g_1^2\right) \\
&= \frac{(1-\lambda q)^2}{q^2\lambda(1-q)^2}\operatorname{tr}\left(g_1^3 g_2 - 2\frac{(1-q)^2}{(1-\lambda q)}\operatorname{tr}(g_1^2) + (1-q)^2\operatorname{tr}(g_1^2)\right) \\
&= \frac{1}{q^2\lambda}\left(-\operatorname{tr}(g_1^3)\frac{1-\lambda q}{1-q} + (1-\lambda q)(1-\lambda q-2)\operatorname{tr}(g_1^2)\right) \\
&= \frac{1}{q^2\lambda}\left(\frac{1-q}{1-\lambda q}\,\frac{1-\lambda q}{1-q}(1+q^2-\lambda q^2)\right. \\
&\qquad \left. - (1-\lambda q)(1+\lambda q)\left(\frac{(1-q)^2}{1-\lambda q}+q\right)\right) \quad \text{(by (6.4))} \\
&= \frac{1}{q^2\lambda}\left(1+q^2-\lambda q^2-(1+\lambda q)(1-q+q^2-\lambda q^2)\right) \\
&= \frac{1}{q\lambda}\left(1-\lambda(1-q+q^2)+\lambda^2 q^2\right) \\
&= \frac{1}{t^2}\left(1-t^2(x^2+1)+t^2\right) = -1-x^2+\frac{1}{t^2}+t^2.
\end{aligned}$$

Example 6.6. The n-component unlink is given by the closure of the braid $1 \in B_n$ so that

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}$$

and

$$P_L(t,x) = \left(\frac{t-t^{-1}}{x}\right)^{n-1}.$$

The above calculations are probably no simpler than those of a skein induction but they illustrate an important point. If one is interested in closed braids for a fixed B_n, the invariants can be computed quite rapidly no matter how many crossings there are. Just write the braid as a word on the σ_i's and expand the product in terms of the basis (4.4). The number of algebraic operations will be proportional to the length of the word. In practice the space required for the basis elements is quite large and 7 or 8 strings seems to be as far as one can go on a present-day average-sized computer. But this already allows one to compute the invariant for many knots and links whose computation is impossible by the direct skein theoretic method for which calculation time grows exponentially with the number of crossings. A program such as the one outlined above has been implemented by Morton and Short who have thus been able to answer in the negative the important question: Is the invariant of $\hat{\alpha}$ determined by the characteristic polynomial of the Burau matrix of α?

Example 6.7. *Connected sums.* We claim that if L_1 and L_2 are oriented links then $X_{L_1 \# L_2}(q, \lambda) = X_{L_1}(q, \lambda) X_{L_2}(q, \lambda)$. This is regardless of which components of L_1 and L_2 one chooses to perform the connected sum.

The claim follows trivially from two facts: (i) If $L_1 = \hat{\alpha}_1$, $\alpha_1 \in B_n$, and $L_2 = \hat{\alpha}_2$, $\alpha_2 \in B_m$, then $L_1 \# L_2 = \alpha_1 \Sigma^{n-1}(\alpha_2) \in B_{n+m}$ where Σ is the shift map on the inductive limit of the B_n's, $\Sigma(\sigma_i) = \sigma_{i+1}$. For instance, the connected sum of the figure-8 and the trefoil is $\hat{\alpha}$ where $\alpha = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_3^3 \in B_4$ (see [6]). (ii) If w_1 is a word on $1, g_1, \ldots, g_n$ and w_2 is a word on $1, g_{n+1}, \ldots, g_{n+m}$ then $\operatorname{tr}(w_1 w_2) = \operatorname{tr}(w_1)\operatorname{tr}(w_2)$. The easiest way to prove this is to note that $x \mapsto \operatorname{tr}(x w_2)$ defines a non-normalized Markov trace on $H_{n+1}(q)$. Normalizing it one obtains the result by the uniqueness of a Markov trace. See [22] for the skein theoretical proof.

Example 6.8. *Reversing the orientation.* Reversing all the arrows in Figure 6.3 preserves the diagram; so by (6.2) we may conclude that $P_L = P_{L'}$ if L' is the oriented link obtained from L by reversing all the orientations. Note though that if the orientations of individual components are reversed, P_L may change drastically. Compare this with Corollary 13.16.

The result $P_L = P_{L'}$ can be seen on the Hecke algebra level as well. The symmetry of the presentation of the braid group (1.2) and (1.3) implies the existence of an antiautomorphism θ of B_n which sends σ_i to σ_i. It is geometrically trivial that $\widehat{\theta(\alpha)} = (\hat{\alpha})'$. By formulae (4.1), (4.2) and (4.3), θ also defines an antiautomorphism of $H(q, n)$ which preserves the trace (uniqueness of a Markov trace). Thus $X_L = X_{L'}$.

This result may be construed as a negative result about the polynomial: It cannot detect (at least in any simple way) the difference between two orientations of a knot. But it is to be expected that a more refined analysis of the Markov relation in the Hecke algebra will be more successful. One might try, for instance, to let q and z be elements of a finite field so that $\pi(B_n)$ is finite for all n.

Example 6.9. *Mirror images.* One of the useful features of P_L is that it is very sensitive to taking the mirror image of a link.

If L is an oriented link let $\tilde{L}$ be the oriented link obtained by viewing L in a mirror or equivalently, reversing all the crossings in some projection of L. It is obvious from Figure 6.3 that $P_{\tilde{L}}(t, x) = P_L(t^{-1}, -x)$. We shall show this using the Hecke algebra. Although this method is less easy, it is quite revealing. If $\alpha \in B_n$ then the mirror image of $\hat{\alpha}$ is $\theta(\alpha^{-1}) \in B_n$ but as we have seen we may take $\alpha^{-1} \in B_n$. Thus if $L = \hat{\alpha}$ and e is the exponent sum of α then

$$X_{\tilde{L}}(q, \lambda) = \left(- \frac{(1 - \lambda q)}{\sqrt{\lambda}\,(1 - q)} \right)^{n-1} (\sqrt{\lambda})^{-e} \operatorname{tr}(\pi(\alpha^{-1})).$$

PROPOSITION 6.10. $X_{\tilde{L}}(q, \lambda) = X_L(1/q, 1/\lambda)$.

Proof. Let $f(q, \lambda)$ be $\operatorname{tr}(\pi(\alpha))$. Write $\pi(\alpha)$ as a product with each g_i written $(q+1)e_i - 1$ where $e_i = (1 + g_i)/(1 + q)$. The e_i satisfy

(6.11) $$e_1^2 = e_i$$

(6.12) $$e_i e_{i+1} e_i - q/(1+q)^2 e_i = e_{i+1} e_i e_{i+1} - q/(1+q)^2 e_{i+1}$$

(6.13) $$e_i e_j = e_j e_i \quad \text{if } |i - j| \geq 2$$

(6.14) $$\operatorname{tr}(e_i) = \frac{q(1-\lambda)}{(1+q)(1-\lambda q)}.$$

Then $\pi(\alpha^{-1})$ will be the same expression on the e_i's simply with q replaced by q^{-1}. Both the expressions $q/(1+q)^2$ and $q(1-\lambda)/(1+q)(1-\lambda q)$ are invariant under the change of variables $q \mapsto 1/q$, $\lambda \mapsto 1/\lambda$. Moreover, relations (6.11), (6.12) and (6.13) are equivalent, for $q \neq -1$, to (4.1), (4.2) and (4.3), so they suffice to calculate the trace of any word on the e_i's, which will be a sum of powers of q times powers of $q(1-\lambda)/(1+q)(1-\lambda q)$. Thus $\operatorname{tr}(\pi(\alpha)^{-1}) = f(1/q, \lambda) = f(1/q, 1/\lambda)$. Finally, $(1-\lambda q)/\sqrt{\lambda}(1-q)$ is invariant under the change of variables and $(\sqrt{\lambda})^e$ becomes $(\sqrt{\lambda})^{-e}$, which completes the proof.

Q.E.D.

Scholium 6.15 (Morton, Franks-Williams [28], [47]). *If* $|e| > n - 1$, $\hat{\alpha}$ *is not amphicheral* ($\alpha \in B_n$).

Proof. We saw in the proof of Proposition 6.10 that $\operatorname{tr}(\pi(\alpha))$, as a function of q and λ, has a finite limit, for fixed $q \neq 0$, when $\lambda \to 0$ (relation (6.14)).

Suppose $e > 0$ and $e > n - 1$. Then as $\lambda \to 0$ we see that $X_L(q, \lambda) \to 0$. But since X_L is a Laurent polynomial in $\sqrt{\lambda}$, this means that only strictly positive powers of λ can occur; hence $X_L \neq X_{\tilde{L}}$. If $e < 0$, use the same formula for X_L. Q.E.D.

Note 6.16. L. Rudolph has used this result to show that the figure-8 knot cannot be represented by a quasi-positive braid, i.e., one which is a product of conjugates of σ_i's ([38]).

Note 6.17. Bennequin had shown in [5] that $\hat{\alpha}$ is non-trivial whenever $|e| > n - 1$. Scholium 6.15 gives a rather different proof of this.

Scholium 6.18. *Let $\alpha \in B_n$ and let q_+ and q_- be the orders of the poles at infinity and zero respectively in $P_{\hat{\alpha}}(q, \lambda)$. Let e_+ be the sum of the positive exponents of the σ_i's in α and e_- be the sum of the negative ones. Then $e_+ > 0$ implies $q_+ \leq e_+ - 1$, and $e_- < 0$ implies $q_- \leq e_- + 1$.*

Proof. Fix λ and let $q \to \infty$. The expression $(1 - \lambda q)/\sqrt{\lambda}(1 - q)$ of Definition 6.1 has a finite limit as $q \to \infty$. Each of the terms in $\mathrm{tr}(\pi(\alpha))$ is either 1 or of the form $q^a \mathrm{tr}(w)$ where w is a product of the e_i's and $a \leq e_+$. Calculating the trace of w using the usual inductive procedure will yield a sum of non-empty products of traces of e_i's multiplied by various powers of $q/(1 + q)^2$ using (6.10)–(6.13). All such terms will go to zero as $q \to \infty$ at least as rapidly as $1/q$. Thus the most rapid growth possible at infinity in q is $q^{e_+ - 1}$ ($e_+ \geq 0$) provided $e_+ > 0$. The same argument applied to α^{-1} yields the full result. Q.E.D.

7. The Alexander polynomial

The Alexander polynomial of a link L will, for the purposes of this paper, be defined as the specialization $\Delta_L(t) = X_L(t, 1/t) = P_L(1, \sqrt{t} - 1/\sqrt{t})$. That such a polynomial exists follows from Proposition 6.2. From Definition 6.1 of $X_L(q, \lambda)$ it would appear that one cannot calculate the Alexander polynomial using the trace of Hecke algebra representations. Indeed for these values of the parameters $\mathrm{tr}(g_i)$ does not make sense. But we will show that if one uses the weighted sum of the traces as in Section 4, the singularity at $\lambda = 1/q$ disappears and one may evaluate Δ_L. A bonus of this approach is an alternative proof of the fact that Δ is given by the Burau matrix.

If Y is a Young diagram, the contribution of Y to $X_L(q, \lambda)$ (for $|q| \neq 1$, $\lambda \neq 1/q$) is

$$(\sqrt{\lambda})^e \left(-\frac{1}{\sqrt{\lambda}}\right)^{n-1} \left(\frac{1-q}{1-\lambda q}\right) \frac{R_Y(q, \lambda)}{T_Y(q)} \mathrm{trace}(\pi_Y(\alpha))$$

where $R_Y(q, \lambda)$ is as calculated from Figure 5.6, and "trace" is the usual trace, sum of the diagonal elements. Since the term in the top left corner of Figure 5.6 is $1 - \lambda q$, the limit exists as $\lambda \to 1/q$; so we may evaluate ΔL. (Here $\alpha \in B_n$, $\hat{\alpha} = L$, $e =$ exponent sum of α.) But we also see from Figure 5.6 that if Y is not of the form of Figure 7.1

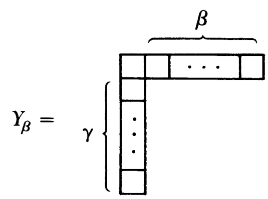


FIGURE 7.1

with $\beta + \gamma + 1 = n$, the contribution is zero since the $(2,2)$ position of Figure 5.6 is $q - q^2\lambda$. We find that for Y as in Figure 7.1, the term

$$(1 - q/1 - \lambda q)R_Y(q, \lambda)/Q_Y(\lambda)$$

is

$$(-1)^\beta \frac{(1-q)(1-q^2)\dots(1-q^\alpha)(1-q)\dots(1-q^\beta)}{(1-q)\dots(1-q^\alpha)(1-q^2)(1-q^3)\dots(1-q^\beta)(1-q^n)}$$

$$= (-1)^\beta \frac{(1-q)}{(1-q^n)}.$$

Thus we obtain the formula

$$(7.2)\quad \Delta_L(t) = (-1)^{n-1}\left(\frac{1}{t}\right)^{(e-n+1)/2} \frac{1-t}{1-t^n} \sum_{\beta=0}^{n-1} (-1)^\beta \operatorname{trace}\left(\pi_{Y_\beta}(\alpha)\right).$$

We shall now identify the representations π_{Y_β}, up to sign, with the exterior powers of π_{Y_1} which we shall call π_1. Consider $f_i = (-1)^{\beta-1}\Lambda^\beta\pi_1(\sigma_i)$. Since $\pi_1(\sigma_i)$ has eigenvalues q and -1, f_i does also; so $f_i^2 = (q-1)f_i + q$. Thus the assignment $\rho(g_i) = f_i$ defines a representation of $H(q, n)$. To see how many irreducible representations it contains it suffices to look at the case $q = 1$, i.e. representations of the symmetric groups. But then we know that π_1 is the tensor product of the signature representation of S_n with the $(n-1)$-dimensional irrep of S_n contained in the n-dimensional permutation representation. But it is well known that the βth exterior powers of this representation are irreducible and have Young diagram $Y_{\beta'}$ for some β'. So ρ is irreducible and corresponds to a Young diagram $Y_{\beta'}$. To find out the value of β' it suffices to count the multiplicities of the eigenvalues q and -1. One deduces that ρ is equivalent to π_{Y_β} (see Lemma 9.1).

Thus we can rewrite (7.2) as

$$(7.3)\qquad \Delta_L(t) = (-1)^{n-1}\left(\frac{1}{t}\right)^{(e-n+1)/2} \times \frac{1-t}{1-t^n}\sum_{\beta}(-1)^{\beta}(-1)^{e(\beta-1)}\operatorname{trace}\left(\Lambda^{\beta}\pi_1(\alpha)\right).$$

But by (5.7), up to equivalence of representations, $\pi_1(\sigma_i) = -\psi(\sigma_i)$, ψ being the reduced Burau representation. So (7.3) becomes

$$(7.4)\qquad \Delta_L(t) = (-1)^{e-n+1}\left(\frac{1}{t}\right)^{(e-n+1)/2}\left(\frac{1-t}{1-t^n}\right)\sum_{\beta}(-1)^{k}\operatorname{trace}\left(\Lambda^{k}\psi(\alpha)\right)$$

or

$$\Delta_L(t) = \left(-1/\sqrt{t}\right)^{e-n+1}\left(\frac{1-t}{1-t^n}\right)\det(1-\psi(\alpha)).$$

This also makes clear the normalization of the Alexander polynomial as calculated from the Burau representation. For this see also [8].

8. Special formulae for closed 3 and 4 braids

For 3 and 4 braids, Figure 3.3 and Notes 4.6 and 5.6 show that almost all the Hecke algebra representations are essentially Burau representations. (Indeed, even the two-dimensional irrep of B_4 corresponding to the Young diagram $(2,2)$ is the composition of the irrep $(2,1)$ of B_3 with the map $\sigma_1 \to \sigma_1$, $\sigma_2 \to \sigma_2$, $\sigma_3 \to \sigma_1$ of B_4 onto B_3). Thus by Section 7 we expect close relationships for closed 3 and 4 braids between the polynomials of Section 6 and the Alexander polynomial. We shall derive these relations for knots only, leaving the case of links up to the reader.

8.1. *Closed 3-braids*. The Hecke algebra $H(q,3)$ is, for generic q, the direct sum of three components corresponding to Young diagrams (3), $(2,1)$ and $(1,1,1)$. So (5.5) becomes, for $x \in H(q,3)$,

$$(8.2)\qquad \operatorname{tr}(x) = \left(\frac{1-q}{1-\lambda q}\right)^3\left[\frac{(1-\lambda q)(q-\lambda q)(q^2-\lambda q)}{(1-q)(1-q^2)(1-q^3)}\operatorname{tr}_{(3)}(x) + \frac{(1-\lambda q)(1-\lambda q^2)(q-\lambda q)}{(1-q)(1-q)(1-q^3)}\operatorname{tr}_{(2,1)}(x) + \frac{(1-\lambda q)(1-\lambda q^2)(1-\lambda q^3)}{(1-q)(1-q^2)(1-q^3)}\operatorname{tr}_{(1,1,1)}(x)\right].$$

Now if $\alpha \in B_3$ has exponent sum e, and $\hat{\alpha}$ is a knot, then e is even. Also the reduced Burau representation ψ is two-dimensional, so that we have

$$\text{(8.3)} \qquad \det(1 - \psi(\alpha)) = 1 - \operatorname{trace}(\psi(\alpha)) + \det(\psi(\alpha)).$$

Moreover, $\det \psi(\alpha) = (-q)^e$ so that by (7.4), $\operatorname{trace}(\psi(\alpha)) = 1 + q^e - q^{e/2}(1 + q + q^{-1})\Delta_{\hat{\alpha}}(q)$. Combining (8.3), (8.2), (5.7), (4.7) and (6.1) we obtain

$$\text{(8.4)} \quad X_{\hat{\alpha}}(q, \lambda) = (\sqrt{\lambda})^{e-2}\Bigg[q^{e+2}\frac{(1-\lambda)(q-\lambda)}{(1-q^2)(1-q^3)} + q\frac{(1-\lambda)(1-\lambda q^2)}{(1-q)(1-q^3)} \times \left(1 + q^e - q^{e/2}(1 + q + q^{-1})\Delta_{\hat{\alpha}}(q)\right) + \frac{(1-\lambda q^2)(1-\lambda q^3)}{(1-q^2)(1-q^3)}\Bigg].$$

In particular $\Delta_{\hat{\alpha}}$ and e determine $X_{\hat{\alpha}}(q, \lambda)$ and hence $P_{\hat{\alpha}}(t, x)$. Birman shows in [8] that $\Delta_{\hat{\alpha}}$ and e do not suffice to determine the type of the knot $\hat{\alpha}$. We invite the reader to try out (8.4) on a few test cases.

8.5. *Closed 4-braids.* For closed 4-braids, $P_{\hat{\alpha}}$ is no longer determined by e and the Alexander polynomial but there is a relation for knots which is sometimes useful. Note now that for $\alpha \in B_4$, e will be odd and there will be a sign difference between $\psi(\alpha)$ and $\pi_{\yng(2,1,1)}(\alpha)$.

Let $w_1, \ldots, w_5$ be the weights for $Y = \yng(4)$, $\yng(3,1)$, $\yng(2,2)$, $\yng(2,1,1)$ and $\yng(1,1,1,1)$, respectively, using (4.6), (4.7), (5.7) and (5.5), and (6.1) becomes, after multiplication by $\lambda^{(1-e)/2}$,

$$\text{(8.6)} \quad \left(\frac{1}{\lambda}\right)^{(e-1)/2} X_{\hat{\alpha}}(q, \lambda) = -\frac{1}{\lambda}\left(\frac{1-\lambda q}{1-q}\right)^3\left(q^e w_1 + q^e \operatorname{Tr}(\psi(\alpha^{-1}))w_2\right) + \operatorname{Tr}(\pi_{\yng(2,2)}(\alpha))w_3 - \operatorname{Tr}(\psi(\alpha))w_4 - w_5.$$

Doing the same for α^{-1} and using Proposition 6.9 we obtain

$$\text{(8.7)} \quad (\lambda)^{(e+1)/2} X_{\hat{\alpha}}\left(\frac{1}{q}, \frac{1}{\lambda}\right) = -\frac{1}{\lambda}\left(\frac{1-\lambda q}{1-q}\right)^3 \times \left(q^{-e}w_1 + q^{-e}\operatorname{Tr}(\psi(\alpha))w_2 + \operatorname{Tr}(\pi_{\yng(2,2)}(\alpha))w_3 - \operatorname{Tr}(\psi(\alpha^{-1}))w_4 - w_5\right).$$

Multiplying (8.7) by q^e and adding we get

$$(8.8)\quad \left(\frac{1}{\lambda}\right)^{(e-1)/2} X_{\hat{\alpha}}(q,\lambda) + (\lambda)^{(e+1)/2} X_{\hat{\alpha}}\left(\frac{1}{q},\frac{1}{\lambda}\right)$$
$$= -\frac{1}{\lambda}\left(\frac{1-\lambda q}{1-q}\right)^3 \left[(1+q^e)w_1 + (w_2 - w_4)\operatorname{Tr}(\psi(\alpha) + q^e\psi(\alpha_{-1}))\right.$$
$$\left. + w_3 \operatorname{Tr}(\pi_{\boxplus}(\alpha) + q^e \pi_{\boxplus}(\alpha^{-1})) - w_5(1+q^e)\right].$$

But the coefficient of w_3 vanishes since $\pi_{\boxplus}$ is two-dimensional and one may easily check that $\det(\pi_{\boxplus}(\sigma_i)) = -q$ so that $\det(\pi_{\boxplus}(\alpha)) = (-q)^e$. And for any 2×2 invertible matrix A, $\operatorname{Tr}(A) = \det(A)\operatorname{Tr}(A^{-1})$.

Also for any invertible 3×3 matrix A we have $\det(1 - A) = 1 - \operatorname{Tr}(A) + \det(A)\operatorname{Tr}(A^{-1}) - \det A$, so that we may rewrite the right-hand side of (8.8) using (7.4) as (remember e is odd and $\det \psi(\sigma_i) = -q$),

$$(8.9)\quad -\frac{1}{\lambda}\left(\frac{1-\lambda q}{1-q}\right)^3 \left[(w_1 - w_5)(1+q^e) + (w_2 - w_4)\right.$$
$$\left. \times\left(1 + q^e - \left(1 + q + q^2 + q^3\right)q^{(e-3)/2}\Delta_{\hat{\alpha}}(q)\right)\right]$$

Finally, using Figure 5.6 we obtain, for a 4-braid knot $\hat{\alpha}$, $\alpha \in B_4$,

$$(8.10)\quad \left(\frac{1}{\lambda}\right)^{(e-1)/2} X(q,\lambda) + \lambda^{(e+1)/2} q^e X\left(\frac{1}{q},\frac{1}{\lambda}\right)$$
$$= \frac{-1}{\lambda(1-q^2)(1-q^3)(1-q^4)}$$
$$\times\left[\{q^3(1-\lambda)(q-\lambda)(q^2-\lambda) - (1-\lambda q^2)(1-\lambda q^3)(1-\lambda q^4)\}(1+q^e)\right.$$
$$+ q(1+q+q^2)(1-\lambda q^2)(1-\lambda)(q^2-1)(1+\lambda q)$$
$$\left.\times\{1 + q^e - (1+q+q^2+q^3)q^{(e-3)/2}\Delta_{\hat{\alpha}}(q)\}\right].$$

This rather complicated formula has a more manageable form in certain specializations (see §12).

Formulae (8.4) and (8.10) lend some weight to the possibility that the exponent sum in a minimal braid representation is a knot invariant.

9. Torus knots

We shall calculate $X_L(q,\lambda)$ for torus knots. It will be clear from the calculation how one can also write down a formula for torus links but the answer is already rather complicated for knots, so we shall content ourselves with that.

A *torus link* of type (m, n) is the closure of the braid $(\sigma_1\sigma_2 \dots \sigma_{n-1})^m \in B_n$ (by various symmetries we may suppose $m, n \in \mathbf{N}$). The link is a knot if and only if m and n are relatively prime. A torus link of type (n, n) is represented by $\Delta^2 = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n \in B_n$. The calculation of the polynomial for torus knots relies heavily on the easy fact that Δ^2 is in the center of B_n.

Let us assume that $q \in \mathbf{R}^+$. This allows us to use the structure of $H(q, n)$ outlined in Section 4. If Y is a Young diagram with n nodes, let π_Y be as usual, let $h_i = \pi_Y(\sigma_i)$ and $e_i = (1 + h_i)/(1 + q)$ and let $d = \dim \pi_Y$.

LEMMA 9.1. *We have* $e_i^2 = e_i$ *and the rank of the idempotent* e_i *is the number of descending paths on Figure* 3.3 *from the diagram* $\square\square$ *to* Y. *In particular, for* Y_β *of Figure* 7.1,

$$\operatorname{rank}(e_i) = \binom{\beta + \gamma - 1}{\gamma}.$$

Proof. All the generators $\sigma_i \in B_n$ are conjugate in B_n which means that all the e_i's have the same rank ($=$ trace) in π_Y. So it suffices to calculate the rank of e_1. But the lines in Figure 3.3 denote the restriction of the representation B_n to B_{n-1}. Thus the assertion about $\operatorname{rank}(e_i)$ as the number of descending paths follows by induction and our convention that $\square\square$ corresponds to the representation $\pi_{\square\square}(\sigma_1) = q$ so that $\pi_{\square\square}(e_1) = 1$. The explicit formula is obtained by counting. Q.E.D.

LEMMA 9.2.

$$\dim(\pi_{Y_\beta}) = \binom{\gamma + \beta}{\gamma}.$$

Proof. Use the hook length formula. Q.E.D.

LEMMA 9.3. *If* Y *is a Young diagram with* $\dim \pi_Y = d$ *and* $\operatorname{rank} e_i = r$, *then*

$$\pi_Y\left((\sigma_1\sigma_2 \dots \sigma_{n-1})^n\right) = q^{rn(n-1)/d}\mathrm{id}_{\pi_Y}.$$

Proof. The representation π_Y is irreducible and $(\sigma_1\sigma_2 \dots \sigma_{n-1})^n = \Delta^2$ is in the center of B_n; so its image is a scalar. Moreover, since $h_i = qe_i - (1 - e_i)$ it follows that $\det(h_i) = \pm q^r$, hence $\det(\pi(\Delta^2)) = q^{rn(n-1)}$. Thus $\pi_Y(\Delta^2) = \omega q^{rn(n-1)/d}$ for some dth root of unity ω. But by [46] or [20] we know that one may write down explicit formulae for $\pi_Y(\sigma_i)$ which depend continuously on q. Thus the root of unity ω varies continuously with q; so we may evaluate it by putting $q = 1$. But then π_Y is a representation of S_n and $\pi_Y(\sigma_1 \dots \sigma_{n-1})$ represents an n-cycle, so that $\omega = 1$. Q.E.D.

Remark. It is also clear from the proof of Lemma 9.3 that $rn(n-1)/d$ is an integer.

LEMMA 9.4. *The matrix* $q^{-r(n-1)/d}\pi_Y(\sigma_1 \dots \sigma_{n-1})$ *is diagonalizable and has the same eigenvalues, with the same multiplicities, as an n-cycle in the representation of* S_n *corresponding to* Y.

Proof. By Lemma 9.3 we know that $q^{-r(n-1)/d}\pi_Y(\sigma_1 \dots \sigma_{n-1})$ is an nth root of unity so it is diagonalizable. The multiplicities of the various nth roots of unity, ω, as eigenvalues, depend continuously on q since they can be expressed in the form $\mathrm{Trace}((1/n)\Sigma_{i=0}^{n-1}\omega^i\pi_Y(\sigma_1 \dots \sigma_{n-1})^i)$. As before they can be evaluated by putting $q = 1$. Q.E.D.

COROLLARY 9.5. *If m and n are relatively prime and Y gives an irrep of* S_n *for which the trace of an n-cycle is zero, then* $\mathrm{trace}(\pi_Y(\sigma_1 \dots \sigma_{n-1})^m) = 0$.

Proof. All n-cycles in S_n are conjugate; so if A is a matrix representing an n-cycle in the irrep of S_n, $\mathrm{trace}(A^m) = 0$. But by Lemma 9.4, $\pi_Y(\sigma_1 \dots \sigma_{n-1})$ is conjugate to a scalar multiple of A. Q.E.D.

The following formula now follows immediately from Remark 3.7, (5.5) and Figure 5.6 (remember $(m, n) = 1$):

$$(9.6)\quad \mathrm{tr}\left((g_1 \dots g_{n-1})^m\right) = \sum_{\substack{\gamma+\beta+1=n\\ \alpha,\beta\ge 0}} (-1)^\gamma q^{\beta m}\frac{q^{\gamma(\gamma+1)/2}}{[\gamma]![\beta]!(1-q^{\gamma+\beta+1})} \times\left(\frac{1-q}{1-\lambda q}\right)^n \prod_{i=-\gamma}^{\beta}(q^i - \lambda q)$$

(we have used the fact that the trace of $\pi_Y(\sigma_1 \dots \sigma_{n-1})^m$ is $(-1)^\gamma q^{r(n-1)m/d}$).

From (9.6) and Definition 6.1 we get the following:

THEOREM 9.7. *Let K be a torus knot of type m, n. Then*

$$X_K(q, \lambda) = \left(\frac{1-q}{1-q^n}\right)\frac{\lambda^{(n-1)(m-1)/2}}{1-\lambda q}\sum_{\substack{\gamma+\beta+1=n\\ \gamma,\beta\ge 0}}(-1)^\beta q^{\beta m+\gamma(\gamma+1)/2} \times\frac{\prod_{i=-\gamma}^{\beta}(q^i-\lambda q)}{[\gamma]![\beta]!}.$$

Since the formula is so involved, let us do a few explicit checks.

Check 9.8. The unknot: $n = 2$, $m = 1$.

$$X_K(q, \lambda) = \frac{1}{(1+q)(1-\lambda q)}\left(\frac{q(q^{-1}-\lambda q)(1-\lambda q)}{1-q} - q\frac{(1-\lambda q)(q-\lambda q)}{1-q}\right) = \frac{1}{1-q^2}(1-\lambda q^2 - q^2 + \lambda q^2) = 1.$$

Check 9.9. The trefoil: $n = 2$, $m = 3$.

$$X_K(q, \lambda)$$
$$= \frac{\lambda}{(1+q)(1-\lambda q)}\left(\frac{q(q^{-1}-\lambda)(1-\lambda q)}{1-q} - \frac{q^3(1-\lambda q)(q-\lambda q)}{1-q}\right)$$
$$= \frac{\lambda}{1-q^2}(1-\lambda q^2 - q^4 + \lambda q^4) = \lambda(1+q^2-\lambda q^2)$$

which checks with Example 6.4.

Thus we can be confident of the following formula for 3, m torus knots:

(9.10)
$$X_K(q,\lambda) = \frac{\lambda^{m-1}}{(1-q^3)(1-q^2)}\big((1-\lambda q^3)(1-\lambda q^2) - q^{m+1}(1+q)(1-\lambda q^2)(1-\lambda) + q^{2m+2}(1-\lambda)(q-\lambda)\big).$$

Note. The formula of (9.7) is not obviously symmetric in m and n. However, professor G. Andrews has kindly supplied me with a combinatorial proof of this symmetry using the Heine transform.

10. Mapping class groups

The problem of classification of closed 3-manifolds can be reduced via Heegard decompositions to the study of the mapping class groups (= diffeomorphism groups modulo the connected component of the identity) of closed orientable surfaces of arbitrary genus. It would be significant if one could find representations of these groups and an invariant via the Reidemeister–Singer theorem ([36]) as we have done for links via Markov's theorem. We have not yet succeeded but we would like to describe some progress towards that goal.

Presentations for the mapping class groups are known (see [6] and [45]). Let us at least explain a known set of generators which owe a lot to Lickorish.

Figure 10.1 depicts a surface of arbitrary genus n. There are $2n+1$ broken curves $c_1, \ldots, c_{2n+1}$ and one exceptional curve, called d. It is known that Dehn twists (see [6]) about these curves generate the mapping class groups. Let θ_i denote the (isotopy class of a) Dehn twist about c_i. It is not hard to see that $\theta_i\theta_{i+1}\theta_i = \theta_{i+1}\theta_i\theta_{i+1}$ and $\theta_i\theta_j = \theta_j\theta_i$ if $|i-j| \geq 2$. Thus the mapping class group is generated by a homomorphic image of the $2n+2$-string braid group and one further element. An obvious question is whether any of the Hecke algebra representations give representations of this homomorphic image. The answer is yes as we shall see.

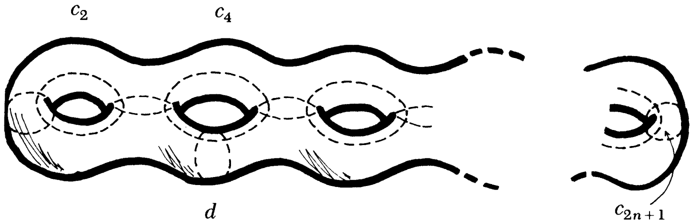


FIGURE 10.1

One may consider the surface of genus n as a branched cover of the 2-sphere, branched at $2n$ points, via an obvious involution of the surface. It is a result of Birman and Hilden in [9] that this branched cover actually gives rise to a homomorphism from the mapping class group generated by the above θ_i's to the mapping class group $M(0, 2n)$ of the sphere minus $2n$ points. A natural presentation of $M(0, m)$ is known. It is (see [6], p. 164) $M(0, m) = \langle \omega_1, \ldots, \omega_{m-1} | \omega_i\omega_j = \omega_j\omega_i$ if $|i-j| \geq 2$, $\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}$, $\omega_1\omega_2 \ldots \omega_{m-2}\omega_{m-1}^2\omega_{m-2} \ldots \omega_1 = 1$, $(\omega_1\omega_2 \ldots \omega_{m-1})^m = 1\rangle$. The homomorphism from the subgroup described above is obtained by sending θ_i to ω_i. The kernel of this map is of order 2, so a powerful representation of $M(0, m)$ will be useful. Now our question becomes: For which Young diagrams on m nodes does the corresponding Hecke algebra representation of B_m pass to $M(0, m)$ (viewed as a quotient, $\sigma_i \to \omega_i$)?

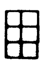
The clue is that in the braid group B_m one has $\sigma_1\sigma_2 \ldots \sigma_{m-2}\sigma_{m-1}^2 \ldots \sigma_1 = (\sigma_1\sigma_2 \ldots \sigma_{m-1})^m(\sigma_2 \ldots \sigma_{m-1})^{-(m-1)}$ so that the relation $\omega_1\omega_2 \ldots \omega_{m-1}^2\omega_{m-2} \ldots \omega_1 = 1$ can be replaced by $(\omega_2\omega_3 \ldots \omega_{m-1})^{m-1} = 1$. Now we know that for a fixed Young diagram Y, π_Y can be adjusted to π'_Y by an appropriate power of q so that $\pi'_Y(\sigma_1\sigma_2 \ldots \sigma_{m-1})^m = 1$. This was done explicitly in Section 9. The essential observation now is that if Y is rectangular the restriction of π_Y to B_{m-1} is still irreducible. This follows immediately from the interpretation of the lines on Figure 3.3. But the element $(\sigma_2\sigma_3 \ldots \sigma_{m-1})^{m-1}$ is in the center of B_{m-1} (thought of as the subgroup of B_m generated by $\sigma_2, \sigma_3, \ldots, \sigma_{m-1}$). Thus $\pi_Y(\sigma_2 \ldots \sigma_{m-1})^{m-1}$ will be a multiple of the identity. Let us check that in fact $\pi'_Y(\sigma_2 \ldots \sigma_{m-1})^{m-1} = 1$. By Lemma 9.3 we see that $\pi'_Y(\sigma_i)$ is $q^{-r/d}\pi_Y(\sigma_i)$. But r and d are the same for π_Y and its restriction to B_{m-1} since Y is rectangular, so that $\pi'_Y(\sigma_2 \ldots \sigma_{m-1})^{m-1} = 1$. Now we have the following:

THEOREM 10.2. *Let Y be a Young diagram and let π'_Y be the corresponding representation of B_m, adjusted as above so that $\pi'_Y(\sigma_1 \ldots \sigma_{m-1})^m = 1$. Then π'_Y defines a representation of $M(0, m)$ via $\omega_i \mapsto \pi'_Y(\sigma_i)$ if and only if Y is rectangular.*

Proof. We have already proved the "if" part. Now if Y is not rectangular, $\pi_Y|_{B_{m-1}}$ will split as the direct sum of several representations corresponding to Young diagrams $\tilde{Y}$ of the form Y minus one node. Numbering the $\tilde{Y}$s from 1 to k, let r_i and d_i be the corresponding dimensions. Then $d = \sum_{i=1}^{k} d_i$, $r = \sum_{i=1}^{k} r_i$. But the only way to have $\pi'_Y(\sigma_2 \ldots \sigma_{m-1})^m = 1$ is for r_i/d_i to be r/d for all i. A combinatorial argument shows that this is impossible if $k > 1$. Q.E.D.

Thus for every integer m which is not a prime we have constructed non-trivial representations of $M(0, m)$ with a parameter q. The primes seem to be more difficult to deal with. Certain special values of q give representations of $M(0, m)$ in the context of the (k, l) tableaux of [46], but their interest is, at this stage, unclear. It is entirely possible that the representations with parameter q are faithful. This is closely linked to the question of faithfulness of the Burau representation.

As described above we also get representations of subgroups of mapping class groups of closed orientable surfaces, but this begs the important question of whether they can be extended in some way to the whole mapping class group.

However, in genus two, the group generated by the θ_i's *is* the whole mapping class group so that we do obtain representations of this group $M(2, 0)$. Up to symmetry there is only one rectangular tableau on 6 nodes, so in fact there is really only one representation. Here is a choice of matrices corresponding to  which, when multiplied by $q^{-2/5}$, give a representation

$$\theta_1: \begin{pmatrix} -1 & 0 & 0 & 0 & q \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix} \qquad \theta_2: \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & q & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\theta_3: \begin{pmatrix} -1 & 0 & 0 & q & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad \theta_4: \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & q \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}$$

$$\theta_5: \begin{pmatrix} -1 & q & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

One could also have written down matrices using [46] but the ones given above have the advantage of being defined over $\mathbf{Z}[q, q^{-1}]$. They were obtained using the Kazhdan-Lusztig formalism of W-graphs.

An element of interest in this mapping class group is the Dehn twist about the curve in Figure 10.3.

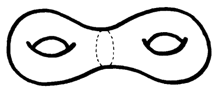


FIGURE 10.3

It is represented by the matrix

$$\begin{pmatrix} q^6 & 9 & 0 & 0 & 0 \\ 0 & q^6 & 0 & 0 & 0 \\ 0 & 0 & q^6 & 0 & 0 \\ -1 + q^2 - q^3 + q^5 & q - q^2 + q^4 - q^5 & -1 + q^2 - q^3 + q^5 & 1 & q - q^2 + q^4 - q^5 \\ 0 & 0 & 0 & 0 & q^6 \end{pmatrix}.$$

This shows that the representation is highly non-trivial on the Torelli group—the normal closure of the above element. Note that when $q = -1$ this matrix is the identity.

Presumably one may use the above representation to settle most questions about genus 2 Heegaard splittings rather mechanically by specializing q to be in a finite field.

11. The V polynomial

The discovery of $P_L(l, m)$ was preceded by the discovery of one of its specializations $V_L(t)$, different from the Alexander polynomial; see [16]. It satisfies the skein relation

$$1/tV_{L_+} - tV_{L_-} = \left(\sqrt{t} - 1/\sqrt{t}\right)V_{L_0} \tag{11.1}$$

where L_+, L_-, and L_0 are as in Figure 6.3. Thus,

$$V_L(t) = P_L\left(it^{-1}, -i\left(\sqrt{t} - 1/\sqrt{t}\right)\right) = X_L(t, t).$$

However, V_L has retained its interest in spite of its more powerful generalization. Ironically, one reason is simply because it only has one variable which makes it easier to work with. Different models for it have been recently given by Kauffman [19], which have led to solutions of some old problems in knot theory; see [19], [32], [43]. It is also shorter to calculate, for reasons that will become clear.

We will also see in the next section that V has the surprising property of being essentially an invariant of unoriented links. There is every indication that a topological understanding of P will only be found once V has been understood in its own right. For these reasons we would like to devote the next three sections to the study of V_L from the Hecke algebra point of view. In particular we will give full proofs of most of the results of [16].

It is unclear from the skein picture why V_L (or even Δ_L for that matter) is an interesting specialization. For this we look back to the origins of this work. In fact the Hecke algebra relations (4.1), (4.2) and (4.3) were not the author's original motivation. In work on type II_1 factors we discovered $*$-algebras A_n with generators $1, e_1, e_2, \dots, e_n$ and relations

(11.2) $$e_i^* = e_i, \quad e_i^2 = e_i$$

(11.3) $$e_i e_{i\pm 1} e_i = \tau e_i$$

(11.4) $$e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2$$

(where τ is a real number related to the II_1 factors). These relations are not, for all τ, a presentation of A_n, whose actual structure is decided by the existence on it of a trace (which we will call tr by abuse of notation), for which the associated sesquilinear form $\langle a, b\rangle = \mathrm{tr}(ab^*)$ is positive definite. This trace is uniquely determined by the following Markov property:

(11.5) $$\mathrm{tr}(xe_{n+1}) = \tau\, \mathrm{tr}(x) \qquad \text{if } x \in A_n.$$

Now let t be such that $\tau^{-1} = 2 + t + t^{-1}$ and set $g_i = te_i - (1 - e_i)$. Then $e_i^2 = e_i$ is equivalent to $g_i^2 = (t-1)e_i + t$ and $e_i e_{i+1} e_i = \tau e_i$ is equivalent to

(11.6) $$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0.$$

Thus in particular the g_i's satisfy (4.1), (4.2) and (4.3) with $t = q$. But (11.6) does not follow from these relations. Thus the algebra A_n is a quotient of $H(t, n+1)$. To understand which quotient, it suffices, for generic t, to look at the case $t = 1$. It is well known that the representations of the symmetric group for which $s_i s_{i+1} s_i + s_i s_{i+1} + s_{i+1} s_i + s_i + s_{i+1} + 1 = 0$ (the s_i's are transpositions as in Section 4) are those whose Young diagrams have at most two columns. One obtains the following diagram for A_n, whose meaning is the same as Figure 3.3 was for the Hecke algebras, where the Young diagrams are replaced by the dimensions of the corresponding representations.

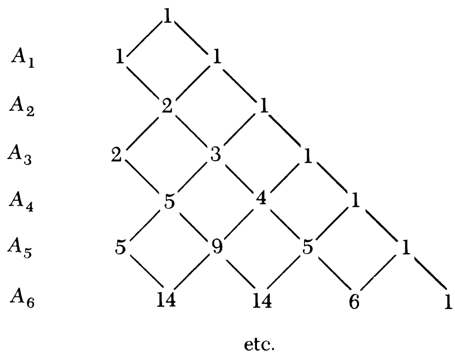


FIGURE 11.7

We see that dim A_n is the Catalan number

$$\frac{1}{n+2}\binom{2n+2}{n+1}.$$

When $t = e^{\pm 2\pi i/n}$ the Hecke algebra is not semisimple, but a C^*-algebra must be, so that some collapsing occurs. The 1's on the extreme right of each level of Figure 11.7 die out, together with their progeny, as explained in [17], [18]. This creates representations of mapping class groups as described in Section 10.

It is clear that the trace of (11.5) is related to the Markov trace of Section 5. In fact for only one value of z does the trace of Section 5 pass to the quotient A_n. This value may be evaluated in two different ways. The first is to take the trace of (11.6). One finds $z\,\mathrm{tr}(g_1^2) + 2z\,\mathrm{tr}(g_i) + 2z + 1 = 0$ or $(z(1+t)+1)(z+1) = 0$. The value $z = -1$ is of little interest so we have $z = -1/(1+t)$. This gives $\mathrm{tr}(e) = t/(1+t)^2 = \tau$. The other way to derive this is by noting that the trace must be zero on simple Hecke algebra quotients whose Young diagrams have more than two columns. By Figure 5.6 we see that this will be the case if $q^2 - \lambda q = 0$, or $\lambda = q$, which is the same as $z = -1/(1+t)$.

Thus we are led to consider $V_L(t) = X_L(t,t)$ which satisfies (11.1). Of course V_L can be defined entirely within the A_n's. We will use π_0 to be the representation of B_n into A_{n-1} given by $\pi_0(\sigma_i) = te_i - (1 - e_i)$. Then if $\alpha \in B_n$ has exponent sum e one has

$$V_L(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^e \mathrm{tr}(\pi_0(\alpha)). \tag{11.8}$$

Relations (11.2)–(11.5) suffice to calculate $\mathrm{tr}(\pi_0(\alpha))$ but since

$$\frac{1}{(n+2)}\binom{2n+2}{n+1}$$

grows more slowly than $(n+1)!$, computer calculation of V_L, as outlined in Section 6, is feasible for closed braids on more strings than for P_L.

Let us record some specializations to V_L of results obtained earlier for P_L.

Proposition 11.9. *Let n and m be relatively prime and let K be an (n, m) torus knot. Then*

$$V_K(t) = \frac{t^{(n-1)(m-1)/2}}{(1-t^2)}(1 - t^{m+1} - t^{n+1} + t^{n+m}).$$

Proof. Putting $q = t$ in the formula of Theorem (9.7) causes the sum to collapse to only two terms, those with $\beta = 0$ and 1. The answer follows by calculation. Q.E.D.

Note. It would be possible to avoid the algebraic complexity of the proof of (9.7) by rerunning the argument on A_n alone. The two terms in the sum correspond to the two right-hand terms on each line of Figure 11.7. The weights are easily obtained from Figure 5.7 and also appear in [17], [18] and [44]. Moreover, if (9.7) is considered a reasonable formula one could write down a formula for V_L where L is a general torus link.

Proposition 11.10 ([16], Theorem 21). *If $\alpha \in B_3$ is such that $\hat{\alpha}$ is a knot and the exponent sum of α is e then*

$$V_{\hat{\alpha}}(t) = t^{e/2}\big(1 + t^e + t + 1/t - t^{e/2-1}(1 + t + t^2)\Delta_{\hat{\alpha}}(t)\big).$$

Proof. Putting $q = \lambda = t$ in (8.4) gives the result. Q.E.D.

Proposition 11.11 ([16], Theorem 22). *With notation as in Proposition* 11.10 *except that $\alpha \in B_4$,*

$$t^{-e}V_{\hat{\alpha}}(t) + t^{e}V_{\hat{\alpha}}(1/t) = (t^{-3/2} + t^{-1/2} + t^{1/2} + t^{3/2})(t^{e/2} + t^{-e/2}) - (t^{-2} + t^{-1} + 2 + t + t^2)\Delta_{\hat{\alpha}}(t).$$

Proof. Put $t = \lambda = q$ in (8.10), multiply by $(1/t)^{(e+1)/2}$ and simplify. Q.E.D.

We have discussed various explicit formulae for the irreducible representations of the Hecke algebras but there are many other natural representations which are not necessarily irreducible. One which plays a prominent role in the algebraic theory of the V polynomial was first discovered by Temperley and Lieb in [42] in connection with the Potts and ice-type models of statistical mechanics. It was rediscovered by Pimsner and Popa in [34]. We use the Pimsner–Popa formalism. The infinite tensor product of 2×2 matrices, $\otimes_{i=1}^{\infty} M_2(\mathbf{C})$, has an obvious shift endomorphism σ defined by $\sigma(x \otimes 1 \otimes 1 \otimes \dots) = 1 \otimes x \otimes 1 \otimes 1 \dots$. Let

$$e = \left\{ \frac{t}{1+t} e_{11} \otimes e_{22} + \frac{\sqrt{t}}{1+t}(e_{21} \otimes e_{12} + e_{12} \otimes e_{21}) + \frac{1}{1+t} e_{22} \otimes e_{11} \right\} \otimes 1 \otimes 1 \otimes \dots$$

where e_{ij} are matrix units for $M_2(\mathbf{C})$. One may easily check that if $e_i = \sigma^i(e)$ then relations (11.2)–(11.4) hold for $t \in \mathbf{R}^+$ where $*$ is the usual conjugate transpose. The ensuing representation, call it θ, of A_n for all n is faithful for generic t. For positive real t this follows easily from [17].

It is interesting that the trace tr on A_n is *not* given by the restriction of the (normalized) trace to $\theta(A_n)$. In fact, the relevant linear functional on $\otimes_{i=1}^{\infty} M_2(\mathbf{C})$ is the Powers state ϕ_t (see [35]) defined by

$$\phi_t(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \dots) = \mathrm{TR}((h \otimes h \otimes \cdots \otimes h)(x_1 \otimes \cdots \otimes x_n))$$

where TR is the non-normalized trace (sum of diagonal elements) on $M_{2^n}(\mathbf{C})$ and h is the 2×2 matrix

$$\begin{pmatrix} 1/(1+t) & 0 \\ 0 & t/(1+t) \end{pmatrix}.$$

One may check in fact that $\phi_t(xy) = \phi_t(yx)$ for $x \in \bigcup_n \theta(A_n)$ and $y \in T$. The Markov property $\phi_t(\theta(xe_{n+1})) = t/(1+t)^2 \phi_t(\theta(x))$ is straightforward. Let us call this representation (of A_{n+1}, $H(q,n)$ or B_n) the PPTL representation.

Another representation for $\tau = 1/n$, $n = 2, 3, 4, \dots$, can be obtained by iterated crossed products. If $\mathbf{C}\mathbf{Z}_n$ is the group algebra, then $\hat{\mathbf{Z}}_n$ ($\cong \mathbf{Z}_n$) acts on it in the obvious way. So one may form $\mathbf{C}\mathbf{Z}_n \rtimes \hat{\mathbf{Z}}_n$. Iterating this procedure one obtains an algebra generated by u_k's, $k = 1, 2, \dots$ with $u_k^n = 1$, $u_k u_{k+1} = e^{2\pi i/n} u_{k+1} u_k$ and $u_k u_j = u_j u_k$ if $|k - j| \geq 2$. By Takesaki duality [41] one knows that, for even k, the algebra generated by $u_1, \dots, u_k$ is the $p^{k/2} \times p^{k/2}$ matrices. Setting

$$e_k = \frac{1}{n} \sum_{i=0}^{n-1} u_k^i$$

one obtains a faithful representation of A_k. An explicit matrix form may be found in [4]. Here the Markov trace is the usual normalized trace.

12. The values $V_L(e^{2\pi i/n})$, $n = 1, 2, 3, 4, 6, 10$

In [17] it was shown that the only values of τ for which the trace on A_n is positive definite ($\mathrm{tr}(a^*a) > 0$ for $a \neq 0$) are the values $\tau = (4\cos^2\pi/n)^{-1}$, $n = 3, 4, 5, \ldots$. These values correspond to $t = e^{\pm 2\pi i/n}$ so it was expected that these values of V_L should be somehow special, as A_n at these values of τ provided entirely new examples of subalgebras of von Neumann algebras. We remind the reader that these are also values of q (except for $n = 1$) for which the Hecke algebra is not always semisimple.

Although many of the following results can be deduced from the skein relation, we shall indicate the algebraic proofs as well since the skein relation is not very suggestive.

(12.1) $V_L(1)$: When $t = 1$ the PPTL representation of the braid group factors through the symmetric group and is given, up to sign, by permutations of the tensor product components of $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \ldots$. The Powers state is in this case only the normalized trace. Also $\sqrt{t} + 1/\sqrt{t} = 2$ and $\mathrm{tr}(g_i) = 2$. It follows immediately that $V_L(1) = (-2)^{c-1}$ where c is the number of components of the link L (Theorem 15 of [16]). Note that the value $t = 1$ is not allowed in the two variable polynomial, $V_L(1)$ being the limit of $X_L(q, \lambda)$ as $(q, \lambda) \to (1, 1)$ along the curve $q = \lambda$.

(12.2) $d/dt\, V_L(1)$: Here the skein theoretic proof is much simpler and the algebraic proof does not lead to any more understanding. Suffice it to say that the following result may be obtained non-inductively (though not easily) by differentiating the formula for V_L coming from the PPTL representation and the Powers state, putting $t = 1$ and using the permutation method of (12.1). The result is

$$V_L'(1) = -3(-2)^{c-2} \quad \text{(total linking number of } L\text{)}.$$

(This result was also noticed by Murakami in [30].)

Proof. Differentiating the skein relation and putting $t = 1$ gives $V_{L_+}'(1) - V_{L_-}'(1) = V_{L_0}(1) + V_{L_+}(1) + V_{L_-}(1)$. If the crossing involves only one component of the link $V_{L_+}'(1) = V_{L_-}(1) = -(1/2)V_{L_0}(1)$ by (12.1) so that in this case $V_{L_+}'(1) = V_{L_-}'(1)$. If the crossing involves two components, $V_{L_+}(1) = V_{L_-}(1) = -2V_{L_0}(1) = (-2)^{c-1}$ so that in this case $V_{L_+}'(1) - V_{L_-}'(1) = -3(-2)^{c-2}$. The result follows by induction Q.E.D.

The higher derivatives of V_L at 1 seem interesting as the difference between the nth derivatives of V_{L_+} and V_{L_-} will not involve the nth derivative of V_{L_0}. We record here the simple fact that changing a positive crossing of a knot to a negative one will change $V_K''(1)$ by six times the linking number of the two-component link formed by eliminating the crossing. A similar argument with Δ shows $V_K''(1) = -3\Delta_K''(1)$.

(12.3) $V_L(-1)$: The formula $V_L(-1) = \Delta_L(-1)$ follows immediately from their skein relations.

(12.4) $V_L(e^{2\pi i/3})$: Define the representation of A_n when $t = e^{2\pi i/3}$ ($\tau = 1$) by sending e_i to 1. The trace on A_n that this defines satisfies the Markov property trivially; so since $\sqrt{t}\,\sigma_i$ is sent to -1 we have, by (11.8), $V_L(e^{2\pi i/3}) = (-1)^{c-1}$ where L has C components (see 14 of [16]).

Proposition 12.5. *If K is a knot then* $1 - V_K(t) = W_K(t)(1-t)(1-t^3)$ *for some Laurent polynomial* $W_K(t)$.

Proof. By (12.1) and (12.4), $1 - V_K$ is divisible by $1 - t^3$ and by (12.2) it is further divisible by $1 - t$. Q.E.D.

Thus extraneous information is recorded in $V_K(t)$. We have chosen to record the simplified $W_K(t)$ in Table 15.9.

(12.6) $V_K(i)$: It has been shown by Murakami [31] and Lickorish-Millet [23] that

$$V_L(i) = \begin{cases} 0 & \text{unless arf}(L) \text{ exists} \\ (-2\sqrt{2})^{c-1}(-1)^{\text{arf}(L)} & \text{otherwise.} \end{cases}$$

The author first proved the result for knots by observing that the calculus of Lannes in [21] when applied to a braid projection of a knot follows the same pattern as calculating the trace of the image of the braid in the iterated crossed product representation at the end of Section 11 when $n = 2$.

(12.7) $V_L(e^{i\pi/3})$: This value suggested itself as interesting because of the special nature of the algebra A_n when $\tau = 1/3$. It is shown in [18] that the image of B_n is finite, and the groups are identified for all n in [10]. Birman conjectured that $V_L(e^{i\pi/3}) = i^k(\sqrt{3})^{k_2}$ for integers k_1 and k_2 depending on L. The situation is completely clarified in papers by Lickorish-Millet [23] and Lipson [26] where $V_L(e^{i\pi/3})$ is calculated in terms of a Seifert form for L. A braid-representation version is also possible. This has been begun by the author and D. Goldschmidt in [15].

(12.8) $V_L(e^{i\pi/5})$: The values $q = e^{\pm i\pi/5}$ have a peculiar feature in the Hecke algebra picture: $\pi(B_n)$ is finite when $n \le 3$ but infinite otherwise. In fact, it was shown in [18] that $\pi(\sigma_1\sigma_2\sigma_3^{-1})$ has infinite order. Thus $V_L(e^{i\pi/5})$ can be used as a test for being a 3-braid—the set of all possible values for closed 3-braids is finite and could easily be written down.

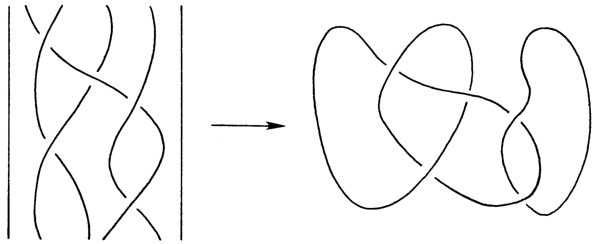
A question posed in [6] is whether every $\alpha \in B_{n+1}$ is conjugate to an element of the form $\alpha_1\sigma_n\alpha_2\sigma_n^{-1}$ with α_1 and $\alpha_2 \in B_n$. For α of this form we have

$\pi_0(\alpha) = \pi_0(\alpha_1)((t+1)e_n - 1)\pi_0(\alpha_2)((t^{-1}+1)e_n - 1)$ and expanding and simplifying we find that $\operatorname{tr}(\pi_0(\alpha)) = 1/\tau \operatorname{tr}(e_n\pi_0(\alpha_1)e_n\pi_0(\alpha_2))$. Letting $n = 3$ and $t = e^{i\pi/5}$ we see that the set of V_L values of elements of the above form is finite, so to answer the question in the negative (even with conjugacy replaced by Markov equivalence), it suffices to exhibit infinitely many values of $V_L(e^{i\pi/5})$ with $L = \hat{\alpha}$, $\alpha \in B_4$. But in [18] it is established that the Burau matrix of $\sigma_1\sigma_2\sigma_3^{-1}$ has an eigenvalue which is not a root of unity. On the other hand, in the representation on the extreme left of the A_3 level of Figure 11.7, $e_1 = e_3$ so that $\sigma_1\sigma_2\sigma_3^{-1}$ is the same as $\sigma_1\sigma_2\sigma_1^{-1}$ which has finite order. Thus in the weighted sum version of the trace, the contributions of the first and third representations on the A_3 line are periodic whereas that of the second is aperiodic. So infinitely many powers of $\sigma_1\sigma_2\sigma_3^{-1}$ are not Markov equivalent to 4-braids of the special form.

Finally, note that for $t = e^{\pm 2\pi i/n}$ except those above, one cannot expect any especially simple behavior. In fact, for each such t, $\{|V_L(t)| \mid L$ is a closed 3-braid$\}$ is dense in the interval $[0, 4\cos^2\pi/n]$, which we shall prove in Section 14. For other roots of unity the situation is unclear at this point.

13. The plat approach

A plat is a closed, *unoriented* link $\tilde{\alpha}$ that is formed when a braid $\alpha \in B_{2m}$ is closed according to the prescription of Figure 13.1.



$\alpha \in B_6$ the plat $\tilde{\alpha}$

FIGURE 13.1

Any unoriented link is of the form $\tilde{\alpha}$ for some (α, m), $\alpha \in B_{2m}$ (see [6]). Birman has proved the following analogue of Markov's theorem which is essentially contained in [7].

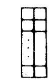
THEOREM 13.2 (Birman). *For each m let C_m be the subgroup of B_{2m} generated by the set $X \cup Y \cup Z$ where*

$$X = \{\sigma_{2i-1}; \quad i = 1, 2, \ldots, m\},$$
$$Y = \{\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}; \quad i = 1, 2, \ldots, m-1\},$$
$$Z = \{\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}; \quad i = 1, 2, \ldots, m-1\}.$$

Also let the map $S_k\colon B_{2k} \to B_{2k+2}$ *be defined by* $S_k(\alpha) = \alpha\sigma_{2k}$. *Then*

(a) *If* $\alpha \in B_{2m}$, $\widetilde{x\alpha y} = \tilde{\alpha}$ *for* $x, y \in C_m$ *and* $\widetilde{S_m(\alpha)} = \tilde{\alpha}$ (= *means isotopic*).

(b) *The equivalence relation on* $\coprod_m B_{2m}$ *generated by the two "moves"* $\alpha \mapsto x\alpha y$ ($\alpha \in B_{2m}$, $x, y \in C_m$) *and* $\alpha \leftrightarrow S_m(\alpha)$, $\alpha \in B_{2m}$ *is the same as the equivalence relation given by isotopy of the plats.*

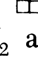
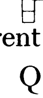
It is not obvious that the plat closure should be related to the Hecke algebra in any way similar to the closure of Section 1. In fact it seems impossible to use all the information of the Hecke algebra in the plat picture. But pp. 192–194 of [6] and Section 10 give a clue as to how to use the Hecke algebra: the plat closure of $\alpha \in B_{2m}$ really only depends on the image of α in the mapping class group of the 2-sphere minus $2m$ points. Thus one should look first at quadratic representations whose Young diagrams are of the form , which we know occur as summands of the representation π_0 of Section 11. The following lemma is thus suggestive (A_n are as in Section 11).

LEMMA 13.3. (i) *In* A_{2m-1}, *the idempotent* $p_m = e_1 e_3 \dots e_{2m-1}$ *is minimal, i.e.* $p_m A_{2m-1} p_m \subseteq \mathbf{C} p_m$.

(ii) *If* ρ_Y *is the irrep of* A_{2m-1} *corresponding to the Young diagram* (*with at most two columns*) Y *then* $\rho_Y(p_m) \neq 0$ *if and only if* Y *is rectangular with two columns* (*i.e. as above*).

These representations are on the extreme left of Figure 11.7.

Proof. (i) It suffices to prove that if x is a word on the e_i's, $i = 1, 2, \dots, 2m-1$, then $p_m x p_m \in \mathbf{C} p_m$. But from Lemma 4.1.2 of [17] and (11.2), (11.3) and (11.4) we have $e_{2m-1} x e_{2m-1} \in A_{2m-3} e_{2m-1}$. The assertion follows by induction.

(ii) Since p_m is minimal there is at most one Y for which $\rho_Y(p_m) \neq 0$. By induction and Figure 11.7 we see that Y must be either $Y_1 =$  or $Y_2 =$  But the Markov trace of p_m is τ^m and the weights for Y_1 and Y_2 are different (use Figure 5.4 with $q = \lambda$), that of Y_1 being τ^m. Q.E.D.

COROLLARY 13.4. *Let* Y *be the Young diagram* . *Then* $\rho_Y(e_1) = \rho_Y(e_3)$.

Proof. Both $\rho_Y(e_1)$ and $\rho_Y(e_3)$ are minimal idempotent in the two-dimensional representation ρ_Y. They both dominate the minimal idempotent $\rho_Y(e_1 e_3) = \rho_Y(p_2)$. Q.E.D.

Remark 13.5. In general the number of nodes in the second (smaller) column of Y is the largest m for which the product of a commuting subset of size m of the e_i's is non-zero in the corresponding representation of A_n. For instance, $e_i e_j = 0$ if $|i - j| \geq 2$ in the Burau representation (tensored with parity) which characterizes it among Hecke algebra representations.

Definition 13.6. The linear functional $\phi\colon \bigcup_m A_{2m-1} \to \mathbf{C}$ is defined by $p_m x p_m = \phi(x) p_m$ for $x \in A_{2m-1}$.

Note that in order for ϕ to be well-defined we need to show that if $p_{m+1} x p_{m+1} = \lambda p_{m+1}$ and $p_m x p_m = \mu p_m$ then $\lambda = \mu$. This is clear from the definition of p_n. Thus ϕ is also defined by $\lim_{m \to \infty}(p_m x p_m - \phi(x) p_m) = 0$ (see [1]).

Note further that the proof of Lemma 13.3(i) was constructive. It gives an algorithm for calculating $\phi(x)$. The following formula will also be useful.

Proposition 13.7. *For* $x \in A_{2m-1}$, $\phi(x) = 1/\tau^m \operatorname{tr}(x p_m)$.

Proof. Just take the trace of the defining relation for ϕ and use $\operatorname{tr}(p_m) = \tau^m$. Q.E.D.

The next result gives the connection of ϕ with plats.

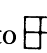
Lemma 13.8. *Let* $g_i = te_i - (1 - e_i) \in A_n$. *Then*

(i) $p_m g_{2i-1}^{\pm 1} = g_{2i-1}^{\pm 1} p_m = t^{\pm 1} p_m$ for $i = 1, 2, \ldots, m$.

(ii) $p_m(g_{2i} g_{2i-1} g_{2i+1} g_{2i}) = (g_{2i} g_{2i-1} g_{2i+1} g_{2i}) p_m = t p_m$ for $i = 1, 2 \ldots m - 1$

(iii) $p_m(g_{2i} g_{2i-1} g_{2i+1}^{-1} g_{2i}^{-1}) = (g_{2i} g_{2i-1} g_{2i+1}^{-1} g_{2i}^{-1}) p_m = p_m$ for $i = 1, 2 \ldots m - 1$.

Proof. (i) is trivial. For (ii) and (iii) note first that it suffices to consider the case $m = 2$ since all the e_i's except two are irrelevant to the calculation.

Let f be the minimal central idempotent of A_3 corresponding to . Then $p_3 = p_3 f$ by Lemma 13.3 so that $p_3 g_2 g_1 g_3^{-1} g_2^{-1} = p_3 f g_2 g_1 g_3^{-1} g_2^{-2} = p_3$ by Corollary 13.4, which proves (iii).

For (ii) note that $(g_2 g_1 g_3 g_2) g_1 (g_2^{-1} g_1^{-1} g_3^{-1} g_2^{-1}) = g_3$ follows from the braid group relations as does the formula with 1 and 3 interchanged. Thus it is immediate that $g_2 g_1 g_3 g_2$ commutes with $e_1 e_3 = p_3$. Hence $p_3 g_2 g_1 g_3 g_2 = p_3 g_2 f g_1 g_3 g_2 p_3 = p_3 g_2 g_1^2 g_2 p_3 = \phi(g_2 g_1^2 g_2) p_3$. Now it remains to calculate $\phi(g_2 g_1^2 g_2) = 1/\tau^2 \operatorname{tr}(e_1 e_3 g_2 g_1^2 g_2) = 1/\tau \operatorname{tr}(e_1 g_2 g_1^2 g_2)$. But $g_1^2 = (t - 1) g_1 + t$

so that

$$\phi(g_2g_1^2g_2) = \frac{t-1}{\tau}\operatorname{tr}(e_1g_2g_1g_2) = \frac{t}{\tau}\operatorname{tr}(e_1g_2^2)$$
$$= (t-1)t^2\operatorname{tr}(g_2) + t\operatorname{tr}((t^2-1)e_2+1)$$
$$= \frac{-t^3+t^2+t^3+t}{1+t} = t. \qquad \text{Q.E.D.}$$

COROLLARY 13.9. *Suppose $\alpha \in B_{2m}$ and $x, y \in C_m$ (the group of Theorem 13.2); then $\phi(x\alpha y) = t^k\phi(\alpha)$ for some $k \in \mathbf{Z}$.*

We now take care of the stabilizing move S_m.

LEMMA 13.10. *Let $x \in A_{2m-1}$. Then*

$$\phi(xg_{2m}) = -\frac{1}{(1+t)}\phi(x).$$

Proof. By Proposition 13.7,

$$\phi(xg_{2m}) = \frac{1}{\tau^{m+1}}\operatorname{tr}(xg_{2m}p_me_{2m+1}) = \frac{1}{\tau^m}\operatorname{tr}(g_{2m})\operatorname{tr}(xp_m) = -\frac{1}{(1+t)}\phi(x).$$

Q.E.D.

Combining Corollary 13.9, Lemma 13.10, and Theorem 13.2 we obtain the following:

THEOREM 13.11. *Let $\alpha \in B_{2m}$ and $\beta \in B_{2n}$ be such that $\tilde{\alpha}$ and $\tilde{\beta}$ are isotopic. Then there exists $k \in \mathbf{Z}$ such that*

$$(-(t+1))^{m-1}\phi(\pi_0(\alpha)) = t^k(-(t+1))^{n-1}\phi(\pi_0(\beta)).$$

Proof. By Corollary 13.9 and Theorem 13.2 it suffices to check that $(-(t+1))^{m-1}\phi(\pi_0(\alpha)) = (-(t+1))^m\phi(\pi_0(S_m(\alpha))$. But this is Lemma 13.10.

Q.E.D.

Thus we have a (potentially) new invariant for unoriented links. We shall show that it is in fact only an unnormalized version of $V_{\hat{\alpha}}$. The idea of the proof is that a closed braid is really a special kind of plat, where the closing scheme is changed as in Figure 13.12.

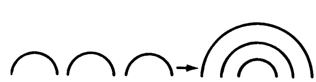


FIGURE 13.12

The two theories corresponding to the different types of closure are equivalent under conjugation by an element Ω which we shall make explicit below. We begin with a simple lemma.

LEMMA 13.13. *Let* $\psi\colon \bigcup_n A_n \to \mathbf{C}$ *be a linear functional with* $\psi(1) = 1$, $\psi(xe_{n+1}y) = \tau\psi(xy)$ *if* $x, y \in A_n$. *Then* $\psi = \mathrm{tr}$.

Proof. By induction assume $\psi = \mathrm{tr}$ on A_n. Then $\psi = \mathrm{tr}$ on the subspace of A_{n+1} spanned by A_n and all elements of the form $xe_{n+1}y$, which is all of A_{n+1} by [17]. Q.E.D.

THEOREM 13.14. *Let L be an oriented link and let* $\alpha \in B_{2m}$ *be such that* $\tilde{\alpha} = L$ *as unoriented links. Then there is a* $k \in \mathbf{R}$, $2k \in \mathbf{Z}$ *with* $V_L(t) = t^k(-(t+1))^{m-1}\phi(\pi_0(\alpha))$.

Proof. We know there is an n and a $\beta \in B_n$ for which $\hat{\beta} = \tilde{\alpha}$. Let

$$\Omega_n = (\sigma_2\sigma_3 \dots \sigma_{2n-1})(\sigma_3\sigma_4 \dots \sigma_{2n-2}) \dots (\sigma_n\sigma_{n+1}) \in B_{2n}.$$

As shown in [6], $\widetilde{\Omega_n\beta\Omega_n^{-1}} = \hat{\beta} = \tilde{\alpha}$ where β is considered as an element of B_{2n} in the usual way (equality meaning isotopy as unoriented links). By Theorem 13.11 and the definition of $V_{\hat{\beta}}$ it suffices to show that $\phi(\pi_0(\Omega_n\beta\Omega_n^{-1})) = \mathrm{tr}(\pi_0(\beta))$. In fact we claim that if ψ on $\bigcup_n A_n$ is defined by

$$\psi(x) = \phi\left(\pi_0(\Omega_n)x\pi_0(\Omega_n)^{-1}\right) \quad \text{for } x \in A_{n-1}, \text{ then } \psi = \mathrm{tr}.$$

The first thing to check is that ψ is well defined; i.e.

$$\phi\left(\pi_0(\Omega_{n+1})x\pi_0(\Omega_{n+1})^{-1}\right) = \phi\left(\pi_0(\Omega_n)x\pi_0(\Omega_n)^{-1}\right) \quad \text{for } x \in A_{n-1}.$$

But the formulae $\Omega_p\sigma_i\Omega_p^{-1} = \sigma_{2i-1}^{-1}\sigma_{2i}\sigma_{2i-1}$ for $1 \le i \le p-1$ are easily checked in the braid groups. They imply

$$(13.15) \qquad \pi_0(\Omega_p)e_i\pi_0(\Omega_p^{-1}) = g_{2i-1}^{-1}e_{2i}g_{2i-1} \quad \text{for } 1 \le i \le p-1.$$

So conjugation by $\pi_0(\Omega_{n+1})$ has the same effect on A_{n-1} as conjugation by $\pi_0(\Omega_n)$, and ψ is well-defined.

Now $\psi(1) = 1$ is obvious so by Lemma 13.13 we only have to show that $\psi(xe_ny) = \tau\psi(xy)$ if $x, y \in A_{n-1}$. We proceed by induction on n. Using Proposition 13.7 we have

$$\begin{aligned}
\psi(xe_ny) &= \phi\left(\pi_0(\Omega_{n+1})xe_ny\pi_0(\Omega_{n+1})^{-1}\right)\\
&= \phi\left(\pi_0(\Omega_n)x\pi_0(\Omega_n)^{-1}g_{2n-1}^{-1}e_{2n}g_{2n-1}\pi_0(\Omega_n)y\pi_0(\Omega_n)^{-1}\right) \quad \text{(by (13.15))}\\
&= \tau^{-n-1}\mathrm{tr}\left(p_{n+1}\pi_0(\Omega_n)x\pi_0(\Omega_n)^{-1}g_{2n-1}^{-1}e_{2n}g_{2n-1}\pi_0(\Omega_n)y\pi_0(\Omega_n)^{-1}\right)\\
&= \tau^{-n+1}\mathrm{tr}\left(p_n\pi_0(\Omega_n)xy\pi_0(\Omega_n)^{-1}\right) \quad \text{(by (11.5) used twice)}\\
&= \tau\left(\tau^{-n}\phi\left(\pi_0(\Omega_n)xy\pi_0(\Omega_n)^{-1}\right)\right)\\
&= \tau\psi(xy).
\end{aligned}$$

Q.E.D.

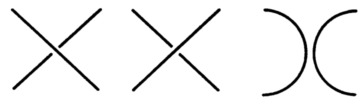


FIGURE 13.18

COROLLARY 13.16 (see [24]). *If L and L' are two oriented links which are isotopic as unoriented links, there is a $k \in \mathbf{Z}$ such that*

$$V_L(t) = t^k V_{L'}(t).$$

Note that the value of k is easily determined in terms of linking numbers by (12.2).

COROLLARY 13.17 (see [11], [25]). *Let L_+, L_- and L_∞ be unoriented links identical except at one crossing where they are as in Figure* 13.18. *Then there are numbers k_1 and k_2, $2k_i \in \mathbf{Z}$, such that*

$$V_{L_+} - t^{k_1} V_{L_-} = t^{k_2}(1-t)V_{L_\infty}.$$

Proof. Changing a link into a plat is rather easy, being achieved simply by threading local maxima and minima through the link. This can be done so as to leave alone any portion of the link that already looks like a plat, in particular a single crossing. Thus there are braids $\alpha_1, \alpha_2 \in B_{2m}$ such that $L_- = \alpha_1\sigma_k\alpha_2$, $L_+ = \alpha_1\sigma_k^{-1}\alpha_2$, $L_0 = \alpha_1\alpha_2$ (look at Figure 13.18 sideways). But by Eq. (4.1), $\pi_0(\alpha_1\sigma_k\alpha_2) - t\pi_0(\alpha_1\sigma_k^{-1}\alpha_2) = (t-1)\pi_0(\alpha_1\alpha_2)$. Taking ϕ on both sides of the equation gives the answer by Theorem 13.14. Q.E.D.

The values of k_1 and k_2 can be determined by calculus. For instance, if L_+ and L_- are knots and L_∞ is also a knot, we have $V'_{L_+}(1) - k_1V_{L_-}(1) - V'_{L_-}(1) = -V_{L_\infty}(1)$ so that by (12.1) and (12.2), $k_1 = 1$ and $V_{L_+} - tV_{L_-} = t^{k_2}(1-t)V_{L_\infty}$. Differentiating again gives $V''(1) - V''(1) = -2k_2$ or $k_2 = -(V''_{L_+}(1) - V''_{L_-}(1))/2$. But by the comments at the end of (12.2), $V''_{L_+}(1) - V''_{L_-}(1)$ is -6 times the linking number of the oriented link L_0 formed by eliminating the crossing according to the orientation. Thus we obtain in the above situation $V_{L_+} - tV_{L_-} = t^{3lk(L_0)}(1-t)V_{L_\infty}$. This is easily checked on the right-handed trefoil, where L_- and L_∞ are both unknots.

We see by induction that V_L is determined by certain linking numbers of two-component links obtained by successively eliminating crossings of L so as to obtain knots. A similar statement is true for links.

Corollary 13.16 is somewhat surprising (Lickorish has shown that Corollaries 13.16 and 13.17 are essentially the same) since the polynomial P_L, and even the Alexander polynomial, usually change wildly if the orientation of a link is altered.

Note 13.19. A plat $\tilde{\alpha}$ with $\alpha \in B_{2m}$ is called an m-bridge link. By (ii) of Lemma 13.3 and Figure 11.7, to calculate V_L for an m-bridge link it suffices to know a

$$\frac{1}{m+1}\binom{2m}{m}$$

matrix representing α and an expression for the matrix p_m. For 2-bridge links, one is dealing with 2×2 matrices and the representation of B_4 is given by

$$\sigma_1, \sigma_3 \to \begin{pmatrix} t & 0 \\ 1 & -1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} -1 & t \\ 0 & t \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 1 & 0 \\ 1/1+t & 0 \end{pmatrix}$$

so that $V_{\hat{\alpha}}$, for $\alpha \in B_4$, is obtained by calculating the 2×2 matrix of $\tilde{\alpha}$ and adding up the terms in the first row with weights $1 + t$ and 1 (the answer is correct up to a power of t which must be determined).

What is more interesting is that we can give the formula for 3-bridge links, since the representation in question is for $Y =$ , which appears explicitly (with $q \leftrightarrow t$ and $\theta_i \leftrightarrow \sigma_i$) in Section 10. One may easily calculate $p_2 = e_1 e_3 e_5 = (1+t)^{-3}(1+g_1)(1+g_3)(1+g_5)$ and one finds that $V_{\hat{\alpha}}$ is, up to a power of t, obtained by calculating the 5×5 matrix representing α and adding up the third row, with weights t, $1 + t$, $(1+t)^2$, $1 + t$, $1 + t$. On a computer this calculation is very rapid for indefinitely complicated 3-bridge links.

Note 13.20. There is a remarkable connection here with statistical mechanics. In fact, the algebras A_n had been used by Temperley and Lieb in [42] to partially solve a statistical mechanical model known as the Potts model. Indeed the linear functional ϕ essentially defines the partition function. A solution of the Potts model would be an explicit expression for $f(x, y)$ defined as

$$\lim_{n \to \infty} \frac{1}{n^2} \log\Big(\phi\big(((1+xe_1)(1+xe_3)\dots(1+xe_{n-1})(y+e_2) \times (y+e_4)\dots(y+e_n))^n\big)\Big).$$

The Potts model is defined for a system of "atoms" arrayed on the vertices of a regular lattice in $\mathbf{R}^2$. This is how the relation with closed braids (especially regular knot projections) comes about. The algebras A_n are a calculational device known as transfer matrices. In [19], Kauffman eliminates the need for braids by defining a Potts model on an arbitrary link projection (see also Chapter 12 of [4] and [42]).

14. Positivity considerations

In the context encountered by the author, the algebras A_n possessed more structure. They themselves were C^*-algebras and the trace tr was the data necessary to complete $\bigcup_n A_n$ to become a von Neumann algebra. More precisely, for each positive real t and $t \in \{e^{2\pi i/n} | n = 3, 4, 5, \dots\}$ there is a unique C^*-algebra A (infinite dimensional if $t \neq e^{\pm 2\pi i/3}$) generated by non-zero projections e_i $(i = 1, 2, \dots)$ satisfying (11.2), (11.3) and (11.4) together with a faithful trace tr uniquely defined by (11.2). The values $t = e^{\pm 2\pi i/n}$ and $t \in \mathbf{R}^+$ are distinguished on the braid group level by the obvious fact that for $t = 1$ or $e^{\pm 2\pi i/n}$, $\pi_0(\sigma_i)$ is unitary, but is not unitary otherwise.

The G.N.S. construction ([3]) is to make A into a pre-Hilbert space with scalar product $\langle a, b\rangle = \text{tr}(ab^*)$. Then A itself acts faithfully by left multiplication on the Hilbert space completion $\mathscr{H}_{\text{tr}}$ and the e_i's are realized as projections onto closed subspaces. The situation is thus very geometric and relations (11.2)–(11.4) can be thought of as defining special configurations of subspaces. Note that they are *not* (for $t \neq 1$) the same configurations as those arising in Coxeter-Dynkin theory, though there are relations; see [12].

The von Neumann algebra in question is the closure of A (on $\mathscr{H}_{\text{tr}}$) in the topology of pointwise convergence. Similar considerations apply to the linear functional ϕ.

Proposition 14.1. *For positive real t and* $t \in \{e^{2\pi i/n} | n = 3, 4, 5, \dots\}$, *the linear functional* ϕ *of Section* 13 *is a* state, *i.e.* $\phi(x^*x) \geq 0$, $\phi(1) = 1$.

Proof. Elements of the form a^*a are positive in a C^*-algebra and since $p_m = p_m^*$, $p_m x^* x p_m$ is positive. Q.E.D.

In fact, it is easy to see that ϕ is a pure state; i.e., in the G.N.S. representation (as above), A acts irreducibly.

While many of the early uses of the C^*-structure have been improved upon (e.g. Theorem 5 of [16] is much weaker than Scholium 6.15), there are still some results that have no other proof and the C^*-techniques are often useful in quick growth estimates. To illustrate the technique we prove an inequality which, while not terribly sharp, is certainly in the right direction.

Proposition 14.2. *Let* $\alpha \in B_n$ *have an exponent sum e and let* e_+ *be the sum of all positive exponents of* σ_i*'s in* α. *Let* V_+ *be the largest power of t in* $V_{\hat{\alpha}}$. *Then* $V_+ \leq (n - 1 + e)/2 + e_+$.

Proof. For $t \in \mathbf{R}$, $t \geq 1$, $\|g_i\| = t$ $(g_i = te_i - (1 - e_i))$ and $\|ab\| \leq \|a\| \|b\|$ so that $\|\pi_0(\alpha)\| \leq t^{e+}$ since $\|g_i^{-1}\| = 1$. When $t \to \infty$ the result follows that from formula (11.8) and $|\text{tr}(x)| \leq \|x\|$. Q.E.D.

Note 14.3. One could deduce something about the degree of the polynomial part of V from Proposition 14.2 but Murasugi [32] and Thistlethwaite [43] have shown that this degree is in fact a lower bound for the crossing number of the link L. The C^*-von Neumann picture is more complicated for the whole Hecke algebra but the theory is adequately explained in [14], [46].

Positivity might be useful in questions of faithfulness of representations by the following results.

PROPOSITION 14.4. *If* $\alpha \in B_n$ *then* $\alpha \in \ker \pi_0$ *for* $t = \pm e^{2\pi i/k}$, $k = 3, 4, 5, \ldots$ *if and only if* $V_{\hat{\alpha}}(e^{2\pi i/k}) = (-2\cos \pi/k)^{n-1}$ (*see Theorem* 10 *of* [16]).

Proof. Saying $V_{\hat{\alpha}}(e^{2\pi i/k}) = (-2\cos \pi/k)^{n-1}$ is the same as saying $\mathrm{tr}(\pi_0(\alpha)) = 1$. But π_0 is unitary, so that by the "equality" part of Cauchy–Schwarz, $\pi_0(\alpha) = 1$. Q.E.D.

COROLLARY 14.5. *If* $\alpha \in B_n$ *then* $\alpha \in \ker \pi_0$ (*for generic* t) *if and only if* $V_{\hat{\alpha}}(t) = (-(t+1)/\sqrt{t})^{n-1}$.

Proof. All entries in the PPTL representation are rational functions of $\sqrt{t}$ and so are determined by their values at $e^{2\pi i/k}$, $k = 3, 4, 5, \ldots$. Q.E.D.

Finally we use positivity to prove an assertion of Section 12.

PROPOSITION 14.6. *For each* t *let* $A_{t,n} = \{ |V_L(t)| \mid L = \hat{\alpha},\ \alpha \in B_n \}$. *Then for* $t = e^{\pm 2\pi i/k}$, $k \in \mathbf{Z}^+$, $k \notin \{1, 2, 3, 4, 6, 10\}$ *and* $n \geq 3$, $\overline{A_{t,n}} = [0, (2\cos \pi/k)^{n-1}]$ (*where the bar denotes closure*).

Proof. That $A_{t,n} \subseteq [0, (2\cos \pi/k)^{n-1}]$ follows from the Cauchy–Schwarz inequality and formula (11.8). Now it is shown in [18] (essentially because of the paucity of finite subgroups of SO(3)) that for t of the above form, $\psi(B_3)$ is infinite, ψ being the Burau representation. Thus also $\psi(\Gamma)$ is infinite for $\Gamma < B_3$, $[B_3 : \Gamma] < \infty$. Now by [40], ψ may be normalized by changing $\psi(\sigma_1)$, $\psi(\sigma_2)$ by a root of unity so that $\psi(B_3) \subseteq \mathrm{SU}(2)$, but then if Γ is the kernel of any map from B_3 to a finite cyclic group, $\psi(\Gamma)$ will be an infinite subgroup of SU(2). Its closure must be a Lie group with non-trivial connected sets. In particular the connected component of the identity of $\overline{\psi(\Gamma)}$ must contain -1. But if Γ is chosen correctly, $\mathrm{tr}(\psi(\gamma))$ will equal $\mathrm{tr}(\pi_{(2,1)}(\gamma))$ for $\gamma \in \Gamma$ and $\mathrm{tr}(\pi_{(1,1,1)}(\gamma)) = 1$. Thus using the weights for the third row of Figure 11.7, which are $(4\cos^2\pi/k)^{-1}$ and $1 - (2\cos^2\pi/k)^{-1}$, we see that the connected component G of the identity of $\overline{\pi_0(\Gamma)}$ must contain an element of (Markov) trace zero. But then $x \mapsto |\mathrm{tr}(x)|$

is a continuous function from G to $[0, 1]$ which contains 1 and 0. This argument proves the assertion for $n = 3$, but for $n \geq 4$ the only change is in the normalization since tr is compatible with the inclusions of the A_n's. Q.E.D.

COROLLARY 14.7. *For t as above, $\overline{\{|V(t)| \mid L \text{ a link}\}}$ is $[0, \infty)$.*

Note. The corollary is also true for $t = e^{2\pi i/10}$ by a special argument; indeed, Proposition 14.6 holds for $n \geq 4$ for $t = e^{2\pi i/10}$.

15. Braid index; bridge number; tables

THEOREM 15.1 (Morton, Franks-Williams [28], [47]). *Let L be an oriented link with polynomial $X_L(q, \lambda)$. Let d_+ be the degree of the largest power of λ in X_L and d_- be the smallest. If $\alpha \in B_n$ has $\hat{\alpha} = L$ then*

(MFW) $$n \geq d_+ - d_- + 1.$$

Proof. Let α have exponent sum e. Then

$$X_L = \left(- \frac{1 - \lambda q}{\sqrt{\lambda}(1 - q)} \right)^{n-1} (\sqrt{\lambda})^e \operatorname{tr}(\pi(\alpha)).$$

By induction on the Markov property $\operatorname{tr}(\pi(\alpha))$ is an honest polynomial in $z = -(1 - q)/(1 - \lambda q)$. In particular for fixed q it has a finite limit as $\lambda \to \infty$. Thus $|X_L| \leq (\text{const})\lambda^{(n+e-1)/2}$ as $\lambda \to \infty$. So $d \leq (n + e - 1)/2$. Considering α^{-1} and by Proposition 6.10, $-d_- \leq (n - e - 1)/2$. Thus $d_+ - d_- \leq n - 1$. Q.E.D.

The *braid index* of L is by definition the smallest n for which there is an $\alpha \in B_n$ with $\hat{\alpha} = L$.

The MFW inequality is a wonderful thing. A careful look at the proof of Theorem 16.1 shows that the inequality, viewed as a lower bound for the braid index, should be fairly good. But the author was totally unprepared for what he found in compiling Table 15.9. Of the more than 270 knots on ten crossings or less up to symmetry, the MFW inequality is sharp on all but five (9_{42}, 9_{49}, 10_{150}, 10_{132} and 10_{156}) of them! In the Lickorish-Millet variables d_+ and d_- are the largest and smallest powers of l^2. The MFW lower bound can be read immediately off the tables, for instance, in the notation of [22],

$$P_{9_{22}} = (-1 - 4[-4] - 2)(2\ 6\ [6]\ 1)(-1 - 4[-2])(1\ [0]),$$

so the braid index of 9_{22} is ≥ 4 since there are four non-zero terms in the first $(\ldots)$. In an interesting improvement of the MFW inequality, Morton shows in [29] that the number $d_+ - d_- + 1$ is in fact a lower bound for the number of Seifert circles in any regular projection of L.

It is useful to have other tools available in case the MFW inequality fails. A convenient one comes from Section 8.

PROPOSITION 15.2. *If K is a knot and $|\Delta_K(i)| > 3$, then K is not a closed 3-braid (see 23 of* [16]).

Proof. By (12.6), $V_K(i) = \pm 1$. So by (8.4) and (11.2), $|1 + i^e - i^{e/2}\Delta(i)| = 1$ if K is a closed 3-braid; so we would have $|\Delta_K(i)| \leq 3$. Q.E.D.

The basic inequality coming from positivity is the following.

PROPOSITION 15.3. *If L is a closed n-braid then*

$$\left|V_L(e^{2\pi i/k})\right| \leq (2\cos \pi/k)^{n-1}, \qquad k = 3, 4, 5, \ldots .$$

Proof. See 14.6.

COROLLARY 15.4. *If $\alpha \in \ker \pi_0$, $\alpha \in B_n$, for $t = e^{2\pi i/k}$, $k = 3, 4, 5, \ldots,$ then the braid index of α is n.*

Proof. $|V_{\hat{\alpha}}(e^{2\pi i/k})| = (2\cos \pi/k)^{n-1}$; so by Proposition 15.3, $\hat{\alpha}$ cannot be a closed p-braid for $p < n$. Q.E.D.

COROLLARY 15.5. *Suppose $\alpha \in B_n$ is a product of conjugates of terms of the form $\sigma_i^{k_i}$ with* GCD$(k_i) \geq 2$. *Then the braid index of $\hat{\alpha}$ is n.*

Proof. If GCD(k_i) is even then $\hat{\alpha}$ is an n-component link, so obviously has braid index $\leq n$; otherwise $\pi_0(\alpha) = \pm 1$. So by 16.3 we are through, as in Corollary 15.4. Q.E.D.

By [6], the minimal bridge number of a link L is the smallest n for which L is of the form $\tilde{\alpha}$ for $\alpha \in B_{2n}$. We have been unable to find sharp general results on the bridge number using V or P but Proposition 14.1 does give results formally the same as Proposition 15.3 and Corollaries 15.4, 15.5.

PROPOSITION 15.6. *Suppose L is a plat on $2n$ strings; then*

$$\left|V_L(e^{2\pi i/k})\right| \leq (2\cos \pi/k)^{n-1}.$$

Proof. Imitate 15.3 using ϕ instead of tr. Q.E.D.

COROLLARY 15.7. *If $a \in B_{2n}$ is in $\ker \pi_0$ for $t = e^{2\pi i/k}$, $k = 3, 4, 5, \ldots,$ then the bridge number of $\tilde{\alpha}$ is n.*

Proof. See Corollary 15.4. Q.E.D.

COROLLARY 15.8. *Suppose $\alpha \in B_{2n}$ is a product of conjugates of terms of the form $\sigma_i^{k_i}$ with* GCD$(k_i) \geq 2$. *Then the bridge number of $\tilde{\alpha}$ is n.*

Proof. See Corollary 15.5. Q.E.D.

TABLE 15.9. The following table gives the braid index, a braid expression, amphicheirality information, and the W polynomial of Proposition 12.5 for all unoriented prime knots up to 10 crossings. The last column may also contain a reference number for any special comments, which appear after the table. For amphicheirality, the last column is left blank if the knot is not amphicheiral and this is detected by V. An entry "A" means the knot is amphicheiral and "N" means that it is not, but V fails to detect it (so also does P for knots on ≤ 10 crossings).

Some care was taken to ensure that the braid words are as simple as possible but no adequate technique is yet available to give proofs except in special cases. Less care was taken with the 10-crossing knots.

The knot enumeration is as in [13], [37] with one or two corrections. Either the knot drawn in [37] or its mirror image is recorded, chosen according to whose V has the least power of t^{-1}. No confusion need arise as the braid word completely specifies the knot. Braid indices come from the MFW inequality unless otherwise indicated.

All coincidences of V are recorded.

The following example explains how to read the table: The knot 8_8 has W polynomial $t^{-3}(1 - t + 2t^2 - t^3 + t^4)$.

Knot		Braid Word	$P_0(W)$	W	A/N
3_1	2	1^3	0	1	
4_1	3	$12^{-1}12^{-1}$	− 2	− 1	
5_1	2	1^5	0	1101	4
5_2	3	$1^2 2^2 1^{-1} 2$	0	101	
6_1	4	$1^{-1}21^{-1}32^{-1}32$	− 2	− 10 − 1	
6_2	3	$1^{-1}21^{-1}2^3$	− 1	− 11 − 1	
6_3	3	$1^{-1}2^2 1^{-2}2$	− 3	1 − 11	A
7_1	2	1^7	0	1111101	
7_2	4	$1^{-1}3^3 21^2 3^{-1}2$	0	10101	
7_3	3	$1^2 21^{-1}2^4$	0	110201	
7_4	4	$1^2 23^2 1^{-1}23^{-1}2$	0	10201	
7_5	3	$1^4 21^{-1}2^2$	0	1102 − 11	
7_6	4	$12^{-1}1^{-2}32^3 3$	− 1	− 12 − 11	
7_7	4	$13^{-1}23^{-1}21^{-1}23^{-1}2$	− 3	1 − 21 − 1	
8_1	5	$1^{-1}232^{-1}1^{-1}4^2 324^{-1}$	− 2	− 10 − 10 − 1	
8_2	3	$1^{-1}2^5 1^{-1}2$	1	1 − 11 − 1	
8_3	5	$1^{-2}2^{-1}14^2 34^{-1}2^{-1}3$	− 4	− 10 − 20 − 1	A
8_4	4	$1^3 32^{-1}3^{-2}12^{-1}$	− 3	− 10 − 21 − 1	
8_5	3	$1^3 2^{-1}1^3 2^{-1}$	1	1 − 21 − 1	
8_6	4	$1^{-1}21^{-1}3^{-1}2^3 3^2$	− 1	− 11 − 21 − 1	
8_7	3	$1^4 2^{-2}12^{-1}$	− 2	1 − 12 − 11	

Knot		Braid Word	$P_0(W)$	W	A/N
8_8	4	$1^{-1}21^{2}3^{-1}2^{2}3^{-2}$	-3	$1-12-11$	5
8_9	3	$1^{-1}21^{-3}2^{3}$	-4	$-11-21-1$	A
8_{10}	3	$1^{-1}2^{2}1^{-2}2^{3}$	-2	$1-13-11$	
8_{11}	4	$1^{-1}2^{2}3^{-1}23^{2}1^{-1}2$	-1	$-12-21-1$	
8_{12}	5	$12^{-1}34^{-1}34^{-1}213^{-1}2^{-1}$	-4	$-11-31-1$	A
8_{13}	4	$1^{2}23^{-1}21^{-1}3^{-2}2$	-3	$1-22-11$	
8_{14}	4	$1^{2}2^{2}1^{-1}3^{-1}23^{-1}2$	-1	$-12-22-1$	
8_{15}	4	$1^{2}2^{-1}13^{2}2^{2}3$	0	$1103-22-1$	
8_{16}	3	$1^{2}2^{-1}1^{2}2^{-1}12^{-1}$	-2	$1-23-21$	
8_{17}	3	$1^{-1}21^{-1}2^{2}1^{-2}2$	-4	$-12-32-1$	A
8_{18}	3	$12^{-1}12^{-1}12^{-1}12^{-1}$	-4	$-13-33^{-1}$	A
8_{19}	3	$1212122^{2}1$	0	11111	
8_{20}	3	$1^{3}21^{-3}2$	-1	101	
8_{21}	3	$12^{-2}1^{2}2^{3}$	0	$1-11-1$	
9_1	2	1^{9}	0	1112111101	
9_2	5	$1234^{3}34^{-1}213^{-1}2^{-1}$	0	1010101	
9_3	3	$12^{-1}1^{6}2^{2}$	0	111120201	
9_4	4	$1^{-1}321^{2}23^{4}2^{-1}$	0	11020201	
9_5	5	$1^{2}21^{-1}32^{-1}34^{2}34^{-1}2$	0	1020201	
9_6	3	$1^{2}2^{2}1^{5}2^{-1}$	0	$11112-12-11$	
9_7	4	$1^{3}23^{2}1^{-1}2^{3}3^{-1}$	0	$1102-12-11$	
9_8	5	$12^{-1}31^{-2}4^{-1}32^{-1}3^{2}43$	-3	$-11-22-11$	
9_9	3	$1^{3}2^{-1}1^{4}2^{2}$	0	$11112-13-11$	
9_{10}	4	$1^{-1}21^{2}2^{3}3^{-1}23^{2}$	0	$1103-1301$	
9_{11}	4	$1^{-1}23^{-1}21^{-1}2^{4}32$	1	$2-13-11$	
9_{12}	5	$12^{-1}1^{-2}32^{3}4^{2}34^{-1}$	-1	$-12-22-11$	
9_{13}	4	$1^{2}23^{-1}21^{-1}3^{2}2^{3}$	0	$1103-13-11$	
9_{14}	5	$14^{2}3^{-1}23^{-1}23^{-1}1^{-1}4^{-1}2^{2}32^{-1}$	-3	$1-22-21-1$	
9_{15}	5	$12^{-1}132^{-1}43^{-1}4^{2}3$	-1	$-12-23-11$	
9_{16}	3	$1^{2}2^{-1}1^{3}2^{4}$	0	$11112-23-21$	
9_{17}	4	$1^{3}32^{-1}12^{-1}3^{-1}2^{-1}12^{-1}$	-3	$-11-32-21$	
9_{18}	4	$1^{2}3^{-1}21^{-1}2^{2}3^{2}2^{2}$	0	$1103-23-11$	
9_{19}	5	$1^{2}21^{-1}3^{-1}4^{-1}3^{-1}23^{-1}243^{-1}$	-4	$-11-32-21$	
9_{20}	4	$12^{2}3^{-1}21^{-1}23^{-1}2^{3}$	1	$2-23-21$	
9_{21}	5	$1^{-2}34^{-1}32^{-1}134^{2}2^{2}$	-1	$-13-23-11$	
9_{22}	4	$12^{-1}3^{3}2^{-1}31^{-1}2^{-1}32^{-1}$	-3	$-11-33-21$	
9_{23}	4	$12^{-1}1^{2}23^{3}23^{-1}2$	0	$1103-23-21$	
9_{24}	4	$13^{2}2^{-1}132^{-3}$	-4	$-12-33-11$	
9_{25}	5	$1^{-1}21^{-1}4^{-1}3^{-1}24^{2}34^{2}2^{2}3^{-1}$	-1	$-12-33-21$	
9_{26}	4	$1^{-1}2^{3}321^{-1}21^{-1}3^{-1}2$	-2	$1-23-32-1$	
9_{27}	4	$1^{-1}21^{-2}3^{-1}21^{-1}2^{2}32$	-4	$-12-33-21$	6
9_{28}	4	$1^{2}3^{2}2^{-2}132^{-1}$	-2	$1-24-32-1$	
9_{29}	4	$1^{-1}23^{-1}21^{-1}23^{-1}2^{2}$	-3	$-12-34-21$	
9_{30}	4	$13^{-2}23^{-1}1^{-1}21^{2}3^{-1}2$	-4	$-12-43-21$	
9_{31}	4	$1^{-1}2^{2}3^{-1}2^{2}1^{-1}231^{-1}2$	-2	$1-24-33-1$	
9_{32}	4	$1^{-1}231^{-1}21^{-1}3^{2}23^{-1}2$	-2	$1-34-42-1$	
9_{33}	4	$1^{-3}21^{-1}2^{2}312^{-1}3$	-4	$-13-44-21$	
9_{34}	4	$12^{-1}32^{-1}12^{-1}312^{-1}$	-4	$-13-54-31$	
9_{35}	5	$13^{2}4^{-1}2^{-1}123^{2}23^{-1}43^{-1}2$	0	1020301	

Knot		Braid Word	$P_0(W)$	W	A/N
9_{36}	4	$1^{-1}2^{3}3^{-1}23^{2}1^{-1}23$	1	2 − 23 − 11	
9_{37}	5	$12^{-1}32^{-1}31^{-1}43^{-1}2^{-1}432^{-1}$	− 4	− 11 − 42 − 21	
9_{38}	4	$1^{2}23^{2}2^{2}1^{-1}23^{-1}2$	0	1104 − 34 − 21	
9_{39}	5	$1^{-1}3^{-1}243^{2}1^{-1}2^{2}342^{-1}$	− 1	− 13 − 34 − 21	
9_{40}	4	$132^{-1}312^{-1}132^{-1}$	− 2	1 − 45 − 53 − 1	
9_{41}	5	$1^{-1}2^{2}4^{-1}34^{-1}2^{-1}13^{-1}234^{-1}32$	− 3	1 − 23 − 32 − 1	
9_{42}	4	$1^{3}32^{-1}31^{-2}2^{-1}$	− 3	− 10 − 1	N, 1
9_{43}	4	$121^{2}2^{2}32^{-1}12^{-1}3^{-1}$	1	1 − 11	
9_{44}	4	$1^{-1}21^{-1}32^{-1}32^{2}3^{-1}$	− 2	− 11 − 11	
9_{45}	4	$12^{-1}132^{3}32^{-1}$	0	1 − 12 − 11	
9_{46}	4	$1321^{-1}3^{-1}2132^{-1}$	0	− 10 − 1	
9_{47}	4	$1^{-1}231^{-1}21^{-1}232$	− 2	1 − 22 − 2	
9_{48}	4	$1^{2}2^{-1}32^{2}1^{-1}3^{-1}23^{-1}2$	− 1	− 13 − 12	
9_{49}	4	$1^{2}2^{2}321^{-1}2^{2}32^{-1}$	0	1103 − 12	1
10_{1}	6	$1^{2}4^{-1}5^{-1}3^{-1}21^{-1}345^{-1}432$	− 2	− 10 − 10 − 10 − 1	
10_{2}	3	$12^{-1}1^{7}2^{-1}$	0	10101 − 11 − 1	
10_{3}	6	$13^{-1}2^{-1}15^{-2}234^{-1}5324^{-1}$	− 4	− 10 − 20 − 20 − 1	
10_{4}	5	$1^{2}4^{-3}232^{-1}1^{-1}4^{-1}32$	− 5	− 11 − 20 − 20 − 1	
10_{5}	3	$1^{-2}21^{-1}2^{6}$	− 1	102 − 12 − 11	
10_{6}	4	$12^{-1}31^{-1}2^{-1}1^{-1}32^{6}$	1	1 − 22 − 21 − 1	
10_{7}	5	$13^{-1}2^{-1}1234^{-1}34^{-1}3^{2}2$	− 1	− 12 − 22 − 21 − 1	
10_{8}	4	$1^{5}32^{-1}13^{-2}2^{-1}$	− 2	− 10 − 11 − 21 − 1	
10_{9}	3	$1^{-3}21^{-1}2^{5}$	− 3	− 11 − 22 − 21 − 1	
10_{10}	5	$1^{-1}3^{-1}43^{2}21^{2}3^{-1}24^{-1}3^{-1}23^{-1}$	− 3	1 − 22 − 22 − 11	
10_{11}	5	$1^{-2}3^{-1}2^{-1}143^{3}2^{-1}34$	− 3	− 10 − 32 − 31 − 1	
10_{12}	4	$1^{-2}21^{-1}3^{2}2^{4}3^{-1}$	− 2	1 − 13 − 23 − 11	
10_{13}	6	$1^{2}3^{-1}21^{-1}3^{-1}45^{2}435^{-1}2^{-2}4^{-1}$	− 4	− 11 − 42 − 31 − 1	
10_{14}	4	$1^{-1}21^{-1}3^{2}2^{4}3^{-1}2$	1	2 − 24 − 32 − 1	
10_{15}	4	$1^{-1}32^{-1}3^{4}2^{-1}1^{-2}2$	− 4	1 − 12 − 23 − 11	
10_{16}	5	$12^{-1}1^{-2}32^{-1}4^{-1}34^{2}3^{2}$	− 3	− 11 − 32 − 31 − 1	
10_{17}	3	$1^{4}2^{-1}12^{-4}$	− 5	1 − 12 − 22 − 11	A
10_{18}	5	$13^{2}4^{2}3^{-1}2^{-1}41^{-2}32^{-1}$	− 3	− 11 − 33 − 32 − 1	
10_{19}	4	$1^{-2}2^{-1}13^{4}2^{-1}32^{-1}$	− 4	1 − 12 − 33 − 21	
10_{20}	5	$1^{-1}3^{-1}2^{3}34^{2}34^{-1}1^{-1}2$	− 1	− 11 − 21 − 21 − 1	
10_{21}	4	$1^{-1}21^{-1}3^{2}23^{-1}2^{4}$	1	2 − 23 − 21 − 1	
10_{22}	4	$1^{-3}23^{2}1^{-1}2^{3}3^{-1}$	− 4	− 11 − 32 − 31 − 1	8
10_{23}	4	$1^{-1}21^{-1}2^{3}3^{-1}21^{-1}3^{2}$	− 2	1 − 24 − 33 − 11	
10_{24}	5	$1^{-1}23^{-1}2^{2}1^{2}234^{-1}34^{-1}$	− 1	− 12 − 33 − 31 − 1	
10_{25}	4	$1^{-1}3^{2}2^{3}3^{-1}2^{2}1^{-1}2$	1	2 − 34 − 42 − 1	7
10_{26}	4	$12^{2}3^{-1}2^{-3}12^{-1}1^{2}32^{-1}$	− 4	− 12 − 43 − 31 − 1	
10_{27}	4	$1^{-2}23^{-1}23^{4}1^{-1}2$	− 2	1 − 24 − 44 − 21	
10_{28}	5	$1^{2}23^{-1}21^{-1}4^{-1}34^{-2}32$	− 3	1 − 23 − 23 − 11	
10_{29}	5	$12^{-1}34^{-1}32^{-1}1^{-1}2^{-1}32^{-1}43^{3}$	− 3	− 11 − 43 − 42 − 1	
10_{30}	5	$1^{2}23^{-1}21^{-1}3^{2}43^{-1}23^{-1}4^{-1}2$	− 1	− 13 − 34 − 32 − 1	
10_{31}	5	$1^{2}2^{2}1^{-1}43^{-1}4^{-2}3^{-1}23^{-1}$	− 5	1 − 13 − 33 − 21	
10_{32}	4	$1^{-1}23^{-1}21^{2}3^{-3}2^{2}$	− 4	− 12 − 44 − 32 − 1	
10_{33}	5	$12^{-1}34^{-1}32^{-1}1^{-2}2^{-1}34^{2}$	− 5	1 − 23 − 43 − 21	A
10_{34}	5	$1^{-1}4^{-2}3^{2}21^{2}4^{-1}3^{-1}23$	− 3	1 − 12 − 12 − 11	
10_{35}	6	$1^{-1}23^{-1}23^{-1}45^{-1}45^{-1}12^{-1}34^{-1}2^{2}$	− 4	− 11 − 32 − 31 − 1	8

Knot		Braid Word	$P_0(W)$	W	A/N
10_{36}	5	$1^23^{-1}2^234^{-1}34^{-1}1^{-1}32$	−1	−12 − 23 − 22 − 1	
10_{37}	5	$143^{-1}4^{-2}23^{-2}21^22^{-1}$	−5	1 − 13 − 33 − 11	A
10_{38}	5	$1^243^{-1}2^{-2}3^34^23^{-1}1^{-1}2$	−1	−12 − 33 − 32 − 1	
10_{39}	4	$1^{-1}21^{-1}2^33^{-1}23^3$	1	2 − 34 − 32 − 1	
10_{40}	4	$1^221^{-1}2^23^{-1}213^{-2}$	−2	1 − 25 − 44 − 21	
10_{41}	5	$12^{-1}32^{-1}1^{-1}4^{-1}32^{-1}3432^{-1}3^2$	−3	−12 − 44 − 42 − 1	9
10_{42}	5	$12^{-1}1^{-2}34^{-1}2^234^{-2}3$	−5	1 − 34 − 54 − 21	
10_{43}	5	$1^{-1}23^{-1}4^{-1}23^{-2}123^{-1}4^{-1}32^2$	−5	1 − 24 − 44 − 21	A, 10
10_{44}	5	$1^{-1}2^{-1}32^{-1}34^2123^{-1}2^{-2}43$	−3	−12 − 45 − 43 − 1	
10_{45}	5	$13^{-1}23^{-1}243^{-1}1^{-1}23^{-1}24^{-1}3^{-1}2$	−5	1 − 34 − 64 − 31	A
10_{46}	3	$1^{-1}2^51^{-1}2^3$	0	101 − 11 − 21 − 1	
10_{47}	3	$1^52^{-1}1^22^{-2}$	−1	103 − 13 − 11	
10_{48}	3	$1^{-2}2^41^{-3}2$	−5	1 − 13 − 23 − 11	N
10_{49}	4	$1^432^3312^{-1}$	0	11113 − 24 − 32 − 1	
10_{50}	4	$1^{-1}2^23^{-1}23^21^{-1}2^3$	1	2 − 33 − 31 − 1	
10_{51}	4	$1^223^{-2}1^{-1}2^23^{-1}2^2$	−2	1 − 25 − 34 − 11	
10_{52}	4	$1^{-2}2^{-1}3^22^{-1}3^312^{-1}$	−4	1 − 13 − 34 − 21	
10_{53}	5	$12^{-1}3^2421^{-1}3^22^{-1}31^{-1}4^22^2$	0	1104 − 35 − 32 − 1	
10_{54}	4	$1^{-2}2^{-1}3^22^{-1}3^31^{-1}2$	−4	1 − 13 − 23 − 11	
10_{55}	5	$1^{-1}43421^323^24^{-1}32^{-1}$	0	1103 − 24 − 32 − 1	
10_{56}	4	$1^{-1}2^23^22^{-1}31^{-1}2^3$	1	2 − 34 − 42 − 1	7
10_{57}	4	$1^323^{-2}1^{-1}2^23^{-1}2$	−2	1 − 25 − 45 − 21	
10_{58}	6	$125^{-1}43^{-1}23^{-1}4^{-1}5^{-1}12^{-1}34$	−4	−11 − 43 − 42 − 1	
10_{59}	5	$12^241^{-2}3^{-1}43^{-1}2^{-1}34^23^{-1}$	−3	−12 − 45 − 42 − 1	11
10_{60}	5	$1^{-2}2^212^{-1}34^{-1}32^{-1}32^{-1}43$	−4	−13 − 55 − 42 − 1	12
10_{61}	4	$12^{-1}1^{-2}3^32^{-1}3^3$	−2	−10 − 21 − 21 − 1	
10_{62}	3	$1^{-2}2^31^{-1}2^4$	−1	103 − 23 − 11	
10_{63}	5	$1^223^21^{-1}43^{-1}2^{-1}34^232$	0	1103 − 24 − 22 − 1	
10_{64}	3	$1^{-3}2^31^{-1}2^3$	−3	−11 − 33 − 31 − 1	
10_{65}	4	$132^231^{-1}2^{-1}3^31^{-1}2^{-2}$	−2	1 − 24 − 34 − 11	
10_{66}	4	$1^223^{-1}2^212^{-1}3^42$	0	11113 − 35 − 43 − 1	
10_{67}	5	$1^{-1}24^{-1}3^{-1}24^{-1}1^23^22^243^{-1}$	−1	−12 − 34 − 32 − 1	
10_{68}	5	$1^{-1}23^{-1}4^21^22^{-1}32^24^{-1}3^{-2}$	−3	1 − 23 − 33 − 11	
10_{69}	5	$13423^{-1}42^{-2}1^{-2}32^3$	−2	1 − 35 − 55 − 21	
10_{70}	5	$12^{-1}3^31^{-2}2^24^{-1}34^{-1}$	−3	−11 − 44 − 42 − 1	
10_{71}	5	$1^24^{-2}2^{-1}3^{-1}2^23^{-1}4^{-1}231^{-1}2$	−5	1 − 24 − 54 − 21	13, N
10_{72}	4	$1^32^{-1}1^223^{-1}23^{-1}2$	1	2 − 35 − 43 − 1	
10_{73}	5	$12^{-1}3^214^{-1}32^{-1}3432^{-1}$	−2	1 − 35 − 54 − 21	14
10_{74}	5	$1^{-1}423^{-1}24^{-1}3^{-1}21^24^{-1}3^22$	−1	−13 − 34 − 31 − 1	
10_{75}	5	$12^{-1}132^{-1}3432^{-1}34^{-1}2^{-1}$	−4	−13 − 45 − 42 − 1	
10_{76}	4	$1^22^{-1}1^{-1}3^32^{-1}12^{-1}3^3$	1	1 − 33 − 42 − 1	
10_{77}	4	$1^{-1}2^33^{-2}1^22^23^{-1}$	−2	1 − 14 − 34 − 21	
10_{78}	5	$1^{-2}432^{-1}13^24^{-1}32^33$	1	3 − 35 − 32 − 1	
10_{79}	3	$1^32^{-2}1^22^{-3}$	−5	1 − 14 − 34 − 11	A
10_{80}	4	$1^223^{-1}2^21^{-1}2^332^2$	0	11113 − 35 − 42 − 1	
10_{81}	5	$1^{-2}2^{-2}1^{-1}3^{-1}24^23^24$	−5	1 − 25 − 55 − 21	15, A
10_{82}	3	$1^42^{-2}12^{-1}12^{-1}$	−3	−12 − 34 − 32 − 1	
10_{83}	4	$1^{-1}23^{-1}23^{-2}21^223^{-1}$	−4	−13 − 55 − 42 − 1	12, 2
10_{84}	4	$1^32^23^{-1}21^{-1}3^{-1}23^{-1}$	−2	1 − 25 − 55 − 31	

Knot		Braid Word	$P_0(W)$	W	A/N
10_{85}	3	$1^{-1}2^{2}1^{-1}21^{-1}2^{4}$	-1	$1-13-33-21$	
10_{86}	4	$1^{2}23^{-1}2^{2}3^{-1}21^{-1}3^{-1}2$	-2	$1-35-54-21$	2, 14
10_{87}	4	$1^{3}3^{-1}23^{-1}1^{-1}23^{-1}23^{-1}$	-4	$-12-45-43-1$	
10_{88}	5	$1^{2}2^{-1}4^{-1}312^{-1}32^{-1}43^{-1}1^{-1}23^{-1}$	-5	$1-35-75-31$	A
10_{89}	5	$123^{-1}21^{-1}423^{-1}243^{-1}4$	-2	$1-46-65-21$	
10_{90}	4	$1^{2}32^{-2}1^{2}2^{-1}12^{-1}31^{-1}2^{-1}$	-4	$-12-54-42-1$	
10_{91}	3	$1^{3}2^{-2}12^{-2}12^{-1}$	-5	$1-24-44-21$	N, 10
10_{92}	4	$1^{3}2^{2}3^{-1}21^{-1}23^{-1}2$	1	$3-46-53-1$	
10_{93}	4	$1^{-2}3^{2}2^{-1}312^{-1}3^{2}2^{-1}$	-4	$1-23-44-21$	
10_{94}	3	$1^{3}2^{-1}1^{2}2^{-2}12^{-1}$	-3	$-12-44-42-1$	9
10_{95}	4	$1^{-2}23^{-1}21^{-1}2^{2}3^{2}2$	-2	$1-36-55-21$	
10_{96}	5	$123^{-1}42132^{-1}3^{-2}43^{-1}$	-4	$-13-65-52-1$	
10_{97}	5	$1^{2}2^{-1}34^{-1}2^{2}1^{-1}3^{-1}23^{2}4^{-1}2$	-1	$-13-46-43-1$	
10_{98}	4	$1^{-1}2^{2}3^{2}21^{-1}2^{2}3^{-1}2$	1	$3-45-52-1$	
10_{99}	3	$1^{-2}21^{-2}2^{2}1^{-1}2^{2}$	-5	$1-25-45-21$	A
10_{100}	3	$1^{3}2^{-1}1^{2}2^{-1}1^{2}2^{-1}$	-1	$1-14-34-21$	
10_{101}	5	$1^{-1}32^{2}1432^{3}3^{2}43^{-1}1^{-1}2^{-1}$	0	$1104-36-43-1$	
10_{102}	4	$12^{2}3^{-1}21^{-1}3^{-1}213^{-2}$	-4	$-12-44-42-1$	
10_{103}	4	$1^{2}2^{2}3^{-1}2^{2}3^{-1}1^{-1}23^{-1}$	-2	$1-25-44-21$	
10_{104}	3	$1^{2}2^{-3}1^{2}2^{-1}12^{-1}$	-5	$1-24-54-21$	13, N
10_{105}	5	$1^{-1}2^{-1}32^{-1}4132^{-1}3^{2}4^{2}3^{-1}2^{-1}$	-3	$-12-56-53-1$	
10_{106}	3	$1^{3}2^{-2}1^{2}2^{-1}12^{-1}$	-3	$-12-45-42-1$	11
10_{107}	5	$1432^{-1}3^{2}43^{-1}2^{-2}1^{-1}2^{-1}32^{-1}$	-5	$1-35-65-21$	
10_{108}	4	$1^{-2}3^{2}2^{-1}13^{2}2^{-1}32^{-1}$	-4	$1-23-43-21$	
10_{109}	3	$1^{-2}2^{2}1^{-2}2^{2}1^{-1}2$	-5	$1-25-55-21$	15, A
10_{110}	5	$13^{-1}23^{-1}4^{-1}32^{3}12^{-1}3^{-1}4^{-1}2$	-3	$-12-55-52-1$	
10_{111}	4	$1^{2}2^{2}3^{-1}2^{2}1^{-1}23^{-1}2$	1	$3-45-42-1$	
10_{112}	3	$1^{3}2^{-1}12^{-1}12^{-1}12^{-1}$	-3	$-13-46-43-1$	
10_{113}	4	$1^{3}23^{-1}21^{-1}23^{-1}23^{-1}$	-2	$1-36-76-41$	
10_{114}	4	$1^{-1}2^{-1}32^{-1}32^{-2}1^{2}3^{-1}23^{2}$	-4	$-13-56-43-1$	
10_{115}	5	$1^{-1}234^{-1}32^{-1}1^{-1}4342^{-1}32^{-2}$	-5	$1-36-76-31$	A
10_{116}	3	$1^{2}2^{-1}12^{-1}12^{-1}1^{2}2^{-1}$	-3	$-13-56-53-1$	
10_{117}	4	$1^{-1}23^{-1}21^{2}23^{-1}2^{2}3^{-1}$	-2	$1-36-66-31$	
10_{118}	3	$1^{2}2^{-1}12^{-2}12^{-1}12^{-1}$	-5	$1-35-65-31$	A
10_{119}	4	$1^{2}23^{-2}21^{-1}3^{-1}23^{-1}2$	-4	$-13-66-53-1$	
10_{120}	5	$12^{2}34^{-1}132^{-1}31^{-1}4^{2}3^{2}23^{-1}$	0	$1105-57-53-1$	
10_{121}	4	$1^{-1}23^{-1}23^{-1}21^{2}23^{-1}2$	-2	$1-47-76-31$	
10_{122}	4	$12^{-1}32^{-1}32^{-1}1^{-1}2^{-1}321^{-1}2^{2}$	-4	$-13-57-54-1$	
10_{123}	3	$12^{-1}12^{-1}12^{-1}12^{-1}12^{-1}$	-5	$1-46-86-41$	A
10_{124}	3	$12^{5}12^{3}$	0	1112111	
10_{125}	3	$1^{-1}2^{-3}1^{-1}2^{5}$	-4	10101	N
10_{126}	3	$12^{-3}12^{5}$	0	20201	
10_{127}	3	$1^{5}21^{-2}2^{2}$	0	$11-12-21-1$	
10_{128}	4	$1^{2}32^{2}32^{2}1^{-1}32$	0	1111201	
10_{129}	4	$1^{-2}32^{-2}32^{2}132^{-1}$	-3	$1-12-11$	5
10_{130}	4	$1^{-1}23^{-3}21^{2}23^{2}$	-1	10201	
10_{131}	4	$1^{-1}23^{3}21^{2}23^{-2}$	0	$1-12-21-1$	
10_{132}	4	$1^{-1}231^{3}2^{-3}32$	0	1101	1, 4, 18
10_{133}	4	$1^{-1}323^{2}21^{2}3^{-1}2^{-1}13^{-1}2$	0	$101-11-1$	

Knot		Braid Word	$P_0(W)$	W	A/N
10_{134}	4	$12^{2}12^{3}321^{-1}2^{2}3^{-1}$	0	11112 − 12 − 1	
10_{135}	4	$12^{2}32^{-1}1^{-3}2^{-1}3^{2}$	− 3	2 − 23 − 11	
10_{136}	4	$1^{2}23^{-1}21^{-1}2^{2}3^{-1}2^{-2}$	− 3	− 11 − 11	
10_{137}	5	$1^{2}32^{-1}12^{-1}31^{-1}4^{-1}32^{-1}32^{-1}4$	− 2	− 11 − 21 − 1	16
10_{138}	5	$1^{2}32^{-1}12^{-1}341^{-1}3^{-1}23^{-1}24^{-1}$	− 3	− 11 − 32 − 2	
10_{139}	3	$1^{2}2^{3}1^{2}212$	0	111211101	
10_{140}	4	$12^{-1}13^{-3}23^{3}2$	1	101	
10_{141}	3	$1^{-3}21^{2}2^{-1}1^{2}2$	− 2	− 11 − 11 − 1	
10_{142}	4	$1^{-1}3^{3}23^{3}1^{2}2$	0	1111202	
10_{143}	3	$1^{-1}2^{2}1^{-2}21^{2}2^{2}$	0	2 − 12 − 11	
10_{144}	4	$1^{2}32^{2}1^{-1}23^{-2}12^{-1}$	− 1	− 22 − 32 − 1	17
10_{145}	4	$1^{2}21^{-1}3^{-1}2132^{2}3$	0	1101101	
10_{146}	4	$1^{2}2^{-1}3^{-2}2^{2}12^{-1}32^{-1}$	− 3	1 − 22 − 21	
10_{147}	4	$1^{-1}2^{-2}3^{2}2^{2}1^{2}2^{-1}1^{-1}3^{-1}2$	− 3	− 11 − 22 − 1	
10_{148}	3	$1^{2}2^{3}1^{-2}21^{-1}2$	0	2 − 13 − 11	
10_{149}	3	$1^{-2}2^{3}1^{3}2^{2}$	0	11 − 13 − 32 − 1	
10_{150}	4	$1^{2}2^{2}321^{-1}2^{-1}3^{2}2^{-1}$	1	2 − 22 − 1	18
10_{151}	4	$1^{2}3^{-1}2^{2}132^{-2}3^{-1}2$	− 2	1 − 24 − 22	
10_{152}	3	$1^{2}2^{3}1^{3}2^{2}$	0	1112111 − 11 − 1	
10_{153}	4	$1^{2}23^{-2}2^{-2}13^{-1}2^{2}$	− 4	101101	
10_{154}	4	$12^{-1}3^{2}2^{2}1^{2}32^{2}$	0	111111 − 11 − 1	
10_{155}	3	$1^{3}21^{-2}21^{-2}2$	− 2	− 11 − 21 − 1	16
10_{156}	4	$1^{-1}232^{2}31^{-2}23^{-1}2$	− 2	1 − 23 − 21	18
10_{157}	3	$1^{-1}2^{2}1^{2}2^{-1}1^{2}2^{2}$	0	11 − 14 − 33 − 1	
10_{158}	4	$1^{2}231^{-1}2^{-2}32^{-2}3$	− 4	− 12 − 42 − 2	
10_{159}	3	$12^{-1}12^{-1}12^{3}12^{-1}$	0	2 − 23 − 21	
10_{160}	4	$123^{2}1^{2}2^{-1}12^{-1}13^{-1}$	1	2 − 12	
10_{161}	3	$1^{-1}21^{2}2^{3}1^{2}2$	0	11111101	3
10_{163}	4	$1^{-2}32^{-1}3^{2}12^{2}32^{-1}$	− 1	− 22 − 31 − 1	
10_{164}	4	$12^{-1}31^{-1}21^{2}2^{-1}12^{-1}3$	− 2	1 − 34 − 32	
10_{165}	4	$1^{-1}23^{-2}12^{-1}12^{-1}132$	− 3	2 − 33 − 21	
10_{166}	4	$1^{2}23^{-1}21^{-1}23^{2}1^{-1}2$	0	1 − 13 − 22 − 1	

Notes:

1. MFW braid index inequality not sharp. Answer, if known, from [33].
2. These two knots are interchanged from Rolfsen's table, to make the Alexander polynomials correct.
3. This is the same as 10_{162} of [37].
4. Compare $5_1 - 10_{132}$ same V, same P.
5. Compare $8_8 - 10_{129}$ same V, same P.
6. Compare $8_{16} - 10_{156}$ same V, same P.
7. Compare $10_{25} - 10_{56}$ same V, same$P$.
8. Compare $10_{22} - 10_{35}$ same V, different Δ.
9. Compare $10_{41} - 10_{94}$ same V, different Δ.
10. Compare $10_{43} - 10_{91}$ same V, different Δ.
11. Compare $10_{59} - 10_{106}$ same V, different Δ.
12. Compare $10_{60} - 10_{83}$ same V, different Δ.
13. Compare $10_{71} - 10_{104}$ same V, different Δ.
14. Compare $10_{73} - 10_{86}$ same V, different Δ.
15. Compare $10_{81} - 10_{109}$ same$V$, differentΔ.
16. Compare $10_{137} - 10_{155}$ same V, different Δ.
17. The uppermost crossing of [37] has been changed.
18. Braid index obtained by Morton by showing braid index of a 2 cable = 8.

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