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THE COLORED JONES POLYNOMIAL AND THE A-POLYNOMIAL OF TWO-BRIDGE KNOTS

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Abstract. We study relationships between the colored Jones polynomial and the A-polynomial of a knot. We establish for a large class of 2-bridge knots the AJ conjecture (of Garoufalidis) that relates the colored Jones polynomial and the A-polynomial. Along the way we also calculate the Kauffman bracket skein module of all 2-bridge knots. Some properties of the colored Jones polynomial of alternating knots are established.

0. Introduction

The Jones polynomial was discovered by Jones in 1994 [Jo] and has made a revolution in knot theory. Despite many efforts little is known about the relationship between the Jones polynomial and classical topology invariants like the fundamental group.

The A-polynomial of a knot, introduced in [CCGLS], describes more or less the representation space of the knot group into $SL(2,\mathbb{C})$, and has been fundamental in geometric topology. The main goal of the paper is to establish for a large class of 2-bridge knots the AJ conjecture (made by Garoufalidis) that relates the colored Jones polynomial and the A-polynomial. Along the way we also calculate the Kauffman bracket skein module of all 2-bridge knots. Some properties of the colored Jones polynomial of alternating knots are established.

0.1. **The colored Jones polynomial and its recurrence ideal.**

0.1.1. *The colored Jones polynomial.* For a knot K in the 3-space $\mathbb{R}^3 \subset S^3$ the colored Jones function (see for example [MM])

$$J_K : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$$

is defined for integers $n \in \mathbb{Z}$; its value $J_K(n)$ is known as the colored Jones polynomial of the knot K with color n. We will recall the definition of $J_K(n)$ in section 1.

In our joint work in S. Garoufalidis [GL] we showed that the function J_K always satisfies a non-trivial recurrence relation as described in the next subsection. Partial results were obtained earlier by Frohman, Gelca, and Lofaro through their theory of non-commutative A-ideal [FGL, Ge], which also plays an important role in the present paper.

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0.1.2. *Recurrence relations and q-holonomicity.* Consider a *discrete function* $f : \mathbb{Z} \longrightarrow \mathcal{R}$, and define the linear operators L and M acting on such functions by:

$$(Mf)(n) = t^{2n} f(n), \qquad (Lf)(n) = f(n+1).$$

It is easy to see that $LM = t^2 ML$, and that $L^{\pm 1}, M^{\pm 1}$ generate the *quantum torus* $\mathcal{T}$, a non-commutative ring with presentation

$$\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM = t^2 ML).$$

We also use the notation $\mathcal{T}_+$ for the subring of $\mathcal{T}$ which consists only of polynomials with non-negative powers of M and L. Traditionally $\mathcal{T}_+$ is called the *quantum plane.*

The *recurrence ideal* of the discrete function f is the left ideal $\mathcal{A}$ in $\mathcal{T}$ that annihilates f:

$$\mathcal{A} = \{P \in \mathcal{T} \mid \quad Pf = 0\}.$$

We say that f is *q-holonomic*, or f satisfies a linear recurrence relation, if $\mathcal{A} \neq 0$. In [GL] we proved that for every knot K, the function J_K is q-holonomic. Denote by $\mathcal{A}_K$ the recurrence ideal of J_K.

0.1.3. *An example.* For the right-handed trefoil, one has

$$J_K(n) = \frac{(-1)^{n-1}\, t^{2-2n}}{1 - t^{-4}} \sum_{k=0}^{n-1} t^{-4nk} \prod_{i=0}^{k} (1 - t^{4i-4n}).$$

The function J_K satisfies $PJ_K = 0$, where

$$P = (t^4 M^{10} - M^6)L^2 - (t^2 M^{10} + t^{-18} - t^{-10} M^6 - t^{-14} M^4)L + (t^{-16} - t^{-4} M^4).$$

Together with the initial conditions $J_K(0) = 0, J_K(1) = 1$, this recurrence relation determines $J_K(n)$ uniquely.

0.1.4. *Generator of the recurrence ideal.* The quantum torus $\mathcal{T}$ is not a principal ideal domain, and $\mathcal{A}_K$ might not be generated by a single element. Garoufalidis [Ga] noticed that by adding to $\mathcal{T}$ all the inverses of polynomials in M one gets a principal ideal domain $\tilde{\mathcal{T}}$, and hence from the ideal $\mathcal{A}_K$ one can define a polynomial invariant. Formally one can proceed as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \{\sum_{k\in\mathbb{Z}} a_k(M) L^k \mid \quad a_k(M) \in \mathcal{R}(M), \;\; a_k = 0 \quad \text{almost everywhere}\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k \cdot b(M)L^l = a(M)\, b(t^{2k}M) L^{k+l}$.

Then it is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}}_K := \tilde{\mathcal{T}}\mathcal{A}_K$ of $\mathcal{A}_K$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_K(t;M,L) = \sum_{i=0}^{n} \alpha_{K,i}(t;M)L^i \in \mathcal{T}_+,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,i}(t;M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. That α_K can be chosen to have integer coefficients follows from the fact that $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$. It's clear that $\alpha_K(t;M,L)$ annihilates J_K, and hence it is in the recurrence ideal $\mathcal{A}_K$. Note that $\alpha_K(t;M,L)$ is defined up to a factor $\pm t^a M^b, a, b \in \mathbb{Z}$. We will call α_K the *recurrence polynomial* of K. For example, the polynomial P in the previous subsection is the recurrence polynomial of the right-handed trefoil.

Remark 0.1. If P is a polynomial in t and M (no L), and $Pf = 0$ then $P = 0$. Hence adding all the inverses of polynomials in M does not affect the recurrence relations.

0.2. **Main results.** Let ϵ be the map reducing $t = -1$. Formally, if V is an $\mathcal{R}$-module, then let $\epsilon(V) = \mathbb{C} \otimes_{\mathcal{R}} V$, where $\mathbb{C}$ is considered as an $\mathcal{R}$-module by setting $t = -1$. Also if $x \in V$ then $\epsilon(x)$ is the image of $1 \otimes x$ in $\epsilon(V)$. Thus $\epsilon(\alpha_K)$ is the polynomial obtained from $\alpha_K(t;M,L)$ by putting $t = -1$. For example, when K is the right-handed trefoil, $\epsilon(\alpha_K) = (M^4 - 1)(L-1)(LM^6 + 1)$.

Suppose $f, g \in \mathbb{C}[M,L]$. We say that f is *M-essentially equal to* g, and write

$$f \overset{M}{=} g,$$

if the quotient f/g does not depend on L. We say that two algebraic subsets of $\mathbb{C}^2$ with parameters (M,L) are *M-essentially equal* if they are the same up to adding some lines parallel to the L-axis. It's clear that if f is M-essentially equal to g, then $\{f = 0\}$ and $\{g = 0\}$ are M-essentially equal. Here $\{f = 0\}$ is the algebraic set of zero points of f.

0.2.1. *The AJ Conjecture.* Denote by $A_K \in \mathbb{Z}[L,M]$ the A-polynomial of K (see [CCGLS, CL]); we will review its definition in section 2. In [Ga] Garoufalidis made the following conjecture.

Conjecture 1. (**The AJ conjecture**) *The polynomials $\epsilon(\alpha_K)$ and $(L-1)A_K$ are M-essentially equal.*

Actually, this is the strong version. The weak version of the conjecture says that $\{\epsilon(\alpha_K) = 0\}$ and $\{(L-1)A_K = 0\}$ are M-essentially equal. The algebraic set $\{(L-1)A_K = 0\}$ is known as the *deformation variety* of the knot group, with the component $\{L-1 = 0\}$ corresponding to abelian representations of the knot group into $SL_2(\mathbb{C})$, and $\{A_K = 0\}$ – to non-abelian ones.

Garoufalidis [Ga] verified the conjecture for the trefoil and the figure 8 knot. Takata [Ta] gave some evidence to support the conjecture for twist knots, but did not prove it. Both works are based heavily on the computer programs of Wilf and Zeilberger. Hikami [Hi] verifies the conjecture for torus knots. In all these works direct calculations with explicit formulas are used.

In the present paper we prove the conjecture for a large class of 2-bridge knots, using a more conceptual approach. Two-bridge knots $\mathfrak{b}(p,m)$ are parametrized by a pair of odd positive integers $m < p$, with $\mathfrak{b}(p,m) = \mathfrak{b}(p,m')$ if $mm' = 1 \pmod p$ (see [BZ] and section 4 below).

Theorem 1. *Suppose $K = \mathfrak{b}(p, m)$ is a 2-bridge knot.*

a) The recurrence polynomial α_K has L-degree less than or equal to $(p+1)/2$.

b) The algebraic set $\{\epsilon(\alpha_K) = 0\}$ is M-essentially equal to an algebraic subset of $\{(L-1)A_K = 0\}$.

c) The AJ conjecture holds true if

(*) the A-polynomial is $\mathbb{Z}$-irreducible and has L-degree $(p-1)/2$.

Here $\mathbb{Z}$-irreducibility means irreducibility in $\mathbb{Z}[M, L]$. There are many 2-bridge knots that satisfy condition (*). For example in a recent work [HS] Hoste and Shanahan proved that all the twist knots satisfy the condition (*). In a separate paper [Le2] we will prove that if both p and $(p-1)/2$ are prime, then $\mathfrak{b}(p, m)$ satisfies the condition (*). Also knot tables show that many 2-bridge knots with small p, m satisfy the condition (*). For all 2-bridge knots the L-degree of A_K is less than or equal to $(p-1)/2$, and for most of them it is exactly $(p-1)/2$.

0.2.2. *The Kauffman Bracket Skein Module of knot complements.* Our proof of the main theorem is more or less based on the ideology that the Kauffman Bracket Skein Module (KBSM) is a quantization of the $SL_2(\mathbb{C})$-character variety (see [Bu2, PS] and section 3.1.1 below), which has been exploited in the work of Frohman, Gelca, and Lofaro [FGL] where they defined the non-commutative A-ideal. The calculation of the KBSM of a knot complement is a difficult task. Bullock [Bu1] and recently Bullock and Lofaro [BL] calculated the KBSM for the complements of $(2, 2p+1)$ torus knots and twist knots. Another main result of this paper is a generalization of these works: We calculate explicitly the KBSM for complements of all 2-bridge knots. We will use another, more geometric approach that allows us to get the results for all 2-bridge knots. In a subsequent paper [Le2] we will give an explicit algorithm for the calculation of the KBSM of 2-bridge knots.

0.3. **Plan of the paper.** In the first section we review the theory of skein modules, the colored Jones polynomial, and prove that for alternating knot the L-degree of the recurrence polynomial is at least 2. In Section 2 we review the theory of the A-polynomial and a closely related polynomial, B_K. Section 3 is devoted to the “quantum” version of B_K, the peripheral polynomial. We will formulate another weaker of version of the AJ conjecture and prove it holds true for 2-bridge knots (in later section 5). In Section 4 we first review the 2-bridge knot theory and then calculate the skein module of its complements. The last section contains a proof of the main theorem.

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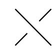
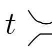
1. On the colored Jones polynomial, Skein modules

We first recall the definition and known facts about the colored Jones polynomial through the theory of Kauffman Bracket Skein Modules which was introduced by Przytycki and Turaev, see the survey [Pr]. Then we prove that the recurrence polynomial of an alternating knot has L-degree greater than or equal to 2.

1.1. **The colored Jones polynomial and skein modules.**

1.1.1. *Skein modules.* Recall that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. A *framed link* in an oriented 3-manifold Y is a disjoint union of annuli in Y. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. Consider the free $\mathcal{R}$-module with basis $\mathcal{L}$, and factor it by the smallest subspace containing all expressions of the form  $- t$  $- t^{-1}$)(and ◯ $+ t^2 + t^{-2}$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of Y.

When $Y = \Sigma \times [0,1]$, the cylinder over the surface Σ, we also use the notation $\mathcal{S}(\Sigma)$ for $\mathcal{S}(Y)$. In this case $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂Y to Y induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(M)$.

1.1.2. *Example:* $\mathbb{R}^3$*, and the Jones polynomial.* When Y is the 3-space $\mathbb{R}^3$ or the 3-sphere S^3, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if L is a framed link in $\mathbb{R}^3$, then its value in the skein module $\mathcal{S}(\mathbb{R}^3)$ is $\langle L \rangle$ times the empty link, where $\langle L \rangle \in \mathcal{R}$, known as the Kauffman bracket of L (see [Ka, Li]), and is just the Jones polynomial of *framed links* in a suitable normalization.

1.1.3. *Example: The solid torus, and the colored Jones polynomial.* The solid torus ST is the cylinder over an annulus, and hence its skein module $\mathcal{S}(ST)$ has an algebra structure. The algebra $\mathcal{S}(ST)$ is the polynomial algebra $\mathcal{R}[z]$ in the variable z, which is a knot representing the core of the solid torus.

Instead of the $\mathcal{R}$-basis $\{1, z, z^2, \dots\}$, two other bases are often useful. The first basis consists of the Chebyshev polynomials $T_n(z)$, $n \geq 0$, defined by $T_0(x) = 2$, $T_1(x) = x$, and $T_{n+1}(x) = xT_n - T_{n-1}$. The second basis consists of polynomials $S_n(z)$, $n \geq 0$, satisfying the same recurrence relation, but with $S_0(x) = 1$ and $S_1(x) = x$. Extend both polynomials by the recurrence relation to all indices $n \in \mathbb{Z}$. Note that $T_{-n} = T_n$, while $S_{-n} = -S_{n-2}$.

For a framed knot K in a 3-manifold Y we define the n-th power K^n as the link consisting of n parallel copies of K. Using these powers of a knot, $S_n(K)$ is defined as an element of $\mathcal{S}(Y)$. In particular, if $Y = \mathbb{R}^3$ one can calculate the bracket $\langle S_n(K) \rangle \in \mathcal{R}$, and it is essentially the colored Jones polynomial. More precisely, we will define the colored Jones polynomial $J_K(n)$ by the equation

$$J_K(n+1) = (-1)^n \times \langle S_n(K) \rangle.$$

The $(-1)^n$ sign is added so that when K is the trivial knot, $J_K(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}$. Then $J_K(1) = 1$, $J_K(2) = -\langle K \rangle$. We extend the definition for all integers n by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$. In the framework of quantum invariants, $J_K(n)$ is the sl_2-quantum invariant of K colored by the n-dimensional simple representation of sl_2.

We will always assume K has 0 framing. In this case $J_K(n)$ contains only even powers of t, i.e. $J_K(n) \in \mathbb{Z}[t^{\pm 2}]$. Hence the recurrence polynomial α_K can be assumed to have only even powers of t.

1.1.4. *Example: cylinder over the torus, and the non-commutative torus.* A pair of *oriented* meridian and longitude on the torus $\mathbb{T}^2$ will define an *algebra* isomorphism Φ between $\mathcal{S}(\mathbb{T}^2)$ and the symmetric part of the quantum torus $\mathcal{T}$ as follows.

For a pair of integers a, b let $(a,b)_T = T_d((a',b')_T)$, where d is the greatest common divisor of a and b, with $a = da', b = db'$, and $(a',b')_T$ is the closed curve without self-intersection on the torus that is homotopic to a' times the meridian plus b' times the longitude. Here T_d is the Chebyshev polynomial; and the framing of a curve on $\mathbb{T}^2$ is supposed to be parallel to the surface $\mathbb{T}^2$. Note that in the definition of skein modules we use *non-oriented links*, hence $(a,b)_T = (-a,-b)_T$. As an $\mathcal{R}$-module, $\mathcal{S}(\mathbb{T}^2)$ is the quotient of the free $\mathcal{R}$-module spanned by $\{(a,b)_T \in \mathbb{Z}^2\}$ modulo the relations $(a,b)_T = (-a,-b)_T$.

Recall that the quantum torus $\mathcal{T}$ is defined as $\mathcal{T} = \mathcal{R}\langle L^{\pm 1}, M^{\pm 1}\rangle/(LM = t^2 ML)$. Let $\mathcal{T}^\sigma$ be the subalgebra of $\mathcal{T}$ invariant under the involution σ, where $\sigma(M^a L^b) = M^{-a}L^{-b}$. Frohman and Gelca in [FG] showed that the map

$$\Phi : \mathcal{S}(\mathbb{T}^2) \to \mathcal{T}^\sigma, \qquad \Phi((a,b)_T) = (-1)^{a+b} t^{ab}\,(M^a L^b + M^{-a}L^{-b})$$

is an isomorphism of *algebras.*

1.1.5. *2-punctured disk.* Let F be the 2-dimensional disk without 2 interior points A and B. Then $\mathcal{S}(F)$ is the polynomial algebra $\mathcal{R}[x, x', y]$, where x is a small loop around A, x' a small loop around B, and y a loop circling both A, B (see Figure 1). Note that in this case $\mathcal{S}(F)$ is a commutative algebra, which is not true in general when F is replaced by an arbitrary surface.

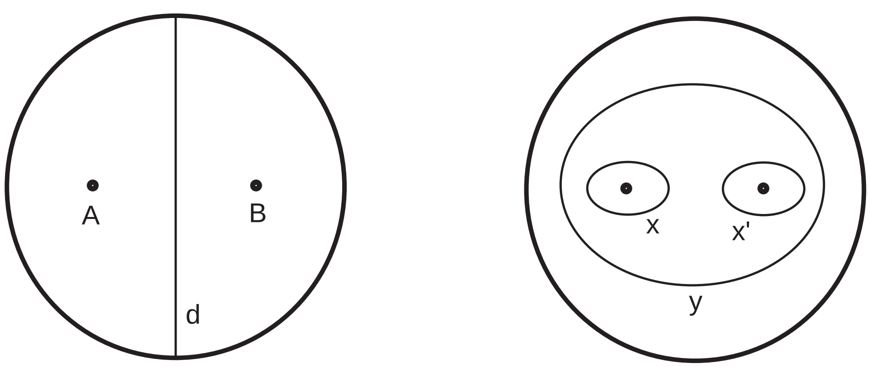


FIGURE 1. The 2-punctured disk

Let d be the line on F separating A and B, as shown in Figure 1. The wall $d \times [0,1]$ divides $F \times [0,1]$ into two halves. An arc in $F \times [0,1]$ is *prime* if it lies in one half with its 2 end points in the wall $d \times [0,1]$ and together with the straight segment connecting the 2 end points it bounds a disk that is pierced by $A \times [0,1]$ or $B \times [0,1]$ exactly once. A link in $F \times [0,1]$ is *monic* if it is transversal to the wall $d \times [0,1]$, and the wall cuts it into prime arcs.

Lemma 1.1. *Suppose* a monic *link* $\ell \subset F \times [0,1]$ *intersects the wall* $d \times [0,1]$ *at* $2k$ *points. Then as an element of the skein module* $\mathcal{S}(F) = \mathcal{R}[x, x', y]$ *the link* ℓ *is a polynomial having* y*-degree* k*, and the coefficient of* y^k *is invertible and of the form* $\pm t^l, l \in \mathbb{Z}$.

Proof. Consider the diagram D of ℓ on F. At every crossing point there are 2 ways to smooth the diagram, one positive that gives coefficient t in the skein relation, and the other is negative. A *state* is the result of smoothing all the crossing; what one has is a bunch of non-intersecting circles, each is one of x, x', y, or the trivial loop. The value of L is the sum over all states, each with coefficient a power of t. It's clear that for a monic link, there is only one state that gives the maximal power y^k. □

1.2. **The colored Jones polynomial of an alternating knot.** One of the best known applications of the Jones polynomial is a proof (Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links, based on an exact estimate of the crossing number using the *breadth* of the Jones polynomial. We will need a generalization of this estimate for the colored Jones polynomial.

For a Laurent polynomial $P(t) \in \mathbb{Z}[t^{\pm 1}]$ let $d_+(P)$ and $d_-(P)$ be respectively the maximal and minimal degree of t in P. The difference $b := d_+ - d_-$ is called the breadth of P.

Proposition 1.2. *Suppose K is a non-trivial alternating knot. Then the breadth of $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ is a quadratic polynomial in n.*

Proof. Choose a reduced alternating diagram D (see [Li]) for the knot K such that the framing is the black-board one. Suppose D has k double points. The n-parallel D^n of D will have kn^2 double points. Let $s_+(D)$ and $s_-(D)$ be the number of circles obtained by positively (respectively, negatively) smoothing all the double points. For an alternating diagram one has $s_+ + s_- = k + 2$. Note that the diagram D^n is adequate in the sense of [Li, Chapter 5], and that $s_\pm(D^n) = ns_\pm(D)$. The exact $d_\pm$ of the Kauffman bracket of adequate diagrams are known (see Lemma 5.7 of [Li]) and one has:

$$d_+\langle D^n\rangle = kn^2 + 2ns_+ - 2,$$
$$d_-\langle D^n\rangle = -kn^2 - 2ns_+ + 2,$$

It follows that $d_+(D^n) > d_+(D^{n-1})$ and $d_-(D^n) < d_-(D^{n-1})$. We have that

$$S_n(K) = D^n + \text{terms of lower degrees in } K,$$

hence $d_\pm(\langle S_n(K)\rangle) = d_\pm(\langle D^n\rangle)$, and

$$b(S_n(D)) = d_+\langle D^n\rangle - d_-\langle D^n\rangle = 2kn^2 + 2n(k+2) - 4.$$

□

1.3. **The L-degree of the recurrence polynomial.**

Proposition 1.3. *Suppose K is a non-trivial alternating knot. Then the recurrence polynomial α_K has L-degree greater than 1.*

Proof. Assume the contrary that $\alpha_K = P(t;M)\,L + P_0(t;M)$, where $P, P_0 \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$. Garoufalidis [Ga] showed that the polynomial $\sigma(\alpha_K) = P(t;M^{-1})\,L^{-1} + P_0(t;M^{-1})$ is also in the recurrence ideal. Since α_K is *the* generator, it follows that for some $\gamma(t;M) \in \mathcal{R}(M)$

$$L\,\sigma(\alpha_K) = \gamma(t;M)\,\alpha_K.$$

One can then easily shows that, after normalized by a same power of M, one has

$$P_0(t;M) = P(t;t^{-2}M^{-1}).$$

The equation $\alpha_K J_K = 0$ can now be rewritten as

$$J_K(n+1) = -\frac{P(t;t^{-2-2n})}{P(t;t^{2n})} J_K(n).$$

It's easy to see that for n big enough, the difference of the breadths $b(P(t;t^{-2-2n})) - b(P(t;t^{2n}))$ is a constant depending only on the polynomial $P(t;M)$, but not on n. From the above equation it follows that the breadth of $J_K(n)$, for n big enough, is a linear function on n. This contradicts Proposition 1.2. □

2. The A-polynomial and and its sibling

We briefly recall here the definition of the A-polynomial and introduce a sibling of its. We will say that f is *M-essentially divisible by g* if f is M-essentially equal to a polynomial divisibly by g.

2.1. **The A-polynomial.**

2.1.1. *The character variety of a group.* The set of representations of a finitely presented group π into $\mathrm{SL}_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation. The naive quotient space, i.e. the set of orbits, does not have a good topology/geometry. Two representations in the same orbit (i.e. conjugate) have the same character, but the converse is not true in general. A better quotient, the algebro-geometric quotient denoted by $\chi(\pi)$ see [CS, LM], has the structure of an algebraic set. There is a bijection between $\chi(\pi)$ and the set of all characters of representations of π into $SL_2(\mathbb{C})$, and hence $\chi(\pi)$ is usually called the *character variety* of π. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$.

Suppose $\pi = \mathbb{Z}^2$, the free abelian group with 2 generators. Every pair of generators λ, μ will define an isomorphism between $\chi(\pi)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (L,M) and τ is the involution $\tau(M,L) = (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with L and M being the upper left entry of λ and μ respectively. The isomorphism does not change if one replaces (λ,μ) with (λ^{-1},μ^{-1}).

2.1.2. *The A-polynomial.* Let X be the closure of S^3 minus a tubular neighborhood $N(K)$ of a knot K. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is 0. The pair provides an identification of $\chi(\pi_1(\partial X))$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depends on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \longrightarrow \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the 1-dimension components is defined by a single polynomial $A'_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that A'_K is defined up to ± 1. It is known that A'_K is divisible by $L - 1$, hence $A'_K = (L - 1)A_K$, where $A_K \in \mathbb{C}[M, L]$ is called the A-polynomial of K. It is known that $A_K \in \mathbb{C}[M^2, L]$. By definition, A_K does not have repeated factor, and is not divisible by $L - 1$.

2.2. **A sibling of the A-polynomial.**

2.2.1. *The dual picture.* It's also instructive and convenient to see the dual picture in the construction of the A-polynomial. For an algebraic set Y let $R[Y]$ denotes the ring of regular functions on Y. For example, $R[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\sigma(M^a L^b) := M^{-a} L^{-b}$.

The map ρ in the previous subsection induces an algebra homomorphism

$$\theta : R[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow R[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*; it is an ideal of $\mathfrak{t}^\sigma$. Let $\hat{\mathfrak{p}} := \mathfrak{t}\,\mathfrak{p}$ be the ideal extension of $\mathfrak{p}$ in $\mathfrak{t}$. The set of zero points of $\hat{\mathfrak{p}}$ is the closure of $\hat{Z}$ in $\mathbb{C}^2$.

2.2.2. *A sibling of the A-polynomial.* The ring $\mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\tilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension of $\hat{\mathfrak{p}}$ in $\tilde{\mathfrak{t}}$, which is $\tilde{\mathfrak{t}}\,\hat{\mathfrak{p}} = \tilde{\mathfrak{t}}\,\mathfrak{p}$, is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^a$ with $a \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$.

From the definitions one has immediately

Proposition 2.1. *The polynomial B_K is M-essentially divisible by $A'_K = (L - 1)A_K$. The two algebraic sets $\{B_K = 0\}$ and $\{A'_K = 0\}$ are M-essentially equal.*

Note that B_K might not be M-essentially equal to A'_K because B_K might contain repeated factors.

3. The quantum peripheral ideal and the peripheral polynomial

3.1. **The quantum peripheral ideal.**

3.1.1. *Skein modules as quantum deformations of character varieties.* Recall that ϵ is the map reducing $t = -1$. One important result (Bullock, Przytycki, and Sikora [Bu1, PS]) in the theory of skein module is that $\epsilon(\mathcal{S}(Y))$ has a natural algebra structure and, when factored by its nilradical, is canonically isomorphic to $R[\chi(Y)]$, the ring of regular functions on the character variety of $\pi_1(Y)$. The product of 2 links in $\epsilon(\mathcal{S}(Y))$ is their union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of the knot in Y. The isomorphism between $\epsilon(\chi(Y))$ and $R[\chi(Y)]$ is given by $K(r) = -\operatorname{tr} r(K)$, where K is a homotopy class of a knot in Y, represented by an element, also denoted by K, of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$.

In many cases the nilradical of $\epsilon(\mathcal{S}(Y))$ is trivial, and hence $\epsilon(\mathcal{S}(Y))$ is exactly equal to the ring of regular functions on the character variety of $\pi_1(Y)$. For example, this is the case when Y is a torus, or when Y is the complement of a 2-bridge knots (see section 4).

In light of this fact, one can consider $\chi(Y)$ as a quantization of the character variety.

3.1.2. *The quantum peripheral ideal.* Recall that X is the closure of the complement of a tubular neighborhood $N(K)$ in S^3. The boundary ∂X is a torus, and using the preferred meridian and longitude we will identify $\mathcal{S}(\partial X)$ with $\mathcal{T}^\sigma$, see subsection 1.1.4.

The embedding of ∂X into X gives us a map $\Theta : \mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma \longrightarrow \mathcal{S}(X)$, which can be considered as a quantum analog of θ. One has the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}^\sigma & \xrightarrow{\Theta} & \mathcal{S}(X) \\ {\scriptstyle\epsilon}\big\downarrow & & {\scriptstyle\epsilon}\big\downarrow \\ \mathfrak{t}^\sigma & \xrightarrow{\theta} & R(\chi(X)) \end{array}$$

The kernel of Θ, denoted by $\mathcal{P}$, is called the *quantum peripheral ideal*; it is a left ideal of $\mathcal{T}^\sigma$ and can be considered as a quantum analog of the classical peripheral ideal $\mathfrak{p} = \ker\theta$. The ideal $\mathcal{P}$ was introduced by Frohman, Gelca, and Lofaro in [FGL] and there it is called simply the peripheral ideal. From the commutative diagram it's clear that $\epsilon(\mathcal{P}) \subset \mathfrak{p}$. The question whether $\epsilon(\mathcal{P}) \neq 0$ is not trivial and still open.

3.2. **The peripheral polynomial.** Let us adapt the construction of the B_K polynomial to the quantum setting. Recall that $\tilde{\mathcal{T}}$ (see Introduction) is a principal left-ideal domain that contains $\mathcal{T}^\sigma$ as a subring. The left-ideal extension $\tilde{\mathcal{P}} = \tilde{\mathcal{T}}\mathcal{P}$ in $\tilde{\mathcal{T}}$ is generated a polynomial

$$\beta_K(t; M, L) = \sum_{i=0}^{s} \beta_{K,i}(t, M)\, L^i \in \mathcal{T}_+,$$

where s is assumed to be minimum and all the coefficients $\beta_{K,i}(t, M) \in \mathbb{Z}[t^{\pm1}, M^{\pm1}]$ are co-prime. We call β_K the *peripheral polynomial* of K, which is defined up to $\pm t^a M^b$ with $a, b \in \mathbb{Z}$.

Proposition 3.1. *$\epsilon(\beta_K)$ is M-essentially divisible by B_K, and hence is M-essentially divisible by $A'_K = (L-1)A_K$.*

Proof. The proposition follows the fact that $\epsilon\mathcal{P} \subset \mathfrak{p}$. □

Proposition 3.2. *Suppose $\epsilon(\mathcal{P}) = \mathfrak{p}$. Then $\epsilon(\beta_K) \overset{M}{=} B_K$.*

Proof. For the extensions $\hat{\mathcal{P}} := \mathcal{T}\mathcal{P}$ and $\hat{\mathfrak{p}} := \mathfrak{t}\mathfrak{p}$ we also have $\epsilon(\hat{\mathcal{P}}) = \hat{\mathfrak{p}}$, since $\epsilon\hat{\mathcal{P}} = \epsilon(\mathcal{T}\mathcal{P}) = \epsilon(\mathcal{T})\epsilon(\mathcal{P}) = \mathfrak{t}\mathfrak{p}$.

From the definition we have that $h(M)\mathcal{B}_K \in \hat{\mathfrak{p}}$ for some non-zero polynomial $h(M) \in \mathbb{Z}[M]$. Hence $\epsilon^{-1}(h(M)B_K) \subset \hat{\mathcal{P}}$ is not empty. Take an element $u \in \epsilon^{-1}(h(M)B_K)$; it is M-essentially divisible by β_K, the generator. Applying the map ϵ, one gets B_K is M-essentially divisible by $\epsilon(\beta_K)$. Combining with Proposition 3.1 one has $\epsilon(\beta_K) \overset{M}{=} B_K$. □

Conjecture 2. *For every knot we have $\epsilon(\mathcal{P}) = \mathfrak{p}$, and hence $\epsilon(\beta_K) \overset{M}{=} B_K$.*

Later in section 5 we will show that the conjecture holds true for *all* 2-bridge knots. This conjecture is closely related the the AJ conjecture.

3.2.1. *The orthogonal ideal and recurrence relations.* There is a pairing between $\mathcal{S}(N(K))$ and $\mathcal{S}(X)$ defined by

$$\langle \ell, \ell' \rangle := \langle \ell \cup \ell' \rangle \in \mathcal{S}(S^3) = \mathcal{R},$$

where ℓ and ℓ' are framed links in $N(K)$ and X respectively.

The *orthogonal ideal* $\mathcal{O}$ is defined by

$$\mathcal{O} := \{\ell' \in \mathcal{S}(\partial X) \quad | \quad \langle \ell, \Theta(\ell')\rangle = 0 \quad \text{for every } \ell \in \mathcal{S}(N(K))\}.$$

It's clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [FGL] $\mathcal{O}$ is called the formal ideal. What is important for us is the following

Proposition 3.3. *The orthogonal ideal is in the recurrence ideal of a knot, $\mathcal{O} \subset \mathcal{A}_K$. As a consequence, $\mathcal{P} \subset \mathcal{A}_K$.*

This was proved by Garoufalidis [Ga]. Frohman, Gelca, and Lofaro [FGL] proved that every element ℓ' in the orthogonal ideal $\mathcal{O}$ gives rise to a linear recurrence relation for the colored Jones polynomial. The idea is simple and beautiful: ℓ' annihilates everything in $\mathcal{S}(N(K)$, in particular, $\langle T_n(z), \Theta(\ell')\rangle = 0$; but this equation, after some calculation, can be rewritten as a linear recurrence relation for the colored Jones polynomial. Garoufalidis, using the Weyl symmetry, further simplified the recurrence relation, and obtained that $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$, which is stronger than the proposition.

3.2.2. *Relation between the peripheral and recurrence polynomials.*

Lemma 3.4. *The peripheral polynomial β_K is divisible by the recurrence polynomial α_K in the sense that there are polynomials $g(t, M) \in \mathbb{Z}[t, M]$ and $\gamma(t, M, L) \in \mathcal{T}_+$ such that*

$$\beta_K(t, M, L) = \frac{1}{g(t, M)}\, \gamma(t, M, L)\, \alpha(t, M, L). \tag{1}$$

Moreover $g(t, M)$ and $\gamma(t, M, L)$ can be chosen so that $\epsilon g \neq 0$.

Proof. From Proposition 3.3 we have that $\mathcal{P} \subset \mathcal{A}$. Hence the left-ideal extension $\hat{\mathcal{P}} := \mathcal{T}\mathcal{P}$ is also a subset of $\mathcal{A}$, since both are left ideals of $\mathcal{T}$. It follows that β_K, as the generator of the extension of $\hat{\mathcal{P}}$ in $\tilde{\mathcal{T}}$, is divisible by the generator of the extension of $\mathcal{A}$, and (1) follows.

We can assume that $t+1$ does not divide both $g(t, M)$ and $\gamma(t, M, L)$ simultaneously. If $\epsilon g = 0$ then g is divisible by $t+1$, and hence γ is not. But then from the equality

$$g\beta_K = \gamma\, \alpha_K,$$

it follows that α_K is divisible by $t+1$, which is impossible, since all the coefficients of powers of L in α_K are suppose to be co-prime. □

4. Two-bridge knots and their skein modules

4.1. **Two-Bridge knots.** A two-bridge knot is a knot $K \subset S^3$ such that there is a 2-sphere $S^2 \subset S^3$ that separates S^3 into 2 balls B_1 and B_2, and the intersection of K and each ball is isotopic to 2 trivial arcs in the ball. The branched double covering of S^3 along a 2-bridge knot is a lens space $L(p, m)$, which is obtained by doing a p/m surgery on the unknot. Such a 2-bridge knot is denoted by $\mathfrak{b}(p, m)$. It is known that both p, m are odd. One can always assume that $p > m \geq 1$. It is known that $\mathfrak{b}(p, m) = \mathfrak{b}(p, m')$ if $mm' \equiv 1 \pmod{p}$.

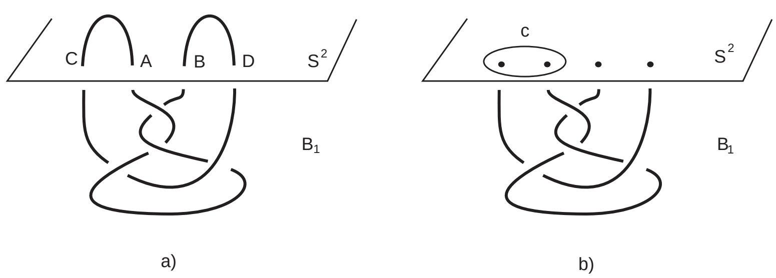


Figure 2. The figure 8 knot, or $\mathfrak{b}(5, 3)$

We will fix once and for all the separating sphere S^2. Let X_i be the ball B_i minus a tubular neighborhood of the knot K, $i = 1, 2$. See Figure 2a for an example in the case of $\mathfrak{b}(5, 3)$, the figure 8 knot. Usually in figures we replace S^3 with $\mathbb{R}^3$ by removing the point at infinity. The sphere S^2, without the infinity point, is depicted by a horizontal plane. We will refer to B_1 as the lower ball (that corresponds to the lower half-space in the figure). Assume that K intersect the 2-sphere S^2 at A, B, C, D, and that in the upper ball B_2 the pair A, B is connected by an arc of K (hence so is the pair C, D). Then A and C are connected by an arc of K in B_1, so are B and D.

Note that both X_1, X_2 are cylinders over a 2-punctured disk. Hence $\mathcal{S}(X_1)$ is isomorphic to the commutative algebra $\mathbb{Z}[x, x', y]$, where x is a small loop on S^2 circling around A, x' a small loop on S^2 circling around B, and y a small loop on S^2 circling around A and B (see subsection 1.1.5).

4.2. **Skein module of complements of 2-bridge knots.** One of our main results is

Theorem 2. *The skein module $\mathcal{S}(S^3 \setminus \mathfrak{b}(p,m))$ is free over $\mathcal{R}$ with basis $\{x^a y^b, 0 \le a, 0 \le b \le (p-1)/2$.*

Proof. Note that X is homeomorphic to X_1 with a 2-handle attached along the curve c encircling A and C, see Figure 2b. If ℓ is a framed link in X_1, we call *a slide $sl(\ell)$ of ℓ* any link in X_1 obtained by taking a band sum of a component of ℓ and c (pushed inside X_1).

The embedding of X_1 into X give rise to a linear map from $\mathcal{S}(X_1) \equiv \mathcal{R}[x, x', y]$ to $\mathcal{S}(X)$. It is known that the map is surjective, and its kernel is generated by all $\ell - sl(\ell)$, for all framed links ℓ and all possible slides $sl(\ell)$, see [Pr]. For example, one can easily slide x to get x', hence $\mathcal{S}(X)$ is equal to $\mathcal{R}[x, y]$ factored by a submodule $\mathcal{K}$, the *kernel submodule*.

4.2.1. *The kernel.* The kernel submodule can be described in a more manageable way as follows. Suppose a set of knots $\{\ell_i\}$ spans $\mathcal{S}(X_1)$ over $\mathcal{R}[x, x']$, i.e. $\{x^a (x')^b \ell_i\}$ form an $\mathcal{R}$-basis of $\mathcal{S}(X_1) \equiv \mathcal{R}[x, x', y]$. For each ℓ_i choose *one* ℓ'_i arbitrarily among the slides of ℓ_i. Then it is known that the kernel submodule $\mathcal{K}$ is generated over $\mathcal{R}[x]$ by $\{\ell_i - \ell'_i\}$.

Although the set $\{1, y, y^2, \dots\}$ is a basis of $\mathcal{S}(X_1)$ over $\mathcal{R}[x, x']$, it is not good enough: its elements are not *knots*; moreover, their slides are difficult to work with. In what follows we will find a basis $\{1, y_1, y_2, \dots, \}$, replacing $\{1, y, y^2, \dots\dots\}$ that has the following properties

(i) each y_i is presented by a knot and as an element of $\mathcal{S}(X_1) = \mathbb{C}[x, x', y]$ has y-degree i, and moreover the coefficient of y^i is invertible and of the form $\pm t^k, k \in \mathbb{Z}$.

(ii) for $0 < i < p$ one of the slides of y_i is y_{p-i}. We will use $y'_i = y_{p-i}$ for $0 < i < p$.

(iii) for $i \ge p$, the y-degree of a slide y'_i of y_i is strictly less than the y-degree of y_i.

Now let $z_i := y_i - y'_i$. The kernel $\mathcal{K}$ is spanned over $\mathcal{R}[x]$ by $\{z_i, i \ge 1\}$. But $z_i = -z_{p-i}$ for $1 \le i \le \frac{p-1}{2}$, hence $\mathcal{K}$ is spanned over $\mathcal{R}[x]$ by $\{z_i, i \ge \frac{p+1}{2}\}$. When $i \ge \frac{p+1}{2}$, z_i is a polynomial of y-degree i with the leading coefficient invertible (of the form $\pm t^k$). Hence $\mathcal{K}$, just like the span over $\mathcal{R}[x]$ of $\{y^i, i \ge \frac{p+1}{2}\}$, is a complement in $\mathcal{R}[x, y]$ of the $\mathcal{R}[x]$-module spanned by $\{y^i, 0 \le i \le \frac{p-1}{2}\}$. The theorem would follow immediately. It remains to construct y_i.

4.2.2. *The construction of y_i.* One way to draw a diagram of the 2-bridge knot $\mathfrak{b}(p,m)$ is the following. The separating sphere S^2, without the infinity point, is depicted by the 2-space $\mathbb{R}^2$, which in the next figures is the plane of the page, with the usual Cartesian axes, a horizontal (the 1-st) one and a vertical (the 2-nd) one. The ball X_1 is now behind the page.

On the left half of the horizontal axis choose $p+1$ points. Mark the left most by A_0 and the right most by A_p. Connect A_0 with A_p by a line segment, which will be part of the knot. At each of the interior $p-1$ points, draw a small vertical line segment which, just like in a link diagram, underpasses the horizontal axis; mark the bottom end of this underpass by A_i and the top end point by A_{2p-i}, $i = 1, 2, \dots, p-1$ (from left to right). See Figure 3a for the example of $\mathfrak{b}(5,3)$, the figure 8 knot.

Do a mirror reflection in the vertical axis we get a similar picture on the right half of the horizontal axis, only the markings are B_i instead of A_i. On the vertical axis choose $2p$

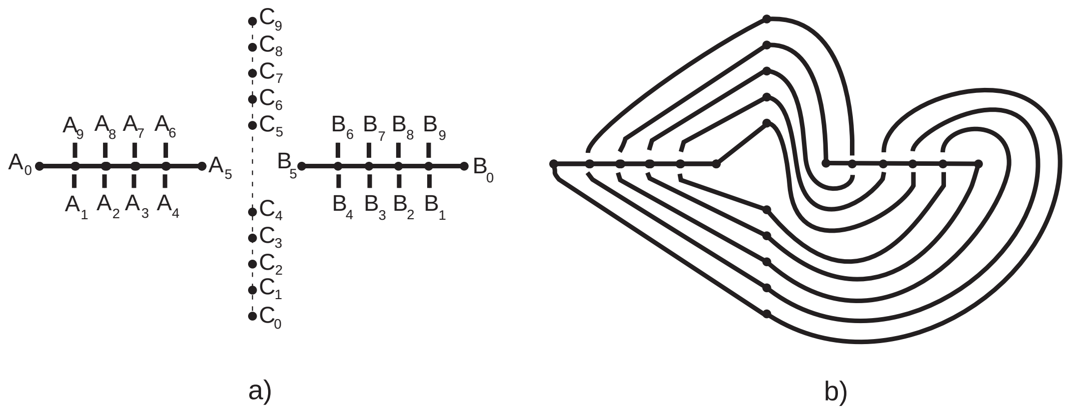


FIGURE 3. The figure 8 knot, or $\mathfrak{b}(5,3)$

points, with p of them under the horizontal axis, and the remaining p above. Mark them from bottom to top by $C_0, C_1, \dots, C_{2p-1}$, see Figure 3a.

All the indexes will be taken modulo $2p$. On the left half-plane connect C_i with A_i by straight line segment. On the right half-plane connect C_i with B_{i-m} using non-intersecting arcs; up to isotopy there is only one way to do this. The result is a diagram of the 2-bridge knot $\mathfrak{b}(p,m)$. See Figure 3b for the example of the knot $\mathfrak{b}(5,3)$, the figure 8 knot.

Now let us return to the 3-space. The 3-rd axis of $\mathbb{R}^3$ is perpendicular to the plane $\mathbb{R}^2$, i.e the page. In the plane formed by the 2-nd axis and the 3-rd axis, choose a small regular neighborhood of the line segment from C_i to C_j, and denote its boundary by c_{ij}, which is a loop in the knot complement. See Figure 4a for an example of c_{03}, c_{49}, and $c_{0,9}$ with the knot $\mathfrak{b}(5,3)$. By pushing the vertical axis into the lower ball B_1 we also assume that c_{ij} is in X_1.

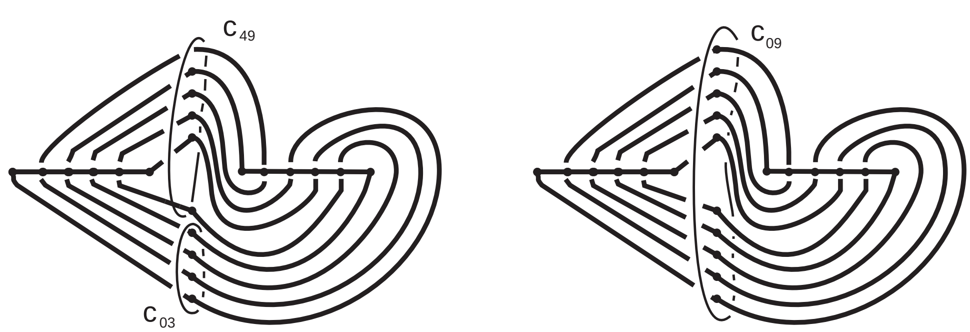


FIGURE 4. The curves $c_{03} = y_2, c_{49} = y_{p-2}$ and $c_{09} = y_p$ for $\mathfrak{b}(5,3)$

For $1 \le i \le \frac{(p-1)}{2}$, let $y_i = c_{0,2i-1}$ and for $\frac{(p+1)}{2} \le i \le p$ let $y_i = c_{2p-2i,2p-1}$. For example, y_2, $y_{p-2} = y_3$, and $y_p = y_5$ for the knot $\mathfrak{b}(5,3)$ are depicted in Figure 4. Let e_k, $k > 0$, be the curve that circles both C_0 and C_1 k times as in Figure 5 (the left part, with $k = 3$), and let y_{p+k} be the connected sum of y_p and e_k as in Figure 5.

One can see that each y_i is monic (see subsection 1.1.5), and hence for some $k_i \in \mathbb{Z}$

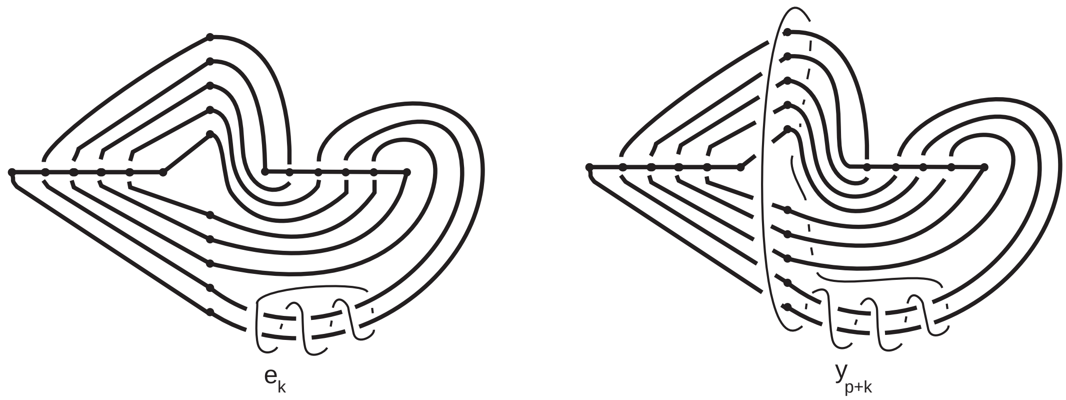


FIGURE 5. The curves e_k and y_{p+k} with $k = 3$ for $\mathfrak{b}(5, 3)$

$$y_i = \pm t^{k_i} y^i + (\text{terms of } y\text{-degree less than } i).$$

This means from $\{1, y, y^2, \dots\}$ one can go to $\{1, y_1, y_2, \dots\}$ by an upper-diagonal matrix with invertible entries on the diagonal, hence the set $\{1, y_1, y_2, \dots\}$ is also a basis of $\mathbb{C}[x, x', y]$ over $\mathbb{C}[x, x']$. This proves (i). It's clear from the construction that for $1 \le i \le p-1$ one can slide y_i into y_{p-i}. An example for the knot $\mathfrak{b}(5, 3)$ is in Figure 4 where one can see immediately that y_2 can slide into y_{p-2}. Thus we have (ii). It's also clear that one can slide y_{p+k} into e_k, which, by Lemma 1.1, is monic and has y-degree k, less than the y-degree of y_{p+k} which is $p + k$. This establishes (iii) and completes the proof of Theorem 2. □

Remark 4.1. The same proof shows that the theorem still holds true if we replace the ground ring $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$ by $\mathcal{R}_{\mathbb{Z}} := \mathbb{Z}[t^{\pm 1}]$.

Corollary 4.2. *For 2-bridge knots one has $\epsilon(\mathcal{S}(X)) = R(\chi(X))$, the ring of regular functions on the character variety.*

Proof. By the result of [Le1], the ring $R(\chi(X))$ is $\mathbb{C}[\bar{x}, \bar{y}]/(\varphi(\bar{x}, \bar{y}))$, where $\varphi(\bar{x}, \bar{y})$ is a polynomial of $\bar{y}$-degree $(p+1)/2$, with leading coefficient 1. Here $\bar{x}$, $\bar{y}$ are respectively the traces of the loop x, y. The corollary follows immediately. □

5. PROOF OF THEOREM 1

5.1. **More on the recurrence polynomial of knots.** The following lemma was known to Garoufalidis.

Lemma 5.1. *When reduced by $t = -1$, the recurrence polynomial α_K is divisible by $L - 1$. In other words,* $\dfrac{\epsilon(\alpha_K)}{L-1} \in \mathbb{Z}[M, L]$.

Proof. For a function $f(t^2, n)$ of two variables $t^2 \in \mathbb{C}$ and $n \in \mathbb{Z}$ let $\overline{f}(z)$ be the limit of $f(t^2, n)$ when

(†) $t^2 \to 1$ and t^{2n} is kept equal to z all the time.

The Melvin-Morton conjecture [MM], proved by Bar-Natan and Garoufalidis [BG], showed that $h(z) := \overline{J_K(n)}$ is the inverse of the Alexander polynomial. In particular, $h(z) \neq 0$.

It is easy to check that $\overline{J_K(n+k)} = \overline{J_K(n)}$ for any fixed k. Hence the operator L becomes the identity after taking the limit (†). Thus applying the limit (†) to the equation $\alpha_K J_K = 0$ we see that

$$\alpha_K|_{t^2=1,M=z,L=1}\, h(z) = 0.$$

Since $h(z) \neq 0$, one has $\alpha_K|_{t^2=1,M=z,L=1} = 0$, which is equivalent to the conclusion of the lemma. □

5.2. **The peripheral polynomial of 2-bridge knots.** Theorem 2 about the structure of the skein module of complements of 2-bridge knots is used to prove the following.

Proposition 5.2. *For the 2-bridge knot $K = \mathfrak{b}(p,m)$ the peripheral polynomial β_K is never 0 and has L-degree less than or equal to $(p+1)/2$.*

Proof. By definition one has the following exact sequence of $\mathcal{R}[x]$-modules

$$0 \to \mathcal{P} \hookrightarrow \mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}(X) \tag{2}$$

When $x = M + M^{-1}$, the field $\mathcal{R}(M)$ of rational functions in M is a flat $\mathcal{R}[x]$-module, since $\mathcal{R}(M)$ contains the fractional field of $\mathcal{R}[x]$ as a subfield. Hence the following sequence, which is obtained from (2) by tensoring with $\mathcal{R}(M)$, is exact

$$0 \to \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{P} \hookrightarrow \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{T}^\sigma \xrightarrow{id\otimes\Theta} \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{S}(X) \tag{3}$$

Note that the first module $\mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{P}$ is exactly $\tilde{\mathcal{P}}$, the left-ideal extension of $\mathcal{P}$ from $\mathcal{T}^\sigma$ to $\tilde{\mathcal{T}}$. It's easy to check that the second module $\mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{T}^\sigma$ is $\tilde{\mathcal{T}}$. One can now rewrite (3) as

$$0 \to \tilde{\mathcal{P}} \hookrightarrow \tilde{\mathcal{T}} \xrightarrow{id\otimes\Theta} \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{S}(X) \tag{4}$$

The third module is a finite-dimensional $\mathcal{R}(M)$-vector space; in fact, its basis is $\{y^i, 0 \le i \le \frac{p-1}{2}\}$, since $\mathcal{S}(X)$ is $\mathcal{R}[x]$-free with the same basis, by Theorem 2. The middle module $\tilde{\mathcal{T}}$ is an $\mathcal{R}(M)$-vector space of infinite dimension; in fact, its basis is $\{L^a, a \in \mathbb{Z}\}$. Thus the kernel $\tilde{\mathcal{P}}$ is never 0, and hence it generator β_K is not 0. More over the image of $(p+1)/2+1$ elements $1, L, L^2, \dots, L^{(p+1)/2}$ are linearly dependent. Hence there must be a non-trivial element in the kernel of L-degree less than or equal to $(p+1)/2$. □

Proposition 5.3. *Suppose for a knot K the skein module $\mathcal{S}(X)$ is free over $\mathcal{R}$ and $\epsilon(\mathcal{S}(X)) = R(\chi(X))$. Then $\epsilon(\mathcal{P}) = \mathfrak{p}$, and hence $\epsilon(\beta_K) \stackrel{M}{=} B_K$.*

Proof. Consider again the exact sequence (2), but now as sequence of modules over $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$, a principle ideal domain. By assumption, the last module $\mathcal{S}(X)$ is free over $\mathcal{R}$. Hence when tensoring (2) with any $\mathcal{R}$-module, one gets an exact sequence. In particular, tensoring with $\mathbb{C}$, considered as $\mathcal{R}$-module by putting $t = -1$, one has the exact sequence

$$0 \to \epsilon(\mathcal{P}) \hookrightarrow \epsilon(\mathcal{T}^\sigma) \xrightarrow{\epsilon(\Theta)} \epsilon(\mathcal{S}(X)).$$

Notice that $\epsilon(\mathcal{T}^\sigma) = \mathfrak{t}^\sigma$, $\epsilon(\mathcal{S}(X)) = R(\chi(X))$, and $\epsilon(\Theta) = \theta$. Thus $\mathfrak{p}$, being the kernel of θ, is equal to $\epsilon(\mathcal{P})$. The second statement follows from Proposition 3.2. □

From Theorem 2, Corollary 4.2 and Proposition 5.3 we get

Theorem 3. *Conjecture 2 holds true for 2-bridge knots:* $\epsilon(\mathcal{P}) = \mathfrak{p}$ *and* $\epsilon(\beta_K) = B_K$.

5.3. **Proof of Theorem 1.** (a) is Proposition 5.2.

(b) One has $\epsilon(\beta_K) \overset{M}{=} B_K$ by Theorem 3. Thus the algebraic set $\{\epsilon(\beta_K) = 0\} \overset{M}{=} \{B_K = 0\}$ is M-essentially equal to $\{A'_K = 0\}$, by Proposition 2.1. Applying ϵ to (1) we get

$$\epsilon(\beta_K) \overset{M}{=} \epsilon(\gamma)\,\epsilon(\alpha_K). \tag{5}$$

which means $\epsilon(\beta_K)$ is M-divisible by $\epsilon(\alpha_K)$. Hence $\{\epsilon(\alpha_K) = 0\}$ is M-essentially an algebraic subset of $\{\epsilon(\beta_K) = 0\} \overset{M}{=} \{A'_K = 0\}$.

(c) Suppose A_K has L-degree $(p-1)/2$. Then $A'_K = (L-1)A_K$ has L-degree $(p+1)/2$. By Proposition 5.2, β_K has L-degree less than or equal to $(p+1)/2$. But $\epsilon(\beta_K)$is M-essentially divisible by A'_K. Hence the L-degree of β_K must be exactly $(p+1)/2$, and also

$$\epsilon(\beta_K) \overset{M}{=} (L-1)A_K.$$

Combining with (5) we have

$$A_K \overset{M}{=} \epsilon(\gamma)\,\frac{\epsilon(\alpha_K)}{L-1}. \tag{6}$$

Recall that $\frac{\epsilon(\alpha_K)}{L-1}$ is a polynomial by Proposition 5.1. From (1) and the fact that the L-degree of α_K is bigger than 1 (Proposition 1.3) it follows that the L-degree of γ is less than $(p-1)/2$, which is the L-degree of A_K. Hence if A_K is irreducible, from (6) one must have $\epsilon(\gamma) \overset{M}{=} 1$ and $\frac{\epsilon(\alpha_K)}{L-1} \overset{M}{=} A_K$. This completes the proof of Theorem 1.

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