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Three-manifolds and the Temperley-Lieb algebra

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1 Introduction

An array of invariants for closed 3-manifolds and for links in 3-manifolds has been revealed by Witten [17] using the inspiration of quantum field theory. When the 3-manifold is the 3-sphere, these link invariants are essentially the Jones polynomial (or one of its generalisations) of the link, evaluated at various complex roots of unity. A proof of the existence of such invariants has been given by Reshetikhin and Turaev [14]. Building on Kirby's theorem [7] concerning the different ways of obtaining a 3-manifold via surgery on the 3-sphere, they use deep results from the theory of quantum groups and the representation theory of Lie algebras. This paper gives an alternative approach, based on only the general outline of their method. The result obtained here establishes those new invariants that, in other interpretations, correspond to the Lie group $SU(2)$. This proof of the invariants' existence, which also starts with Kirby's theorem, uses Kauffman's (easy) bracket invariant [5] of regular isotopy classes of planar link diagrams. The behaviour of this invariant at roots of unity is explored using the discipline of the Temperley-Lieb algebra [1, 6] (that is also used in statistical mechanics in the calculation of the partition function of the Potts model), and the results are blended with some of the elementary tricks of linear skein theory [11]. The results needed (and here proved) from the Temperley-Lieb algebra are implicit in work of Jones [4] which appeared before the advent of his Jones polynomial (see also [3]). Of course, the Kauffman bracket is but a clever reformulation of the Jones polynomial. The nature of the 3-manifold invariants is described in a fairly simple way, but calculations are by no means easy and will not be attempted here. Some of these calculations have been performed and discussed by Kirby and Melvin [8] using some of the few (sixteen) roots of unity at which the Jones polynomial can be expressed in terms of more classical invariants; they do at least show that the invariants are not trivial.

The paper is, apart from its use of Kirby's surgery theorem, intended to be entirely elementary and self-contained. Much of it is but an exercise in elementary

linear algebra. It is a pleasure to record gratitude to V. F. R. Jones for correspondence, to H. Wenzl for very helpful conversation, and to K. H. Ko and L. Smolinsky for their result [9] that confirmed the conviction that the general methods of this paper would work. An early version of this paper using their work has appeared elsewhere [13].

2 The invariant described

The statement of the main result of this paper will be in terms of the bracket polynomial invariant of Kauffman [5, 12]. The bracket is a function

$$\langle\ \rangle : \{\text{Diagrams in } \mathbb{R}^2 \cup \infty \text{ of unoriented links}\} \to \mathbb{Z}[A^{\pm 1}]$$

that is defined by three properties:

(i) $\langle \emptyset \rangle = 1$;

(ii) $\langle D \cup U \rangle = \delta \langle D \rangle$, where U is a component with no crossing at all and $\delta = -A^{-2} - A^{2}$;

(iii) $\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$, where this refers to three diagrams identical except where shown.

[The normalisation of (i) is not entirely standard.] It is very easy to show (see [5] or [12]) that $\langle D \rangle$ is a regular isotopy invariant (that is, it is unchanged by both the second and third type of Reidemeister move shown in Fig. 1) and that its interaction with the third type of move is described by the equation $\langle\ \rangle = -A^{3}\langle\ \rangle$. Further, if D_1 and D_2 are disjoint diagrams it is clear that $\langle D_1 \cup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$.

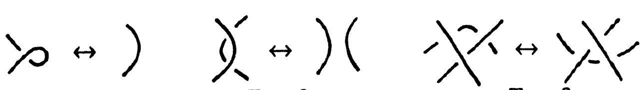


Fig. 1

If L is an oriented link in S^3 represented by the diagram D, let $w(D)$, the writhe of D, be the sum of the signs (± 1) of the crossings. For reference only, the Jones polynomial $V(L)$, usually expressed with variable $t = A^{-4}$, can be defined by the equation

$$\delta V(L) = (-A)^{-3w(D)} \langle D \rangle .$$

This is invariant under all three types of Reidemeister move and so is an invariant of the oriented link.

Various planar diagrams of links and parts of links will appear in what follows. A non-negative integer i beside a curve will signify the presence of i copies of that curve, all parallel *in the plane*. That is illustrated in Fig. 2.

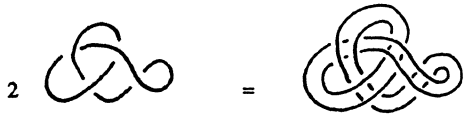


Fig. 2

Figure 3 shows a pair of diagrams (of the Hopf link of $i+j$ components) which are regularly isotopic and so have the same bracket polynomial; let T_{i+j} denote the bracket polynomial of either of these diagrams.

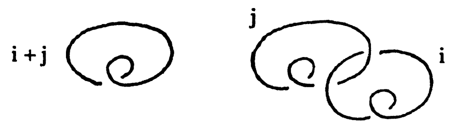


Fig. 3

The bracket takes values in the Laurent polynomial ring $\mathbb{Z}[A^{\pm 1}]$, but in what follows it will be evaluated when A is a specific root of unity, so that the bracket may be thought of as having complex number values.

It is well known [10] that any closed oriented 3-manifold M^3 can be obtained from the 3-sphere S^3 by surgery on a framed (unoriented) link (L, f). Thus, to each component L_s of L is assigned a "framing" which is an integer $f(s)$. M^3 can be constructed by the following process. Remove a small open solid torus neighbourhood of each L_s. On each resulting toral boundary component consider the simple closed curve that represents $f(s)$ meridians and one longitude of L_s; attach new solid tori so that each of these (framing) curves now bounds a disc. A link diagram D in $\mathbb{R}^2 \cup \infty$ will be said to *represent* (L, f) if D is a diagram for L in the usual sense, and, denoting by D_s the part of D corresponding to the component L_s, $w(D_s) = f(s)$ for each s. [Note that $w(D_s)$ is independent of any choice of orientation as D_s represents a single link-component.] Thus for each component the sum of the signs of the crossings encodes the framing of that component. Of course, any diagram for L can be modified, by inserting small "kinks" in its components, to represent (L, f). Framed links (L, f) and (L', f') in S^3 are *ambient isotopic* if there is an ambient isotopy (that is, a movement of the "strings" in 3-space) sending L to L' such that the framings on corresponding components are equal. The following useful result follows at once from a theorem of Trace [15].

Proposition 1. *Suppose that framed links (L, f) and (L', f') are represented by diagrams D and D'. Then (L, f) and (L', f') are ambient isotopic in S^3 if and only if D and D' are regularly isotopic in $\mathbb{R}^2 \cup \infty$.*

Hence the bracket polynomial of a diagram of (L, f) is an ambient isotopy invariant of the framed link (L, f).

It is convenient to explain some notation before stating the theorem. If D is a link diagram with $D_1, D_2, \ldots, D_n$ corresponding to the link's components, and c is a function, $c: \{1, 2, \ldots, n\} \to \mathbb{Z}_+$, let $c * D$ be the diagram in which each D_s has been replaced by $c(s)$ copies all parallel *in the plane* to D_s. Note that if D and D' are regularly isotopic then so are $c * D$ and $c * D'$. Usually c will be restricted to $C(n, r)$, the set of all functions $c: \{1, 2, \ldots, n\} \to \{0, 1, \ldots, r-2\}$. If the framed link (L, f) is given an orientation, the linking numbers of the pairs of its components form a symmetric matrix in which $f(s)$ is taken to the linking number of L_s with itself. The signature and nullity of this matrix are independent of the choice of orientations. The nullity of the matrix is, in fact, the first Betti number of the 3-manifold obtained by surgery along (L, f). Recall that T_{i+j} is the bracket of the diagram shown twice in Fig. 3.

The following theorem is, then, a version of part of the results of Witten [17] as interpreted by Reshetikhin and Turaev. For any primitive $4r^{\text{th}}$ root of unity it produces an invariant of 3-manifolds, a complex number that is, in fact, in the field over the rational numbers generated by the $4r^{\text{th}}$ roots of unity.

Theorem. *Let r be a (fixed) integer, $r \geqq 3$, and let $\delta = -A^{-2} - A^2$ where A is a primitive $4r^{th}$ root of unity.*

(i) *There is a unique solution $\lambda_0, \lambda_1, \ldots, \lambda_{r-2}$ in the complex numbers to the set of linear equations*

$$\sum_{i=0}^{r-2} \lambda_i T_{i+j} = \delta^j, \qquad j = 0, 1, \ldots, r-2.$$

(ii) *Suppose that M^3 is obtained from S^3 by surgery on an n-component framed link (L, f), for which σ and v are the signature and nullity of the linking matrix; suppose that (L, f) is represented by the diagram D. Then the expression*

$$\kappa^{\frac{\sigma+v-n}{2}} \sum_{c \in C(n,r)} \lambda_{c(1)} \lambda_{c(2)} \ldots \lambda_{c(n)} \langle c * D \rangle,$$

where $\kappa = \sum_{i=0}^{r-2} \lambda_i \bar{T}_i$, is an invariant of the 3-manifold, a complex number independent of the choice of (L, f) or of D.

Notice that, in calculating the invariant as described in this theorem, it is necessary to calculate $\langle c * D \rangle$. If $r = 6$ and D is just the 3-crossing knot, a diagram of four parallel copies of the knot has at least 48 crossings; naïve calculation of the bracket polynomial (direct from its definition) would then involve 2^{48} operations. Of course, for particular types of link, and also for special values of r, there are more subtle methods; nevertheless, calculation often poses considerable problems.

The theorem of Kirby [7] describes how framed links are related if they represent (via surgery) the same 3-manifold. That theorem, as refined by Fenn and Rourke [2], will now be interpreted by means of diagrams. It asserts that framed links correspond to the same 3-manifold if and only if any diagrams that represent them are related by regular isotopy and by the equivalence relation generated by moves of two kinds. In the first move diagram D is related to D' as shown in Fig. 4i; D' is obtained from D by inserting an extra unknotted component D'_{n+1}, with $w(D'_{n+1}) = 1$, and adding a positive twist in the strands linked by this component in the way depicted. In the second kind of move, shown in Fig. 4ii, D is related to D'' where D'' is D together with an extra component D''_{n+1} that is unknotted and disjoint from D, with $w(D''_{n+1}) = -1$. (The moves of [2] include a "negative" version of the first move; a proof of the above simplification, due to Turaev, appears in [13].) Employing this result, a 3-manifold invariant comes at once from any quantity associated to link diagrams in S^2 that is invariant under regular isotopy and under the two types of move described above. The proposed invariant of the theorem is not changed by regular isotopy as the bracket is invariant under regular isotopy. Thus, invariance under the two (Kirby) moves is all that has to be proved. This will be established in the final section of the paper. The proof will use a combination of linear skein theory [11] and some (possibly surprising) facts about the bracket polynomial when the variable A is a root of unity. These ideas will be developed in the next section.

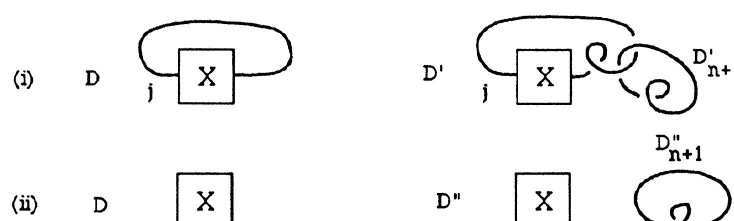


Fig. 4

3 The Temperley-Lieb algebra

A simple version of linear skein theory (see [11]) will now be described. Consider a square in $\mathbb{R}^2 \cup \infty$ with m specified points on its left edge and m such points on its right edge. Consider all tangle diagrams in the square (i.e. link diagrams in which components may be arcs) with the specified points as boundary. Figure 5 shows an example when $m=3$.

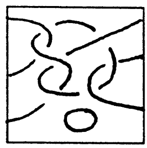


Fig. 5

Let V_m be the module over $\mathbb{Z}[A^{\pm 1}]$ freely generated by all such diagrams quotiented by relations of the form

(i) $D \cup U = \delta D$, where as usual $\delta = -A^{-2} - A^2$,

(ii) $(\times) = A(\asymp) + A^{-1}(\supset \subset)$.

As before, in (i) U is a closed component of the diagram that contains no crossing (one such component is in Fig. 5), and in (ii) the diagrams in parentheses are the same except where shown. Of course, equalities in this module are thought of as partial calculations of bracket polynomials, and regularly isotopic diagrams (keeping the boundary fixed) represent the same element of V_m. When a non-zero complex number is substituted for A, V_m becomes a vector space over the complex numbers. The placing of one diagram beside another, as in Fig. 6, produces at once a third diagram of the same nature.

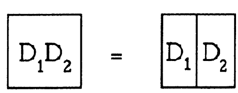
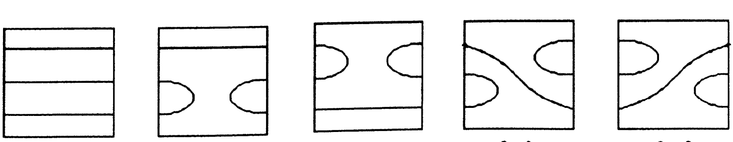


Fig. 6

This operation induces a bilinear map $V_m \times V_m \to V_m$ so that, with respect to this product, V_m becomes an algebra. Now as a vector space, V_m has a base consisting of all (elements represented by) diagrams in the square with no crossing and no closed component. Although it is not needed here, observe that the dimension d_m of V_m is the Catalan number $\frac{1}{m+1}\binom{2m}{m}$. That is so since $d_{m+1} = \sum_{i=0}^{m} d_i d_{m-i}$ and hence, if $\phi(z)$ denotes the generating function $\sum_{i=0}^{\infty} d_i z^i$, then $z\phi(z)^2 = \phi(z) - 1$. Thus $2z\phi(z) = 1 - \sqrt{1-4z}$ and the formula follows from the binomial expansion. For V_3 the base is shown in Fig. 7. V_0 has dimension one, the empty diagram, denoted 1_0, being a base; any link diagram in $\mathbb{R}^2$ can be regarded as being an element of V_0.

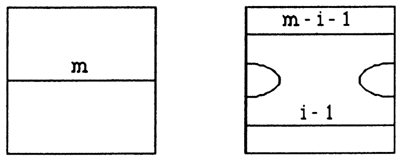
Fig. 7



1_3 e_1 e_2 e_1e_2 e_2e_1

As an *algebra* V_m is generated by the elements $1_m, e_1, e_2, \ldots, e_{m-1}$ that are shown in

Fig. 8



1_m e_i

Figure 8, so every element of V_m is expressible as a linear combination of products of these elements.

For $i, j = 1, 2, \ldots, m-1$, these generators satisfy the relations

$$1_m e_i = e_i = e_i 1_m,$$
$$e_i^2 = \delta e_i,$$
$$e_i e_j = e_j e_i \quad \text{if} \quad |i-j| \geqq 2,$$
$$e_i e_{i\pm 1} e_i = e_i \quad \text{provided } e_{i\pm 1} \text{ is defined}.$$

Up to scaling, V_m is the m^{th} *Temperley-Lieb algebra*, δ being a parameter in $\mathbb{C}$.

Following Wenzl [16] and Jones [4] (see also [3]) and guided also by Ko and Smolinsky [9], consider for each $n \geqq -1$, the polynomial function Δ_n of degree n in δ defined recursively by

$$\Delta_n = \delta \Delta_{n-1} - \Delta_{n-2}, \quad \Delta_0 = 1 \quad \text{and} \quad \Delta_{-1} = 0.$$

This Δ_n is, in fact, the n^{th} renormalised Chebyshev polynomial of the second kind. For each $n \leqq m-1$, provided A (and hence δ) is chosen in $\mathbb{C}$ so that $\Delta_1 \Delta_2 \ldots \Delta_n \neq 0$, define $f_n \in V_m$ by $f_0 = 1_m$ and

$$f_n = f_{n-1} - (\Delta_{n-1}/\Delta_n) f_{n-1} e_n f_{n-1}.$$

It follows easily, by induction, that $f_n - 1_m$ is in the (proper) subalgebra $\mathfrak{A}(e_1, e_2, \ldots, e_n)$ generated by $e_1, e_2, \ldots, e_n$. In particular, f_n commutes with each of $e_{n+2}, e_{n+3}, \ldots, e_{m-1}$.

Lemma 2 [16]. *Suppose δ is such that, for some $n \leqq m-1$, $\Delta_1 \Delta_2 \ldots \Delta_n \neq 0$. Then in V_m,*

(1_n) $$f_n^2 = f_n,$$

(2_n) $$e_i f_n = 0 \quad \textit{for all} \quad i \leqq n,$$

(3_n) $$(e_{n+1} f_n)^2 = (\Delta_{n+1}/\Delta_n) e_{n+1} f_n \quad \textit{provided} \quad n \leqq m-2.$$

Proof. The lemma is clearly true when $n = 0$. Inductively suppose it is true for a given n, and then suppose that $n+1 \leqq m-1$ and $\Delta_1 \Delta_2 \ldots \Delta_{n+1} \neq 0$. This means that $f_1, f_2, \ldots, f_{n+1}$ are all defined.

$$\begin{aligned}(1_{n+1}) f^{n+1} &= (f_n - (\Delta_n/\Delta_{n+1}) f_n e_{n+1} f_n)^2 \\ &= f_n - 2(\Delta_n/\Delta_{n+1}) f_n e_{n+1} f_n + (\Delta_n/\Delta_{n+1})^2 f_n e_{n+1} f_n e_{n+1} f_n \quad (\text{by } 1_n) \\ &= f_n - (\Delta_n/\Delta_{n+1}) f_n e_{n+1} f_n \quad (\text{using } 3_n) \\ &= f_{n+1}.\end{aligned}$$

(2_{n+1}) If $i \leqq n$, then

$$e_i f_{n+1} = e_i(f_n - (\Delta_n/\Delta_{n+1}) f_n e_{n+1} f_n) = 0 \quad (\text{from } 2_n),$$

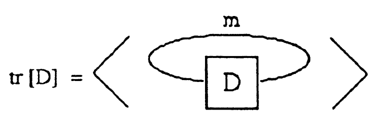
$$e_{n+1} f_{n+1} = e_{n+1} f_n - (\Delta_n/\Delta_{n+1})(e_{n+1} f_n)^2 = 0 \quad (\text{from } 3_n).$$

(3_{n+1}) Suppose that $n+1 \leqq m-2$. Recall that $e_{n+2} f_n = f_n e_{n+2}$ and note that (by 1_n) $f_n f_{n+1} = f_{n+1}$. Then

$$\begin{aligned}(e_{n+2} f_{n+1})^2 &= e_{n+2}\{f_n - (\Delta_n/\Delta_{n+1}) f_n e_{n+1} f_n\} e_{n+2} f_{n+1} \\ &= \delta e_{n+2} f_{n+1} - (\Delta_n/\Delta_{n+1}) f_n e_{n+2} e_{n+1} e_{n+2} f_{n+1} \\ &= (\delta - (\Delta_n/\Delta_{n+1})) e_{n+2} f_{n+1} \\ &= (\Delta_{n+2}/\Delta_{n+1}) e_{n+2} f_{n+1}, \quad \text{using the definition of } \Delta_n.\end{aligned}$$

In this context, the above lemma might be regarded as technical. Its usefulness will be seen in its interaction with the *Markov trace* on the Temperley-Lieb algebra V_m defined in the following simple way. If D is a tangle diagram in the square having m endpoints on each of the left and right edges of the square, D represents an element $[D]$ of V_m. Then $\operatorname{tr}[D]$ is the bracket polynomial, evaluated at the chosen value of A in $\mathbb{C}$, of the link diagram formed from D by joining the points on the left edge of the square to those on the right by arcs outside the square that introduce no new crossing. This idea (analogous to the closure of a braid) is illustrated in Fig. 9.

Fig. 9



This clearly induces a well-defined linear map on the vector space V_m, because the relations used to define V_m are essentially the formulae that characterise the bracket polynomial. It is then clear that this trace function has the following properties.

$$\operatorname{tr}: V_m \to \mathbb{C} \quad \text{is linear},$$

$$\operatorname{tr}(xy) = \operatorname{tr}(yx),$$

$$\operatorname{tr}(1_m) = \delta^m,$$

$$\delta \operatorname{tr}(x e_n) = \operatorname{tr}(x) \quad \text{if} \quad x \in \mathfrak{A}(1_m, e_1, e_2, \ldots, e_{n-1}).$$

The following lemma calculates the trace of the element f_n constructed above.

Lemma 3 [16]. *Suppose δ is such that, for some $n \leqq m-1$, $\Delta_1 \Delta_2 \ldots \Delta_n \neq 0$, and f_n is the element of V_m defined above. Then*

$$\operatorname{tr}(f_n) = \delta^{m-n-1} \Delta_{n+1}.$$

Proof.

$$\begin{aligned}\operatorname{tr}(f_n) &= \operatorname{tr}(f_{n-1}) - (\Delta_{n-1}/\Delta_n) \operatorname{tr}(f_{n-1} e_n f_{n-1}) \\ &= \operatorname{tr}(f_{n-1}) - (\Delta_{n-1}/\Delta_n) \operatorname{tr}(f_{n-1}^2 e_n) \\ &= \{1 - (\Delta_{n-1}/\delta \Delta_n)\} \operatorname{tr}(f_{n-1}), \quad \text{as} \quad f_{n-1} \in \mathfrak{A}(1_m, e_1, e_2, \ldots, e_{n-1}).\end{aligned}$$

So

$$\begin{aligned}\operatorname{tr}(f_n) &= (\Delta_{n+1}/\delta\Delta_n)\operatorname{tr}(f_{n-1})\\ &= (\Delta_{n+1}/\delta^{n+1})\operatorname{tr}(f_0) = \delta^{m-n-1}\Delta_{n+1}.\end{aligned}$$

It is now time to check up on when Δ_n is zero, so the following elementary lemma is included here for completeness (it is also proved and used in [9] and [3]).

Lemma 4. $\Delta_n = \prod_{k=1}^{n}\left(\delta - 2\cos\dfrac{k\pi}{n+1}\right).$

Proof. Recall that $\Delta_n = \delta\Delta_{n-1} - \Delta_{n-2}$, $\Delta_{-1} = 0$ and $\Delta_0 = 1$. Clearly Δ_n is a monic polynomial of degree n in δ. The substitution $\delta = x + x^{-1}$ makes Δ_n a rational function of x, and

$$\Delta_n - x\Delta_{n-1} = x^{-1}(\Delta_{n-1} - x\Delta_{n-2}).$$

Repeating this n times gives

$$\Delta_n - x\Delta_{n-1} = x^{-n}.$$

Using the symmetry between x and x^{-1}

$$\Delta_n - x^{-1}\Delta_{n-1} = x^n,$$

so that $(x - x^{-1})\Delta_n = x^{n+1} - x^{-(n+1)}$. Thus Δ_n is certainly zero when $x^{n+1} = x^{-(n+1)}$ but $x \neq x^{-1}$. This occurs when $x = e^{i\pi k/n+1}$, and hence when $\delta = 2\cos k\pi/n+1$, for $k = 1, 2, \ldots, n$. (Note that

$$\Delta_n(\delta) = \det\begin{pmatrix} \delta & 1 & & & & \\ 1 & \delta & 1 & & & \\ & 1 & \delta & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & \delta \end{pmatrix} \quad \text{and} \quad \Delta_n(-2\cos\theta) = \frac{(-1)^n \sin(n+1)\theta}{\sin\theta}.)$$

The above lemmas now combine to give the result needed in the linear skein theory applications. It will be important to consider $1'_m\colon V_m \to \mathbb{C}$ the element in the dual space of V_m dual to the base element 1_m; so that, in particular, $1'_m$ maps any diagram in the square to the coefficient of 1_m in its expansion as a linear sum of diagrams with no crossing and no loop. Also of key significance is the bilinear form

$$\langle\,,\rangle\colon V_m \times V_m \to \mathbb{C}$$

defined from the trace in the usual way by $\langle x, y\rangle = \operatorname{tr}(xy)$. This, in the case of diagrams in the square, can be thought of as the operation of placing one diagram inside a square, the other outside the square, and calculating the bracket polynomial of the resulting link diagram using the prescribed complex value for A.

Proposition 5. *Suppose that, for some $r \geq 2$, A is a primitive $4r^{th}$ root of unity.*
(i) *For $m \leq r-2$, let $\mathfrak{p}(m) = (\Delta_m)^{-1} f_{m-1} \in V_m$ and $\mathfrak{p}(0) = 1_0$. Then*

$$\langle\,,\mathfrak{p}(m)\rangle = 1'_m.$$

(ii) *If* $m \geqq r-1$ *let* $\mathfrak{q}(m)=(1_m-f_{r-2}) \in V_m$. *Then* $\mathfrak{q}(m)$ *is in the sub-algebra* $\mathfrak{A}(e_1, e_2, \ldots, e_{r-2})$ *of* V_m, *the bilinear form* $\langle\, , \rangle : V_m \times V_m \to \mathbb{C}$ *is degenerate, and* $\langle\, , \mathfrak{q}(m)\rangle = \langle\, , 1_m\rangle$.

Proof. First note that if $A^2=e^{i\theta}$, $\Delta_n = \prod_{k=1}^{n} -2\left(\cos\theta + \cos\frac{k\pi}{n+1}\right)$. So if $-\pi < \theta \leqq \pi$ Δ_n is zero if and only if $\theta = \pm k\pi/(n+1)$ for $k=n, n-1, \ldots, 1$. Thus, when A^2 is a primitive $2r^{\text{th}}$ root of unity $(r \geqq 2)$ $\Delta_1 \Delta_2 \ldots \Delta_{r-2} \neq 0$ but $\Delta_{r-1}=0$.

(i) For $1 \leqq m \leqq r-2$, consider the element f_{m-1} in V_m defined previously; the definition is valid because $\Delta_1 \Delta_2 \ldots \Delta_{m-1} \neq 0$. From Lemma 2, $e_i f_{m-1} = 0$ for $i=1,2,\ldots,m-1$. If b is any base element other than 1_m of V_m (qua vector space), b is some product of these e_i. Thus $\langle b, f_{m-1}\rangle = \operatorname{tr}(b f_{m-1}) = 0$. On the other hand,

$$\langle 1_m, f_{m-1}\rangle = \operatorname{tr}(1_m f_{m-1}) = \operatorname{tr}(f_{m-1}) = \Delta_m,$$

by Lemma 3, and this is not zero since $m \leqq r-2$. Hence $\mathfrak{p}(m)$ has the required property.

(ii) When $r-1 \leqq m$, $\Delta_1 \Delta_2 \ldots \Delta_{r-2} \neq 0$, so that the element f_{r-2} is defined in V_{r-1}. As before, if b is any base element other than 1_{r-1} of V_{r-1}, $\langle b, f_{r-2}\rangle = 0$ and $\langle 1_{r-1}, f_{r-2}\rangle = \Delta_{r-1} = 0$. Hence $\langle\, , f_{r-2}\rangle$ is the zero map $V_{r-1} \to \mathbb{C}$. If now $m > r-1$, there is a natural inclusion of V_{r-1} into V_m induced by taking a diagram representing an element of V_{r-1} and adding $m-r+1$ parallel horizontal arcs immediately above. In this context the e_i and f_i notations are unambiguous for $i \leqq r-2$. If then D is a diagram representing $[D]$ in V_m, let D' be the diagram shown in Fig. 10 representing the element $[D']$ in V_{r-1}. Then

$$\langle [D], f_{r-2}\rangle = \langle [D'], f_{r-2}\rangle = 0$$

(where the first use of $\langle\, , \rangle$ is in V_m, the second in V_{r-1}).

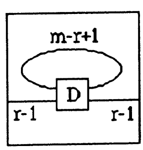


Fig. 10

Hence $\langle\, , f_{r-2}\rangle$ is the zero map $V_m \to \mathbb{C}$ so that, as f_{r-2} is non-zero, the bilinear form is degenerate. Hence $\mathfrak{q}(m) = (1_m - f_{r-2})$ has the required property.

4 The invariant established

There are two parts to the statement of the theorem. The first, concerning the existence of a unique solution to some linear equations, requires that a certain square matrix be shown to be non-singular. This will be proved in Proposition 8. A preparatory lemma now follows. Let x_m be the element of V_m shown at the beginning of Fig. 11.

Lemma 6. *If* $A^2 = e^{i\theta}$, *then, for all* $m \geqq 0$, $1'_m(x_m) = -2\cos(m+1)\theta$.

Proof. Consider the equalities in V_m depicted in Fig. 11.

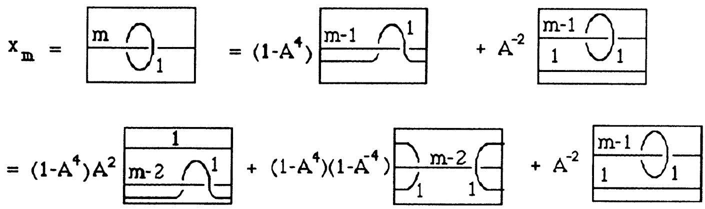


Fig. 11

Now $1'_m$ is clearly zero on the middle of the three terms in the last line of Fig. 11 because any further expansion will never achieve m arcs going from the left edge to the right. Thus there results the following recurrence relation (by repeating, on the first term of the last line of Fig. 11, the last expansion $m-2$ times)

$$1'_m(x_m)=(1-A^4)(A^2)^{m-1}+A^{-2}1'_{m-1}(x_{m-1}),$$

where $1'_0(x_0)$ is to be taken as δ. The solution to this recurrence relation is

$$1'_m(x_m)=-A^{2m+2}-A^{-2m-2}=-2\cos(m+1)\theta.$$

[Note that when $e^{i\theta}$ is a primitive $2r^{\text{th}}$ root of unity the $1'_m(x_m)$ are distinct for $m=0,1,\ldots,r-2$.]

Corollary 7. $1'_m(x^i_m)=(-2\cos(m+1)\theta)^i$.

Proof. This is immediate because an expansion of x^i_m as a linear sum of basis elements can be started by expanding each x_m factor. Only the 1_m-term in x_m can contribute to the 1_m-term in x^i_m.

Proposition 8. *When A is a primitive $4r^{th}$ root of unity the matrix $\{T_{i+j};0\leq i,j\leq r-2\}$ is non-singular.*

Proof. Recall that T_{i+j} is the bracket of either of the link diagrams of Fig. 3. Suppose the matrix $\{T_{i+j};\ 0\leq i,j\leq r-2\}$ is singular. Then there exist $\mu_i\in\mathbb{C}$, not all zero, such that

$$\sum_{i=0}^{r-2}\mu_i T_{i+j}=0 \quad \text{for all} \quad j=0,1,\ldots,r-2.$$

Consider the elements of V_j shown in Fig. 12.

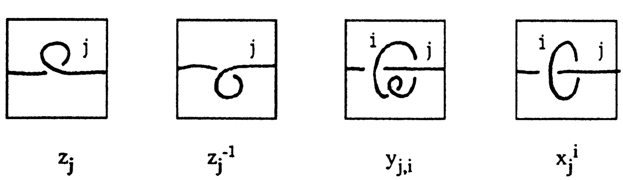


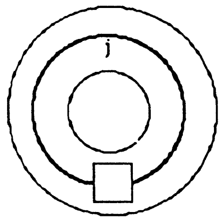
Fig. 12

Now, $T_{i+j}=\langle 1_j,\, z_j y_{j,i}\rangle$, so

$$\sum_{i=0}^{r-2}\mu_i\langle 1_j, z_j y_{j,i}\rangle=0 \quad \text{for all} \quad j=0,1,\ldots,r-2.$$

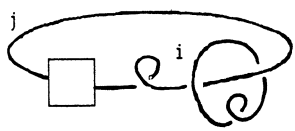
Suppose that b is any *base* element of V_j. Consider the annulus, shown in Fig. 13, containing a square with j parallel arcs joining the left and right sides of the square as shown.

Fig. 13



Inserting b into this square would produce a configuration of disjoint simple closed curves. These may be isotoped in the annulus to α, say, standard mutually parallel curves encircling the annulus and β small nul-homotopic curves (which may be nested). Note that $\alpha \leqq j$. But, $\langle b, z_j y_{j,i}\rangle$ is the bracket of the diagram obtained by inserting b into the square of Fig. 14.

Fig. 14



Then $\langle b, z_j y_{j,i}\rangle = \langle 1_\alpha, z_\alpha y_{\alpha,i}\rangle \delta^\beta$. This is because the isotopy in the above prototype annulus induces a regular isotopy in an immersed annulus (a neighbourhood of the j strands and the square) in the diagram consisting of Fig. 14 with b inserted; at the end of the regular isotopy there are α strands going around the immersed annulus and β small circles with no crossing. (This annulus trick will be used several times more.) Thus

$$\sum_{i=0}^{r-2} \mu_i \langle b, z_j y_{j,i}\rangle = \sum_{i=0}^{r-2} \mu_i \langle 1_\alpha, z_\alpha y_{\alpha,i}\rangle \delta^\beta = 0 .$$

This means that, for all $j=0,1,\dots,r-2$, $\sum_{i=0}^{r-2} \mu_i \langle\ , z_j y_{j,i}\rangle$ is the zero map $V_j \to \mathbb{C}$. But $z_j^{-1} z_j$ and 1_j have regularly isotopic representing diagrams so $z_j^{-1} z_j = 1_j$. Thus

$$0 = \sum_{i=0}^{r-2} \mu_i \langle z_j^{-1}, z_j y_{j,i}\rangle = \sum_{i=0}^{r-2} \mu_i \langle 1_j, y_{j,i}\rangle .$$

This means that $\{\langle 1_j, y_{j,i}\rangle;\ 0 \leqq i, j \leqq r-2\}$ is a singular matrix. But

$$\langle 1_j, y_{j,i}\rangle = \langle 1_i, z_i x_i^j\rangle$$

and so $\{\langle 1_j, z_j x_j^i\rangle;\ 0 \leqq i, j \leqq r-2\}$ is singular. Hence for some $v_i \in \mathbb{C}$, not all zero,

$$\sum_{i=0}^{r-2} v_i \langle 1_j, z_j x_j^i\rangle = 0 .$$

Exactly the same argument as before, using the annulus trick, shows that $\sum_{i=0}^{r-2} v_i \langle\ , z_j x_j^i\rangle$ is the zero map. Hence if a is any element of V_j,

$$0 = \sum_{i=0}^{r-2} v_i \langle a z_j^{-1}, z_j x_j^i\rangle = \sum_{i=0}^{r-2} v_i \langle a, x_j^i\rangle , \quad \text{for all} \quad j = 0,1,\dots,r-2 .$$

So far, no specific value of A has been used. If now A^2 is $e^{i\theta}$, a primitive $2r^{\text{th}}$ root of unity, take for a the element $\mathfrak{p}(j)$ defined in Proposition 5. Then

$$0=\sum_{i=0}^{r-2} v_i(1'_j(x^i_j))=\sum_{i=0}^{r-2} v_i(-2\cos(j+1)\theta)^i$$

for all $j=0,1,\ldots,r-2$, using the result of Corollary 7. Hence the matrix $\{(-2\cos(j+1)\theta)^i; 0\leqq i,j\leqq r-2\}$ is singular. However, as $e^{i\theta}$ is a primitive $2r^{\text{th}}$ root of unity, the $\{\cos(j+1)\theta,\ j=0,1,\ldots,r-2\}$ are all distinct. The matrix $\{(-2\cos(j+1)\theta)^i;\ 0\leqq i,j\leqq r-2\}$ is then a Vandermonde matrix with non-zero determinant

$$\prod_{\substack{0\leqq j,k\leqq r-2\\ j\neq k}} 2(\cos(j+1)\theta-\cos(k+1)\theta).$$

This, of course, contradicts the assertion that this matrix was singular.

Proposition 9. *Suppose that A is a primitive $4r^{\text{th}}$ root of unity ($r\geqq 2$), and that $\lambda_0,\lambda_1,\ldots,\lambda_{r-2}$ are such that*

$$\sum_{i=0}^{r-2}\lambda_i T_{i+j}=\delta^j,\qquad j=0,1,\ldots,r-2.$$

Then, for every $j\geqq 0$, $\sum_{i=0}^{r-2}\lambda_i\langle\ ,z_jy_{j,i}\rangle$ and $\langle\ ,1_j\rangle$ are equal maps $V_j\to\mathbb{C}$.

Proof. As $T_{i+j}=\langle 1_j,z_jy_{j,i}\rangle$, the choice of the λ_i means that for $j=1,2,\ldots,r-2$,

$$\sum_{i=0}^{r-2}\lambda_i\langle 1_j,z_jy_{j,i}\rangle=\delta^j=\langle 1_j,1_j\rangle.$$

Suppose that, for $j\leqq r-2$, b is any base element of V_j and that inserting b into the square in the annulus of Fig. 13 produces α curves encircling the annulus and β small nul-homotopic curves with $\alpha\leqq j$. Then the annulus trick mentioned in the proof of Proposition 8 shows that

$$\sum_{i=0}^{r-2}\lambda_i\langle b,z_jy_{j,i}\rangle=\sum_{i=0}^{r-2}\lambda_i\langle 1_\alpha,z_\alpha y_{\alpha,i}\rangle\delta^\beta=\delta^{\alpha+\beta}=\langle b,1_j\rangle.$$

Thus, $\sum_{i=0}^{r-2}\lambda_i\langle\ ,z_jy_{j,i}\rangle=\langle\ ,1_j\rangle$ as maps of V_j for $j\leqq r-2$.

The proposition is, then, true for $j\leqq r-2$. Suppose that $r-1\leqq m$ and suppose, inductively, that the proposition has been proved for all $j<m$. Let b be a base element of V_m *other than* 1_m. Once again, inserting b into the square in the annulus of Fig. 13 (where j is replaced by m) produces α curves encircling the annulus and β small nul-homotopic curves; here α is strictly less than m (because $b\neq 1_m$). Then, using the annulus trick yet again, and the inductive hypothesis,

$$\begin{aligned}\sum_{i=0}^{r-2}\lambda_i\langle b,z_my_{m,i}\rangle&=\sum_{i=0}^{r-2}\lambda_i\langle 1_\alpha,z_\alpha y_{\alpha,i}\rangle\delta^\beta=\langle 1_\alpha,1_\alpha\rangle\delta^\beta\\&=\delta^{\alpha+\beta}=\langle b,1_m\rangle.\end{aligned}$$

Now, as $r-1\leqq m$, the element $\mathfrak{q}(m)$ of Proposition 5 exists. It is in the sub-algebra $\mathfrak{A}(e_1,e_2,\ldots,e_{r-2})$ and so is a linear sum of base elements other than 1_m. Thus by taking that relevant linear sum, $\mathfrak{q}(m)$ can be substituted for b above. However, from

Proposition 5 $\langle \mathfrak{q}(m), \rangle = \langle 1_m, \rangle$. Hence $\sum_{i=0}^{r-2} \lambda_i \langle 1_m, z_m y_{m,i} \rangle = \langle 1_m, 1_m \rangle$ and so the linear maps $\sum_{i=0}^{r-2} \lambda_i \langle \, , z_m y_{m,i} \rangle$ and $\langle \, , 1_m \rangle$ agree on *all* the base elements of V_m and hence are equal on V_m. This completes the induction step and the proof of the proposition.

Proof of the Theorem. The existence and uniqueness of the λ_i have been proved in Proposition 8. Also it has been noted that the expression for the proposed invariant, given in the statement of the theorem, is indeed unchanged by ambient isotopy of the framed link (L, f). Suppose now that (L, f) and (L', f') are framed links with diagrams D and D' related as in Fig. 4i, L having n components. If $c \in C(n, r)$, let $c_i' \in C(n+1, r)$ be defined by $c_i'(s) = c(s)$ for $s \leqq n$, and $c_i'(n+1) = i$. From Proposition 9

$$\sum_{i=0}^{r-2} \lambda_i \langle c_i' * D' \rangle = \langle c * D \rangle .$$

Multiplying this by $\lambda_{c(1)} \lambda_{c(2)} \ldots \lambda_{c(n)}$ and adding gives

$$\sum_{c' \in C(n+1, r)} \lambda_{c'(1)} \lambda_{c'(2)} \ldots \lambda_{c'(n+1)} \langle c' * D' \rangle = \sum_{c \in C(n, r)} \lambda_{c(1)} \lambda_{c(2)} \ldots \lambda_{c(n)} \langle c * D \rangle .$$

Thus, if $X(D)$ denotes the expression on the right of this equation, $X(D') = X(D)$.

It remains to consider the other basic move on framed link diagrams, namely when D is changed to D'', a new framed link that is equal to D except for the insertion of an extra unknotted component, disjoint in the diagram from the original components, and with writhe -1 (see Fig. 4 ii). However, if diagrams D_1 and D_2 are disjoint in $\mathbb{R}^2 \cup \infty$, then $\langle D_1 \cup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$. Hence

$$X(D'') = X(D) \sum_{i=0}^{r-2} \lambda_i \bar{T}_i .$$

This is because the complex conjugate of T_{i+j} is the bracket of the reflection of the diagram of Fig. 3 (when A is a root unity). Thus $X(D'') = \kappa X(D)$. But the $n \times n$ linking matrix of L has signature σ and nullity ν, so $\frac{1}{2}(n - \sigma - \nu)$ is the number of negative entries in a diagonalisation of the matrix. That number is unchanged when (L, f) with diagram D is changed (as in the first move above) to (L', f') with diagram D' and increases by one if the change is to (L'', f'') with diagram D''. Hence $\kappa^{\frac{\sigma + \nu - n}{2}} X(D)$ is invariant under both types of move, and the proof is complete.

The above ideas generalise at once to give invariants of framed links in the 3-manifold M^3: Suppose that $\hat{L}$ is a framed m-component link in M^3 where, as before, M^3 is obtained by surgery on the framed n-component link (L, f) in S^3. Then $\hat{L}$ can be regarded as a link with framing $\hat{f}$ in $S^3 - L$. Suppose that the diagram $D \cup \hat{D}$ represents the framed link $(L \cup \hat{L}, f \cup \hat{f})$ and $\hat{c}: \{1, 2, \ldots, m\} \to \mathbb{Z}_+$. Then an invariant of the framed coloured link $(\hat{L}, \hat{f}, \hat{c})$ in M^3 is given by the expression

$$\kappa^{\frac{\sigma + \nu - n}{2}} \sum_{c \in C(n, r)} \lambda_{c(1)} \lambda_{c(2)} \ldots \lambda_{c(n)} \langle (c \cup \hat{c}) * (D \cup \hat{D}) \rangle$$

evaluated at a value of A that is a primitive $4r^{\text{th}}$ root of unity. Of course, if $\hat{c}$ has all values 1 and M^3 is S^3, this reduces to $\langle \hat{D} \rangle$.

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