A QUANTUM INTRODUCTION TO KNOT THEORY

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Abstract. This is an introduction to knot theory for non-specialists, focused on quantum invariants. I describe the homologies of covering spaces of a knot, the Alexander polynomial, the HOMFLY polynomial, the colored Jones polynomial, and the volume conjecture. There are essentially no new results in this article.

In this article I would like to introduce knot theory to non-specialists.

In Section 1 I describe how to construct cyclic covering spaces of a knot and show how to calculate their homologies. In Section 2 I define the Alexander polynomial and its relation to the homologies of cyclic coverings. I introduce the HOMFLY polynomial following [38] in Section 3. By using the Jones–Wenzl idempotent, I define the colored Jones polynomial in Section 4. Section 5 is devoted to the volume conjecture.

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1. Branched covering space of S^3 along a knot

For more detail about this and the next sections see [4, 29, 45].

Let K be a knot in the three-sphere S^3, that is, K is an embedded circle in S^3. Since the first homology group $H_1(S^3 \setminus K; \mathbb{Z})$ of the knot complement $S^3 \setminus K$ is $\mathbb{Z}$, there is the n-fold covering space $X_n(K)$ corresponding to the kernel of the following homomorphism:

$$\pi_1(S^3 \setminus K) \xrightarrow{\alpha} H_1(S^3 \setminus K)(\cong \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z},$$

where α is the abelianization map, and the second map is the reduction modulo n. We also allow the case where $n = \infty$ (in this case $\mathbb{Z}/n\mathbb{Z} := \mathbb{Z}$) and $X_\infty(K)$ is called the universal abelian covering space of K.

We can visualize $X_n(K)$ by using a surface bounding K. A compact, connected oriented surface F with $\partial F = K$ is called a *Seifert surface* of K, where ∂F is the boundary of F. See Figures 1 and 2 for the figure-eight knot and one of its Seifert surfaces. One can always construct a Seifert surface for any knot [10, 48]. For a proof see [45] for example. Let Y be the three-manifold obtained from S^3 by cutting along F, whose boundary is $F_+ \cup F_-$, where $F_\pm$ is a copy of F. We make n copies of Y and denote them by Y_i with boundary $F_{i,+} \cup F_{i,-}$ $(i = 1, 2, \dots, n)$. Then X_n is obtained from $\bigcup_{i=1}^n Y_i$ by identifying $F_{1,+}$ and $F_{2,-}$, $F_{2,+}$ and $F_{3,-}$, $\cdots$, $F_{n-1,+}$ and $F_{n,-}$, and $F_{n,+}$ and $F_{1,-}$. See Figures 3–4.

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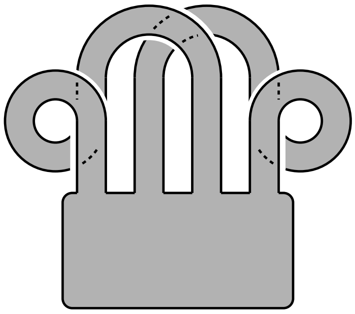


FIGURE 1. The figure-eight knot and its Seifert surface.

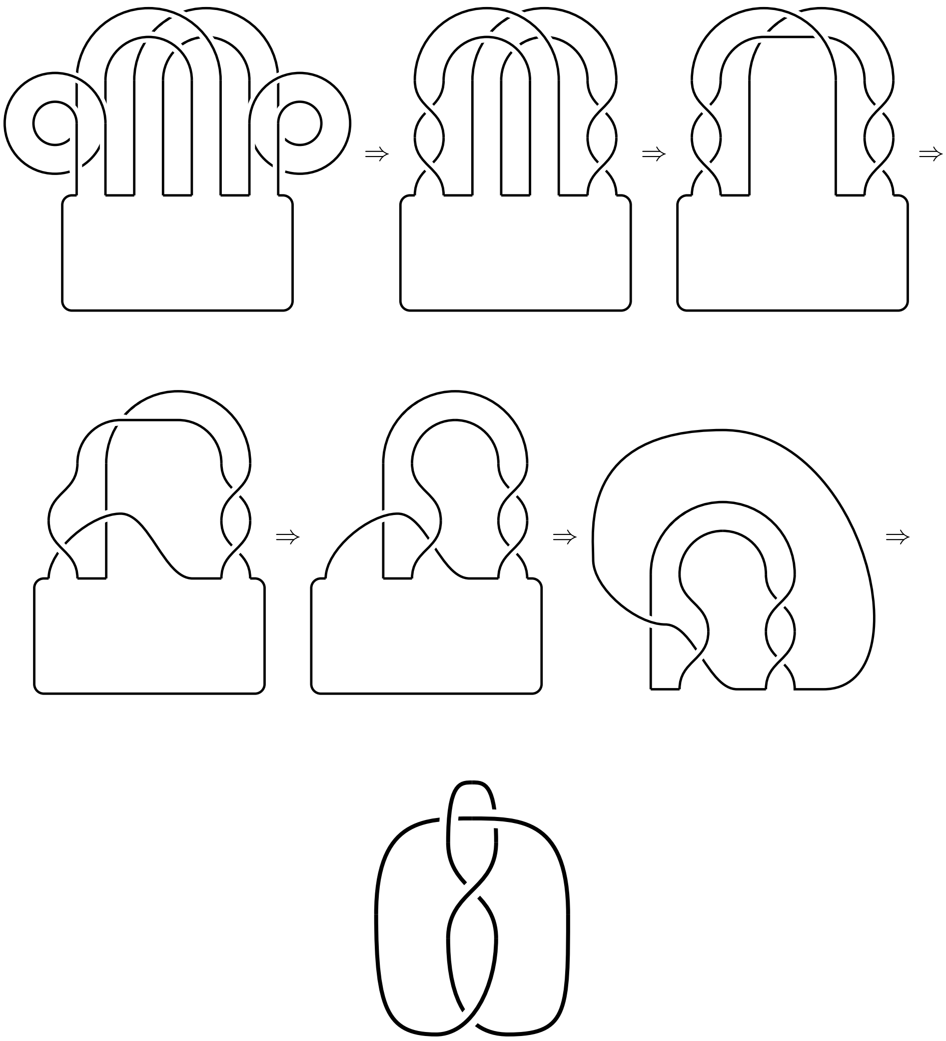


FIGURE 2. All of these represent the figure-eight knot.

Now suppose for simplicity that K is a fibered knot, that is, $S^3 \setminus K$ is a fiber bundle with the interior of an orientable surface $\operatorname{Int} F$ as the fiber and and with S^1 as the base space. Then Y is homeomorphic to the product $\operatorname{Int} F \times I$, where

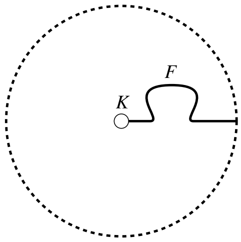


FIGURE 3. A knot K and its Seifert surface F.

I is the unit interval $[0,1]$, and $S^3 \setminus K$ can be reconstructed as $\operatorname{Int} F \times I/(x,0) \sim (f(x),1)$, the space obtained from $\operatorname{Int} F \times I$ by identifying $F \times 0$ with $F \times 1$ such that $(x,0)$ is identified with $(f(x),1)$, for a homeomorphism $f\colon F \to F$. The induced homomorphism $f_*\colon H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$ is called the monodromy map. So $X_n(K)$ is homeomorphic to $\operatorname{Int} F \times I/(x,0) \sim (f^n(x),1)$ and $H_1(X_n;\mathbb{Z}) = \operatorname{Coker}(A^n - E) \oplus \mathbb{Z}$, where A is a matrix presenting f and E is the identity matrix with the same size of A (Figure 4). (The summand $\mathbb{Z}$ comes from a cycle that goes around $X_n(K)$.) If we fill in $X_n(K)$ with a solid torus $D^2 \times S^1$, we have a closed three-manifold $\hat{X}_n(K)$ and $H_1(\hat{X}_n(K);\mathbb{Z}) = \operatorname{Coker}(A^n - E)$ which is called the n-fold cyclic branched covering space of S^3 branched along a knot (Figure 5).

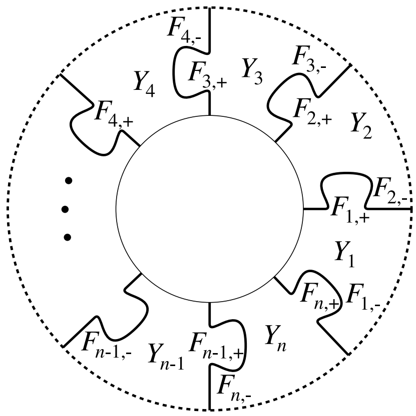


FIGURE 4. The n-fold cyclic (unbranched) covering space of a knot.

2. ALEXANDER POLYNOMIAL

Let $\Delta(K;t)$ be the characteristic polynomial of the monodromy matrix A, that is, $\Delta(K;t) := \det(A - t\,I)$. The polynomial $\Delta(K;t)$ is called the Alexander polynomial of K and is an invariant of knots up to $\pm t^k$ ($k \in \mathbb{Z}$) [2]. The order $|H_1(\hat{X}_n;\mathbb{Z})|$ can be determined by the Alexander polynomial.

Theorem 2.1 ([9]).

$$|H_1(\hat{X}_n(K);\mathbb{Z})| = \left| \prod_{k=1}^{n-1} \Delta\left(K; \exp\left(\frac{2\pi k\sqrt{-1}}{n}\right)\right) \right|,$$

where $|H_1(\hat{X}_n(K);\mathbb{Z})|$ *is the number of elements in* $H_1(\hat{X}_n(K);\mathbb{Z})$ *if it is a finite group and zero otherwise. Note that the right hand side is equal to the resultant* $|R(\Delta(K;t), t^n - 1)|$ *and can be calculated as* $\prod_{j=1}^{2g}({\xi_j}^n - 1)$ *with* ξ_j *zeroes of* $\Delta(K;t)$*, since* $\Delta(K;1) = 1$ *and* $\Delta(K;t)$ *is monic.*

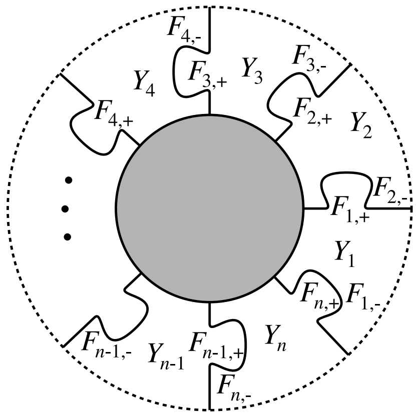


FIGURE 5. The n-fold cyclic branched covering space of a knot. (The shaded disk indicates $D2^t imesS^1$.)

Remark 2.2. For any knot K, the Alexander polynomial $\Delta(K;t)$ is also defined as the 'order' of the first homology group $H_1(\hat{X}_\infty(K);\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module, where t acts as the covering transformation. It is also true that $\Delta(K;1)=1$ for any knot, but in general $\Delta(K;t)$ may not be monic. See Remark 3.6

Example 2.3. Let 3_1 be the trefoil knot. (Knot theorists often use the symbol n_j that means the j-th knot with n minimal crossing in the knot table. For example the figure-eight knot is the unique knot with minimal crossing four, and so it is written as 4_1.) See Figure 6 for diagrams of 3_1 and its Seifert surface. It is well known that 3_1 is a fibered knot with the surface in the picture as a fiber. (See for example [45].) Let $\{a,b\}$ be a generator system of $H_1(F;\mathbb{Z})$ indicated in Figure 7. Let also $\{\alpha,\beta\}$ be its dual generator system of $H_1(S^3\setminus F;\mathbb{Z})$, that is, it is chosen so that $\operatorname{lk}(a,\alpha)=1$, $\operatorname{lk}(a,\beta)=0$, $\operatorname{lk}(b,\alpha)=0$, and $\operatorname{lk}(b,\beta)=1$, where $\operatorname{lk}(x,y)$ denotes the linking number in S^3. If we push off a in the positive direction (to this side of the paper), then we get a 1-cycle $a^+\in H_1(S^3\setminus F;\mathbb{Z})$. We also get $a^-\in H_1(S^3\setminus F;\mathbb{Z})$ pushing off a in the negative side. Similarly we get $b^\pm$. It is not hard to see

$$\begin{cases} a^+ &= -\alpha \\ a^- &= -\alpha+\beta \\ b^+ &= \alpha-\beta \\ b^- &= -\beta. \end{cases}$$

So the monodromy map sends $a\mapsto a^+=-\alpha=a^-+b^-\mapsto a+b$ and $b\mapsto b^+=\alpha-\beta=-a^-\mapsto -a$ it can be presented by the matrix $A=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$. Then its Alexander polynomial is

$$\Delta(3_1;t)=\begin{vmatrix}1-t & -1\\ 1 & -t\end{vmatrix}=t^2-t+1.$$

Therefore the order of the first homology of the branched covering space of 3_1 is

$$|H_1(\hat{X}_n(3_1);\mathbb{Z})|=\left|\left(\exp(\pi\sqrt{-1}/3)^n-1\right)\left(\exp(-\pi\sqrt{-1}/3)^n-1\right)\right|$$

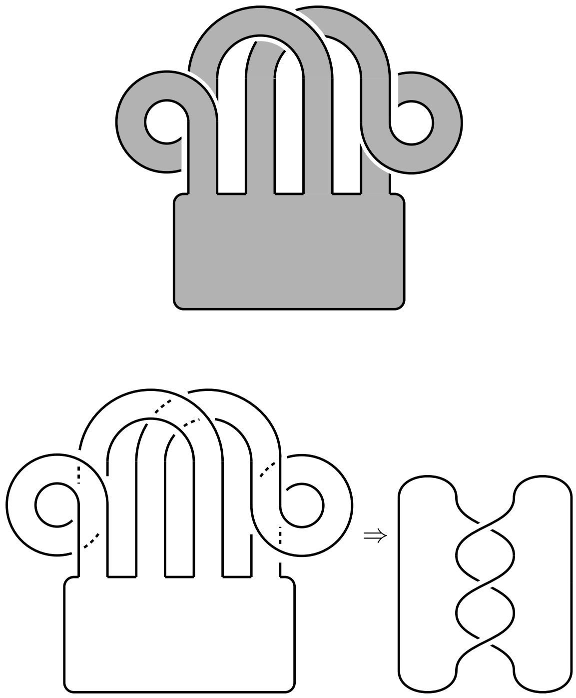


FIGURE 6. Diagrams of the right-hand trefoil (below) and its Seifert surface (above).

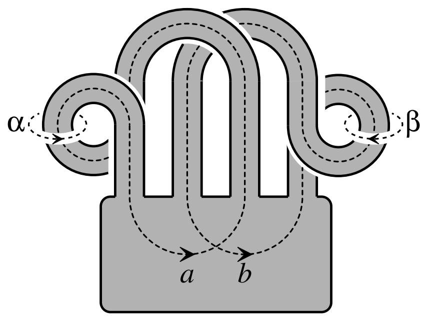


FIGURE 7. $\{a, b\}$ and $\{\alpha, \beta\}$ are generator systems of $H_1(F;\mathbb{Z})$ and $H_1(S^3 \setminus F, \mathbb{Z})$ respectively.

$$= \begin{cases} 0 & \text{if } n \equiv 0 \pmod 6 \\ 1 & \text{if } n \equiv 1 \pmod 6 \\ 3 & \text{if } n \equiv 2 \pmod 6 \\ 4 & \text{if } n \equiv 3 \pmod 6 \\ 3 & \text{if } n \equiv 4 \pmod 6 \\ 1 & \text{if } n \equiv 5 \pmod 6 \end{cases}.$$

In general, $H_1(\hat{X}_n(K)\mathbb{Z})$ is periodic with period k if and only if $\Delta(K;t)$ divides $t^k - 1$ [14].

Example 2.4. Next we consider the figure-eight knot 4_1. Denote by $\{a, b\}$ a generator system of $H_1(F;\mathbb{Z})$ and by $\{\alpha, \beta\}$ a generator system of $H_1(S^3 \setminus F;\mathbb{Z})$.

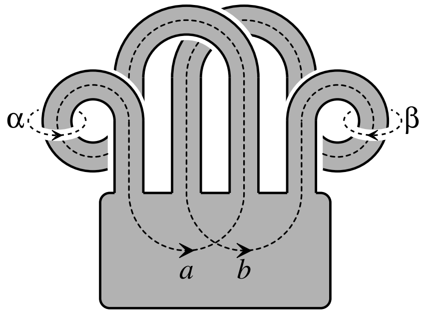


FIGURE 8. $\{a,b\}$ and $\{\alpha,\beta\}$ are generator systems of $H_1(F;\mathbb{Z})$ and $H_1(S^3\setminus F,\mathbb{Z})$ respectively.

In this case since

$$\begin{cases} a^+ &= -\alpha \\ a^- &= -\alpha+\beta \\ b^+ &= \alpha+\beta \\ b^- &= \beta, \end{cases}$$

we have $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $\Delta(4_1;t) = t^2-3t+1$. Since $\xi_1 := \frac{3+\sqrt{5}}{2}$ and $\xi_2 := \frac{3-\sqrt{5}}{2}$ are the roots of $t^2-3t+1=0$, we have

$$|H_1(\hat{X}_n(4_1);\mathbb{Z})| = |(\xi_1^n-1)(\xi_2^n-1)| = |2-\xi_1^n-\xi_2^n|.$$

Observe that

$$\lim_{n\to\infty}\frac{\log|H_1(\hat{X}_n;\mathbb{Z})|}{n} = \log\xi_1$$

since

$$\xi_1^n < |2-\xi_1^n-\xi_2^n| < 4\xi_1^n.$$

Similarly let $\xi_1,\xi_2,\dots,\xi_l,\xi_{l+1},\dots,\xi_{2g}$ be the eigenvalues of a monodromy matrix A of a knot K with $|\xi_i|>1$ for $i=1,2,\dots,l$. Assume that no roots of $\Delta(K;t)=0$ are on the unit circle in the complex plane. Then

$$|H_1(\hat{X}_nK;\mathbb{Z})| = \left|(\xi_1^n-1)(\xi_2^n-1)\cdots(\xi_l^n-1)\left(\xi_{l+1}^n-1\right)\cdots\left(\xi_{2g}^n-1\right)\right|$$

and so

$$\lim_{n\to\infty}\frac{\log|H_1(X_n(K);\mathbb{Z})|}{n} = \sum_{i=1}^{l}\log|\xi_i|$$

since $\prod_{i=1}^{l}|\xi_i|$ is the (unique) biggest term in the expansion of $\left|(\xi_1^n-1)\cdots(\xi_{2g}^n-1)\right|$. See [14] for the case where $\Delta(K;t)$ has zeroes on the unit circle and some of them are not roots of unity.

In general D. Silver and S. Williams proved

Theorem 2.5 ([52])**.** *Let $\hat{X}_n(K)$ be the n-fold branched cyclic cover of a knot K. Then*

$$\lim_{n\to\infty}\frac{\log|\operatorname{Tor}H_1(\hat{X}_n;\mathbb{Z})|}{n} = \log\mathbf{M}(\Delta(K;t)),$$

where $\operatorname{Tor}G$ is the torsion part of a group G, and $\log\mathbf{M}(f(t))$ is the logarithmic Mahler measure of a polynomial $f(t)$:

$$\log\mathbf{M}(f(t)) := \int_{S^1}\log|f(t)|\,dt = \int_0^1\log|f(\exp(2\pi\sqrt{-1}x))|\,dx.$$

Remark 2.6. See also [44, 13, 51].

Remark 2.7. It is well known that

$$\mathbf{M}(f(t)) = |\text{the leading coefficient of } f| \times \prod |\text{zeroes of } f(t) \text{ whose absolute value is bigger than one}|$$

3. HOMFLY polynomial

In this section we follow [38] to define the HOMFLY polynomial for links.

It is well known that the Alexander polynomial for links satisfies the following recursive formula:

$$\Delta(L_+;t) - \Delta(L_-;t) = (t^{1/2} - t^{-1/2})\Delta(L_0;t) \tag{A1}$$

with the initial condition

$$\Delta(O;t) = 1, \tag{A0}$$

where L_+, L_-, L_0 are three links (several circles in $\mathbb{R}^3$) that differ only in a small ball where they are as in Figure 9, and O is the unknot, that is, the knot bounding a disk. The relation (A1) is often called the skein relation and the triple (L_+, L_-, L_0)

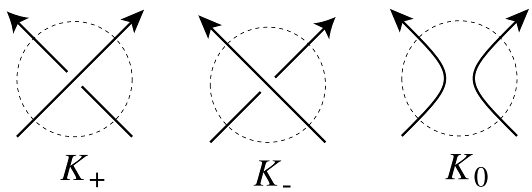


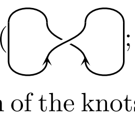
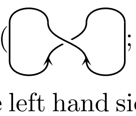
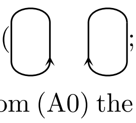
Figure 9. A skein triple L_+, L_-, and L_0.

is called a skein triple. This recursive formula of the Alexander polynomial was introduced by J. H. Conway [6]. The proof of the well-definedness was proved by L. H. Kauffman [26]. See also [12, 17]. Note that here we normalize $\Delta(L;t)$ so that $\Delta(L;t^{-1}) = \Delta(L;t)$ for any link L and that it is defined without ambiguity (not up to $\pm t^k$).

Remark 3.1. Here is a subtle remark. Our skein relation (A1) *differs* from the Alexander polynomial defined by using Seifert surfaces by $(-1)^{\sharp(L)-1}$, where $\sharp(L)$ is the number of components in L.

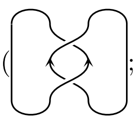
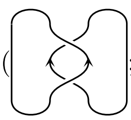
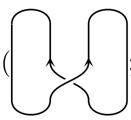
We can use skein relations repeatedly to calculate $\Delta(L;t)$ for any oriented link L as follows.

Example 3.2. First let us consider the following skein relation.

$\Delta($$;t) - \Delta($$;t) = (t^{1/2} - t^{-1/2})\Delta($$;t).$

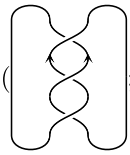
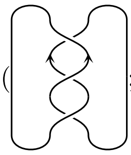
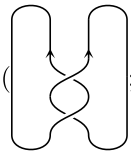
Since both of the knots in the left hand side are the unknot, from (A0) the Alexander polynomial of the trivial two-component link, which is depicted in the right hand side, is 0.

Example 3.3. Next consider the following skein relation.

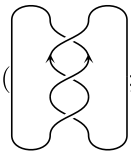
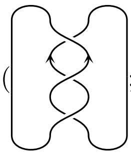
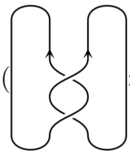
$\Delta($$;t) - \Delta($$;t) = (t^{1/2} - t^{-1/2})\Delta($$;t).$

The second knot in the left hand side presents the two-component trivial link and its Alexander polynomial vanishes. Therefore the Alexander polynomial of the positive Hopf link, the first link in the left hand side, is $t^{1/2} - t^{-/2}$ since the knot in the right hand side is the unknot whose Alexander polynomial is 1 from (A0).

Example 3.4. Now we calculate the Alexander polynomial of the trefoil again. In the skein relation

$\Delta($  $;t) - \Delta($  $;t) = (t^{1/2} - t^{-1/2})\Delta($  $;t),$

the second knot in the left hand side is the unknot and the figure in the right hand side presents the positive Hopf link. Therefore from Examples 3.2 and 3.3, we have

$\Delta($  $;t) = \Delta($  $;t) + (t^{1/2} - t^{-1/2})\Delta($  $;t)$

$$= 1 + (t^{1/2} - t^{-1/2})^2 = t - 1 + t^{-1},$$

which coincides with the Alexander polynomial calculated in the previous section up to a unit in $\mathbb{Z}[t, t^{-1}]$.

Remark 3.5. One can prove combinatorially that the Alexander polynomial is an invariant for links by only using the recursive relations (A1) and (A0). If one does it, one also obtains a generalization of $\Delta(L;t)$, which will be described below.

Remark 3.6. It is an easy exercise to calculate that the Alexander polynomial of the knot 5_2 shown in Figure 10 is $2t - 3 + 2t^{-1}$. Therefore 5_2 is not a fibered knot since the leading coefficient of the Alexander polynomial of any fibered knot, which equals the determinant of the monodromy matrix, is ± 1. See Remark 2.2.

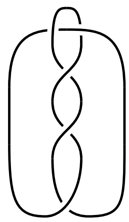


FIGURE 10. The knot 5_2, the second knot with five crossing.

We can generalize the skein relation (A1) to define the HOMFLY (or HOMFLYpt) polynomial $P_n(K;t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$. The HOMFLY polynomial is defined by using the following two axioms.

$$t^{n/2} P_n(L_+;t) - t^{-n/2} P_n(L_-;t) = (t^{1/2} - t^{-1/2}) P_n(L_0;t) \tag{P1}$$

with the initial condition

$$P_n(O;t) = 1, \tag{P0}$$

where L_+, L_-, and L_0 are the skein triple.

The HOMFLY polynomial was defined in [11, 40], generalizing both the Alexander polynomial and the Jones polynomial [20, 21]. Note that $n = 0$ and $n = 2$ give the Alexander polynomial and the Jones polynomial respectively.

Observe that when $n = 1$, $P_n(L) = 1$ for any link L.

As noted in Remark 3.5, one can prove the well-definedness of $P_n(L; t)$ combinatorially by using the skein relation. But I will prove it by defining P_n directly and constructively.

To do that I will first introduce local moves on link diagrams [41, 42]. The Reidemeister move I removes/creates a small kink as described in Figure 11. The

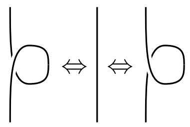


FIGURE 11. Reidemeister move I.

Reidemeister move II changes a pair parallel strands into two strands one of which crosses over the other and vice versa (Figure 12). Figure 13 shows the Reidemeister

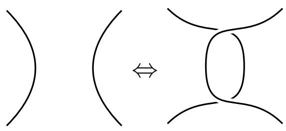


FIGURE 12. Reidemeister move II.

move III, where a strand crosses over a crossing.

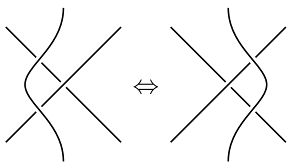


FIGURE 13. Reidemeister move III.

Theorem 3.7 (Reidemeister's Theorem)**.** *Two link diagrams define the same link if and only if one of them can be transformed into the other by a finite sequence of Reidemeister moves I, II, III.*

Proof. For a proof see [4]. (One may need a piecewise-linear topology textbook, for example [46], to understand fully.) Here is a rough sketch.

First we replace links with polygons. Then equivalent polygonal links can be transformed to each other by a composition of *triangle moves*, that is, one replaces a segment of a polygonal link with two sides of a triangle, or vice versa, where the triangle does not intersect other part of the polygonal link. Now observe that the projection of such a triangle can be divided into small triangles so that in each triangle there is either, an arc with both ends on edges of the triangle (Reidemeister move II; see Figure 15), a crossing (Reidemeister move III; see Figure 16), or an arc with an end on a vertex of the triangle (Reidemeister move I; see Figure 14). □

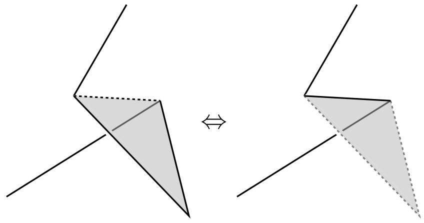


FIGURE 14. Triangle move corresponding to Reidemeister move I.

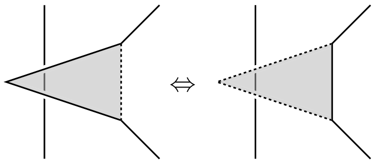


FIGURE 15. Triangle move corresponding to Reidemeister move II.

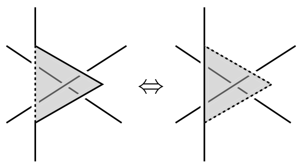


FIGURE 16. Triangle move corresponding to Reidemeister move III.

Now I will explain how to define P_n.

Let n be a positive integer greater than one. Put $\mathcal{N} := \{-(n-1)/2, -(n-3)/2, \dots, (n-3)/2, (n-1)/2\}$. (Note that $\mathcal{N}$ consists of integers if n is odd and half-integers if n is even.) Let G be an oriented trivalent plane graph such that each vertex looks like Figure 17. A *flow* on such a graph is an assignment of a positive

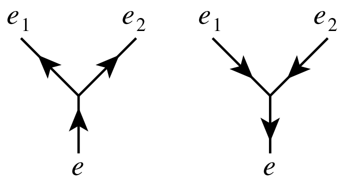


FIGURE 17. Two types of trivalent vertices.

integer less than or equal to n to each oriented edge such that at each vertex the sum of the integers coming in (going out, respectively) is equal to the integer going out (coming in, respectively) as in Figure 18 with $n \geq 5$. See Figure 19 for an example of a graph with flow. For a plane graph with flow f, a *state* σ is an assignment of a subset $\sigma(e) \subset \mathcal{N}$ to each edge e with flow $f(e)$ such that $\sharp\sigma(e) = f(e)$, where $\sharp\sigma(e)$ is the cardinality of $\sigma(e)$. We also require that at each vertex with edges e_1, e_2, and e, $\sigma(e_1) \cap \sigma(e_2) = \emptyset$ and that $\sigma(e_1) \cup \sigma(e_2) = \sigma(e)$, where e_1, e_2 and e are edges in the left, in the right and in the bottom respectively (see Figure 17). See Figure 20 for an example of a state on the graph in Figure 19.

Given a flow f and a state σ on a trivalent plane graph, we define the *weight* $\mathrm{wt}(v;\sigma)$ at a vertex v to be

$$\mathrm{wt}(v;\sigma) := t^{f(e_1)f(e_2)/4 - \pi(\sigma(e_1),\sigma(e_2))/2},$$

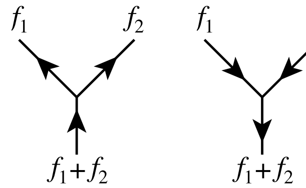


FIGURE 18. Graphs with flows.

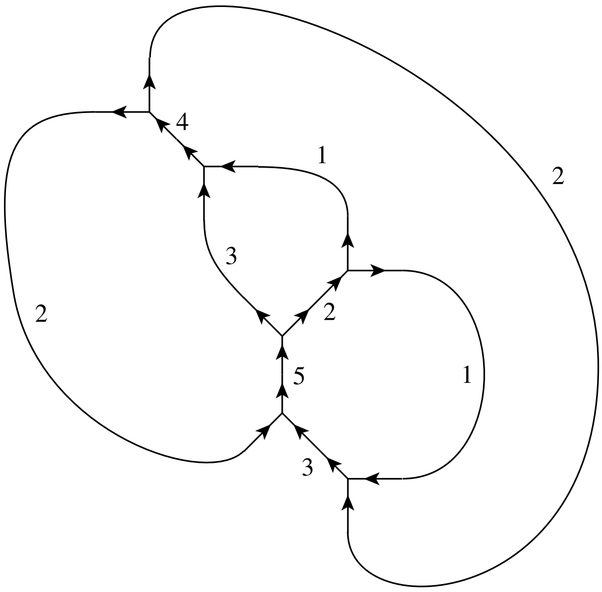


FIGURE 19. A trivalent graph with flow.

where t is an indeterminate, e_1 (e_2, respectively) is the edge in the left (right, respectively) in the Figure 17, and

$$\pi(A_1, A_2) := \sharp\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\}.$$

For example for the upper left vertex of Figure 20, the weight is

$$t^{2\times 2/4-\pi(\{0,2\},\{-2,1\})/2} = t^{1-3/2} = t^{-1/2}.$$

Given a flow f and a state σ, we can decompose each edge e into $f(e)$ parallel segments and connect them to make circles so that each circle carries the same element in $\mathcal{N}$. We define the *rotation number*

$$\operatorname{rot}(\sigma) := \sum_C \sigma(C) \operatorname{rot}(C),$$

where C runs over all the circles defined above, and $\operatorname{rot}(C)$ for an oriented circle C is 1 if it is oriented counterclockwise and -1 otherwise. See Figure 21 how to calculate $\operatorname{rot}(\sigma)$ for the graph with state in Figure 20.

For an oriented, trivalent, plane graph G with flow, we define

$$\langle G \rangle_n := \sum_\sigma \left\{ \prod_v \operatorname{wt}(v;\sigma) \right\} t^{\operatorname{rot}(\sigma)}.$$

We have the following local relations for $\langle G \rangle_n$.

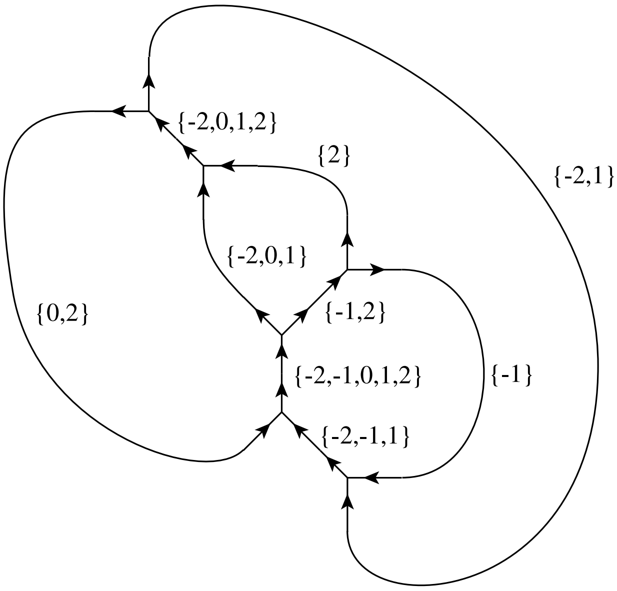


FIGURE 20. A state on the graph in Figure 19

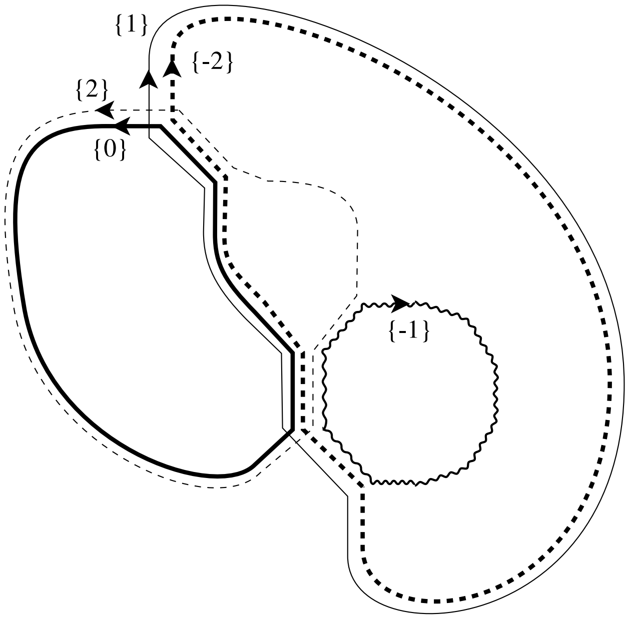
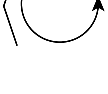
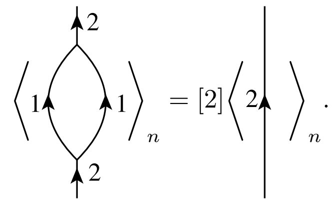
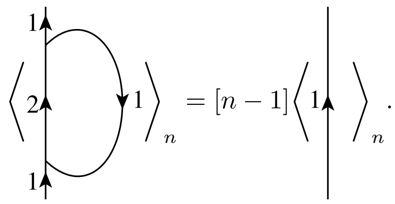


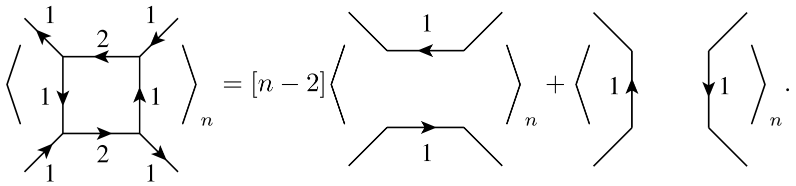
FIGURE 21. The thick circle carries 0, the thin circle carries 1, the thick broken circle carries -2, the thin broken circle carries 2, and the wavy circle carries -1. Therefore the rotation number is $0 \times 1 + 1 \times (-1) + (-2) \times (-2) + 2 \times 1 + (-1) \times (-1) = 6$.

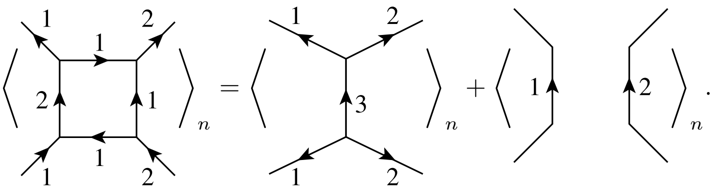
Proposition 3.8 ([38]). *Put* $[k] := (t^{k/2} - t^{-k/2})/(t^{1/2} - t^{-1/2})$. *Then the following formulas hold, where in each formula the diagrams outside the angle brackets are the same.*

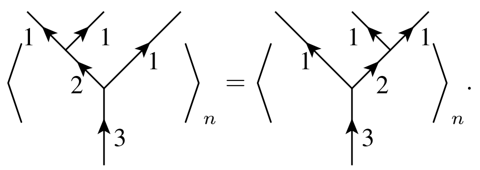
(3.1) $\Big\langle$  $\Big\rangle_n = [n] \Big\langle \quad \emptyset \quad \Big\rangle_n .$

(3.2) 

(3.3) 

(3.4) 

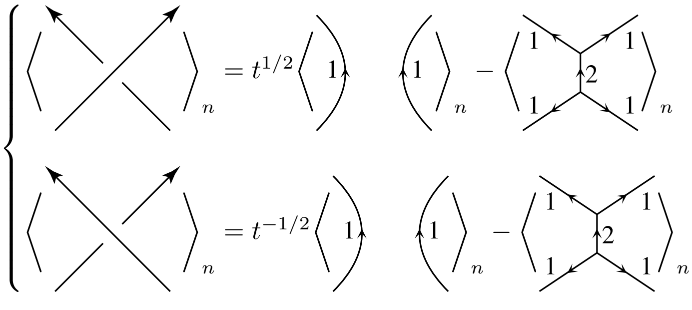
(3.5) 

(3.6) 

Remark 3.9. If we put $t = 1$, then $\langle G \rangle_n$ is equal to the number of colorings of edges with n colors so that each edge e carries $f(e)$ different colors. In this case it is easy to prove the theorem since $[k] = k$ when $t = 1$.

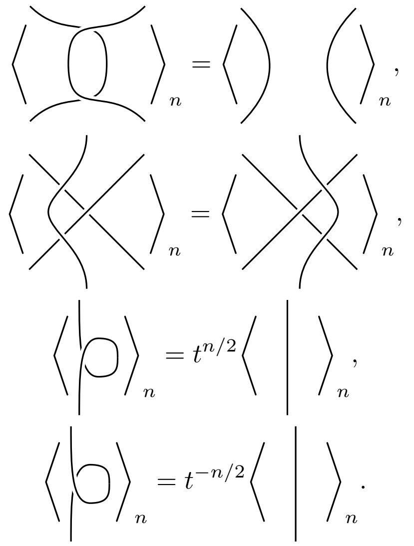
Remark 3.10. In a graph with flow, an edge with flow 1 corresponds to the fundamental (n-dimensional) representation V of the Lie algebra $\mathfrak{sl}_n$. An edge with flow k corresponds to its k-fold exterior power (antisymmetric power) $\wedge^k V$. A vertex in the left hand side of Figure 18 corresponds to the inclusion $\wedge^{f_1+f_2} V \to (\wedge^{f_1} V) \wedge (\wedge^{f_2} V)$ and that in the right hand side corresponds to the projection $(\wedge^{f_1} V) \wedge (\wedge^{f_2} V) \to \wedge^{f_1+f_2} V$.

Now we define $\langle D \rangle_n$ for an oriented link diagram D by



Then we can rather easily prove the following formulas.

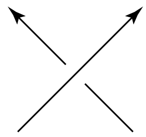
Proposition 3.11 ([38]). *The following formulas hold for any orientations of strings.*



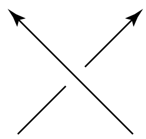
As a corollary if we define

$$\tilde{P}_n(D) := t^{-nw(D)/2}\langle D\rangle_n$$

for an oriented link diagram D, where $w(D)$ the *writhe* of D, that is, the sum of the signs of the crossings in D with



: +1,



: −1,

then $\tilde{P}_n$ is a link invariant. So we can denote $\tilde{P}_n(D)$ by $\tilde{P}_n(L;t)$ if D represents a link L. Note that $\tilde{P}_n(O;t) = [n]$ and so if we normalize $\tilde{P}_n$ to define $P_n(L;t) := \tilde{P}_n(L;t)/[n]$, then $P_n(L;t)$ satisfies the axioms of the HOMFLY polynomial.

It can be easily proved that P_n is a Laurent polynomial in t for knots (and for links with odd number of components), and Laurent polynomial in $t^{1/2}$ for links with even number of components. Note that it is not clear from the graphical definition of P_n that it is a Laurent polynomial in t.

Remark 3.12. One can calculate $\langle G\rangle_n$ for any plane graph G by only using Equations (3.1)–(3.6). But I do not know whether one can prove the well-definedness by only using these axioms.

Remark 3.13. For $n = 2$, we get a version of the Kauffman bracket [27].

Since there is a unique state for an edge with flow 2, we can ignore edges with flow 2. Then a graph is decomposed into circles consisting of segments with arrows. Taking into account the weights, this gives the correspondence between our invariant for $n = 2$ and the Kauffman bracket.

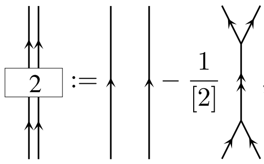
Remark 3.14. For $n = 3$, G. Kuperberg gave a definition via Equations (3.1)– (3.6) [28]. Note that in this case there is a duality changing an edge with flow 2 and an edge with flow 1 with opposite edge. Therefore Equations (3.2) and (3.3) become the same, and so do Equations (3.4) and (3.5). Equation (3.6) becomes trivial since one can ignore an edge with flow 3.

4. Colored Jones polynomial

In this section I will describe another generalization of the Jones polynomial, $P_2(L;t)$ in our notation.

Since $P_2(L;t)$ corresponds to the two-dimensional standard representation V of $\mathfrak{sl}_2$, one can expect a link invariant corresponding to the d-dimensional representation, which can be defined as the $(d-1)$-fold symmetric tensor. It is in fact called the colored Jones polynomial and one can define it graphically as follows.

First let us consider the following graphical object defined by

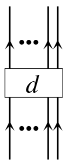


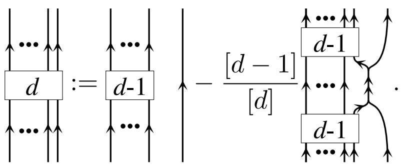
where in the right hand side a segment with single arrow means an edge of flow 1 corresponding to V and a segment with double arrow means an edge of flow 2 corresponding to the antisymmetric tensor product $V \wedge V$ as described in the previous section.

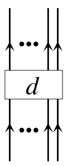
If $t = 1$ then $[2] = 2$ and the meaning of the equation is as follows. The second figure in the right hand side means the composition of the homomorphisms $V \otimes V \to V \wedge V \to V \otimes V$ defined by $a \otimes b \mapsto a \wedge b \mapsto a \otimes b - b \otimes a$. Therefore if we go up from the bottom to the top in the left hand side, we have a homomorphism $V \otimes V \to V \otimes V$ defined by the right hand side, that is:

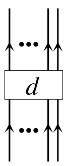
$$(a, b) \mapsto a \otimes b - \frac{1}{2}(a \otimes b - b \otimes a) = \frac{1}{2}(a \otimes b + b \otimes a),$$

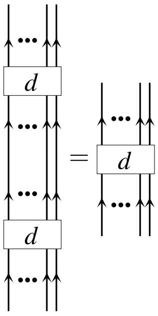
which is nothing but the definition of the symmetric tensor product.

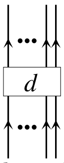
In general we define  by



It can be also proved that  corresponds to the d-fold symmetric tensor of V if

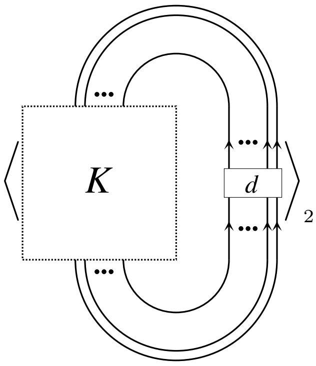
we put $t = 1$. Note that its dimension is $d - 1$. Therefore  can be regarded as a deformation or quantization of d-fold symmetric tensor of V. It is not hard to prove that



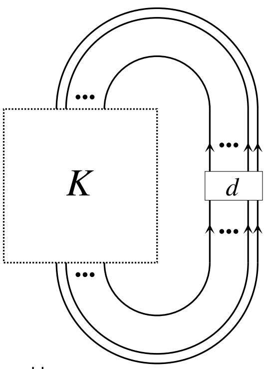
and 

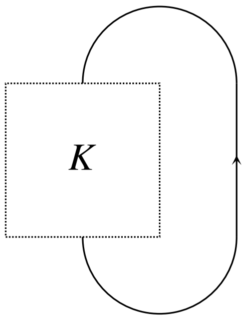
is often called the Jones–Wenzl idempotent [56].

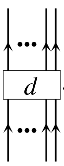
The colored Jones polynomial $J_n(K;t)$ of dimension $d+1$ is defined by



$/[d+1]$,

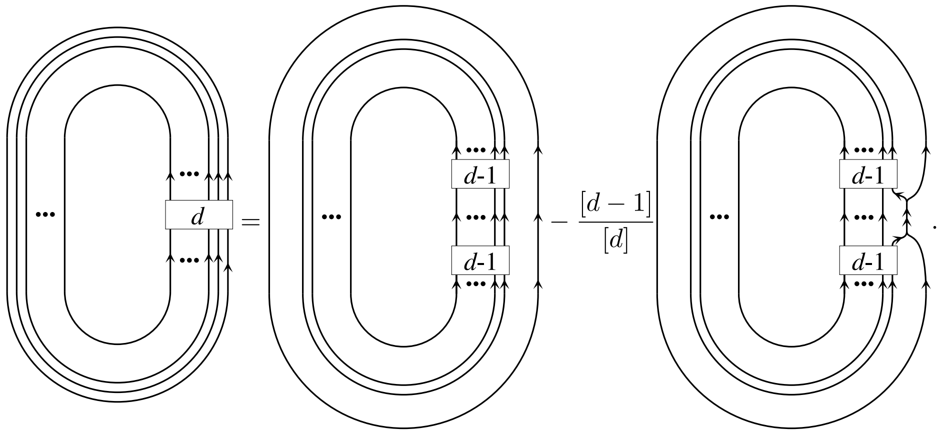
where 

is obtained by replacing the string in 

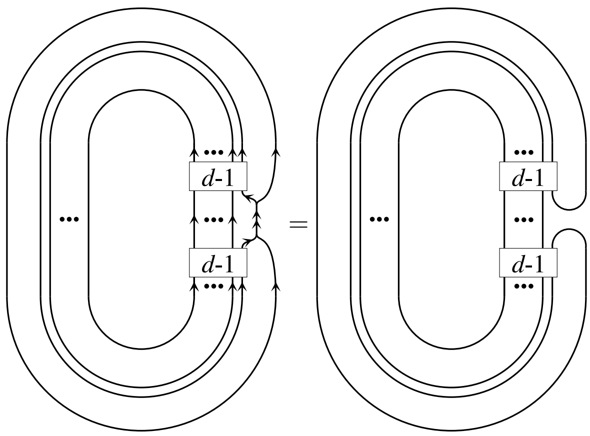
with 

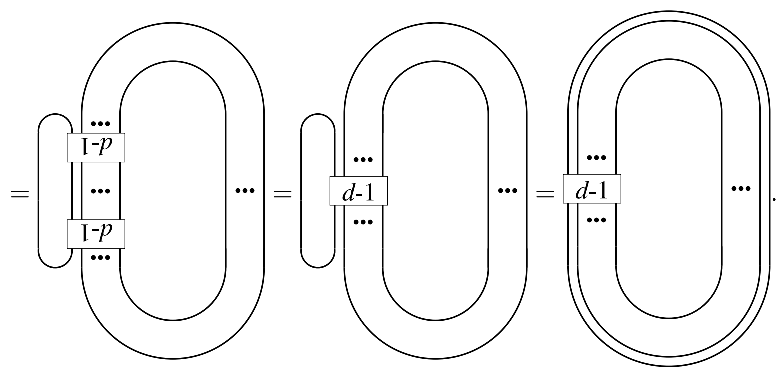
.

For example the $d+1$-dimensional colored Jones polynomial of the unknot $J_{d+1}(O;t)$ can be calculated as follows: First of all



Since we can erase the edges with flow 2, and forget the orientations of other edges, we have





Therefore

$$
\begin{aligned}
J_{d+1}(O;t)[d+1] &= [2]J_d(O;t)[d] - \frac{[d-1]}{[d]}J_d(O;t)[d] \\
&= \frac{(t^{1/2}+t^{-1/2})(t^{d/2}-t^{-d/2}) - (t^{(d-1)/2}-t^{-(d-1)/2})}{t^{1/2}-t^{-1/2}}J_d(O;t) \\
&= [d+1]J_d(O;t),
\end{aligned}
$$

and by induction we have $J_{d+1}(O;t) = 1$. In general it can be proved that $J_d(L;t)$ is a Laurent polynomial in $t^{\pm 1/2}$ for any link L.

For example for the figure-eight knot 4_1, K. Habiro [16] and T. Le proved that

$$J_d(4_1;t)=\sum_{k=0}^{d-1}\prod_{l=1}^{k}\left(t^{(d+l)/2}-t^{-(d+l)/2}\right)\left(t^{(d-l)/2}-t^{-(d-l)/2}\right). \tag{4.1}$$

See also [30] for other examples.

5. Volume Conjecture

We want to investigate the asymptotic behavior of the following quantity:

$$\frac{\log J_N\left(K;\exp(2\pi\sqrt{-1}/N)\right)}{N}. \tag{5.1}$$

Let us calculate the $N\to\infty$ limit of Equation (5.1) for the figure-eight knot 4_1 following T. Ekholm. From Equation (4.1) we have

$$J_N\left(4_1;\exp(2\pi\sqrt{-1}/N)\right)=\sum_{k=0}^{N}\prod_{l=1}^{k}\left(2\sin(l\pi/N)\right)^2.$$

Putting $f(k):=\prod_{l=1}^{k}\left(2\sin(l\pi/N)\right)^2$, we have $J_N\left(4_1;\exp(2\pi\sqrt{-1}/N)\right)=\sum_{k=0}^{N}f(k)$. Now $f(k)$ takes its maximum when $l\pi/N=5\pi/6$, roughly speaking. Therefore we have

$$f(5N/6)\le\sum_{k=0}^{N-1}f(k)\le Nf(5N/6).$$

Taking logarithm, we have

$$\frac{\log f(5N/6)}{N}\le\frac{\log J_N\left(4_1;\exp(2\pi\sqrt{-1}/N)\right)}{N}\le\frac{\log Nf(5N/6)}{N}=\frac{\log N}{N}+\frac{\log f(5N/6)}{N}.$$

Now since $\frac{\log N}{N}\to 0$ when $N\to\infty$, we have

$$\begin{aligned}\lim_{N\to\infty}\frac{\log J_N\left(4_1;\exp(2\pi\sqrt{-1}/N)\right)}{N}&=\lim_{N\to\infty}\frac{\log f(5N/6)}{N}\\&=2\lim_{N\to\infty}\sum_{l=1}^{5N/6}\frac{\log\left(2\sin(l\pi/N)\right)}{N}\\&=\frac{2}{\pi}\int_0^{5\pi/6}\log\left(2\sin(l\pi/N)\right)\,dx.\end{aligned}$$

Here we define the Lobachevsky function $Л(\theta)$ as follows.

$$Л(\theta):=-\int_0^{\theta}\log 2\sin x\,dx.$$

Then we have

$$\lim_{N\to\infty}\frac{\log J_N\left(4_1;\exp(2\pi\sqrt{-1}/N)\right)}{N}=-\frac{2}{\pi}\,Л(5\pi/6).$$

The Lobachevsky function satisfies the following properties (see for example [32] and [54, Chapter 7].)

(1) $Л(\theta)$ is an odd function.
(2) $Л(\theta)$ has period π.
(3) $Л(n\theta)=n\sum_{k \pmod n}Л(\theta+k\pi/n)$ for any integer n.

Therefore

$$Л(5\pi/6) = -Л(\pi/6)$$

and

$$\begin{aligned} Л(\pi/3) &= Л(2\cdot\pi/6) = 2\,Л(\pi/6) + 2\,Л(\pi/6+\pi/2) = 2\,Л(\pi/6) + 2\,Л(2\pi/3) \\ &= 2\,Л(\pi/6) - 2\,Л(\pi/3) \end{aligned}$$

and so we have

$$Л(5\pi/6) = -3\,Л(\pi/3)/2.$$

Now we finally have

$$\lim_{N\to\infty} \frac{\log J_N\left(4_1; \exp(2\pi\sqrt{-1}/N)\right)}{N} = -\frac{2}{\pi}\,Л(5\pi/6) = \frac{6\,Л(\pi/3)}{2\pi}.$$

What is this?

Here I will explain a little about hyperbolic geometry. See for example [54, 55, 3]. for more details.

Let us consider the upper half space model $\mathbf{H}^3$, which is topologically same as the upper half Euclidean xyz-space. A geodesic plane in $\mathbf{H}^3$ is either a (Euclidean) upper half plane perpendicular to the xy-plane or an upper hemisphere with center on the xy-plane. A geodesic line is the intersection of two geodesics planes; either upper half line perpendicular to the xy-plane or an semicircle perpendicular to the xy-plane. A hyperbolic tetrahedron has geodesic faces and an ideal hyperbolic tetrahedron means a tetrahedron with all the vertices on the 'infinite sphere', that is, xy-plane $\cup\,\infty$. By isometry of $\mathbf{H}^3$ one can assume that a vertex of any ideal hyperbolic tetrahedron is ∞. Then it looks like Figure 22. If we see from the above, it becomes a triangle (Figure 23). Two ideal hyperbolic tetrahedra are isometric

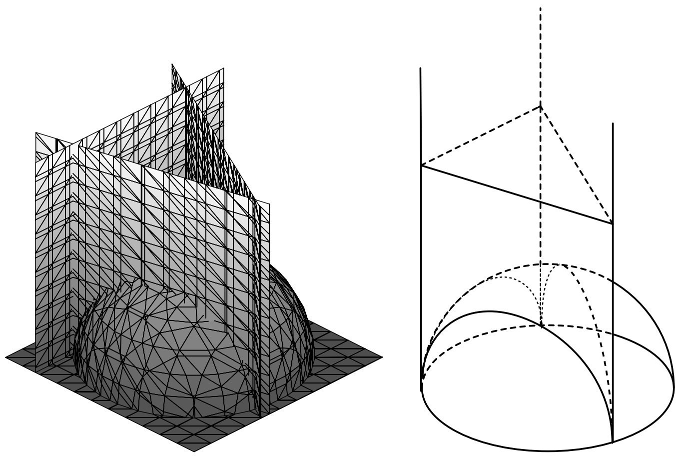


FIGURE 22. Ideal hyperbolic tetrahedron

if and only if their associated triangles are similar. Therefore ideal hyperbolic tetrahedra are parameterized by three interior angles α, β and γ with $\alpha+\beta+\gamma = 2\pi$, which are the dihedral angles of tetrahedra. (There are six dihedral angles but the opposite angles are the same.) It can be proved that the volume of the ideal hyperbolic tetrahedron with dihedral angles α, β, γ is given by $Л(\alpha)+Л(\beta)+Л(\gamma)$.

Roughly speaking a hyperbolic three-manifold can be obtained by pasting hyperbolic tetrahedra. In particular, it is well known that the complement of the figure-eight knot can be obtained from two regular ideal hyperbolic tetrahedra [43].

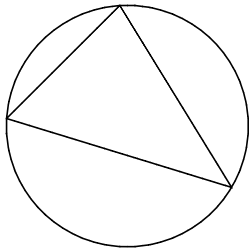


FIGURE 23. Top view of the ideal hyperbolic tetrahedron

Remark 5.1. Note that since an ideal tetrahedron is a tetrahedron with vertices removed, if we paste ideal tetrahedra we have a non-compact three-manifold without boundary. If we cut out small tetrahedra near the vertices of the ideal tetrahedron, we have a truncated tetrahedron. If we paste these truncated tetrahedron together in the same way as the ideal tetrahedra, we get a compact three-manifold with boundaries, each of which is homeomorphic to a torus.

Thus the volume of the complement of the figure-eight $\mathrm{Vol}(S^3 \setminus 4_1)$ is equal to $6\,Л(\pi/3)$ and so we conclude that

$$2\pi \lim_{N\to\infty} \frac{\log J_N\left(4_1; \exp(2\pi\sqrt{-1}/N)\right)}{N} = \mathrm{Vol}(S^3 \setminus 4_1). \tag{5.2}$$

Remark 5.2. R. Kashaev observed by numerical calculations that for the knots 4_1, 5_2, and 6_1 (5.1) is equal to the volume of the complement divided by 2π. He also conjectured that this also holds for any hyperbolic knot (a knot whose complement has complete hyperbolic structure) [24].

He calculated the limits of these invariants by first approximating the summations by meromorphic functions and then applying the saddle point method. He consulted the table of [1] and found out that these limits coincide with the volumes. Unfortunately his computation was not rigorous but efficient enough to be applied to other knots and links [37].

Remark 5.3. To be more precise, Kashaev conjectured that the limit of his link invariant defined in [22, 23] determines the hyperbolic volume. In [36], J. Murakami and I proved that Kashaev's invariant is the absolute value of the colored Jones polynomial evaluated at a root of unity.

In [36] J. Murakami and I conjectured that (5.2) holds for any knot if we replace the hyperbolic volume with a version of simplicial volume (or Gromov norm) [15]. Here the simplicial volume can be calculated as follows.

It is known [18, 19] that any link complement space can be uniquely decomposed by tori into hyperbolic spaces and Seifert fibered spaces (JSJ decomposition):

$$S^3 \setminus K = (H_1 \cup_{\text{torus}} H_2 \cup_{\text{torus}} \cdots \cup_{\text{torus}} H_k) \bigcup_{\text{torus}} (F_1 \cup_{\text{torus}} F_2 \cup_{\text{torus}} \cdots \cup_{\text{torus}} F_l),$$

where H_i is hyperbolic and F_j is Seifert fibered.

A Seifert fibered space is a generalization of a circle bundle over a surface with (possibly) singularities [47] (English translation can be found in [50]). Here a neighborhood of a 'singular fiber' is obtained from a cylinder $D^2 \times [0,1] \subset \mathbb{C} \times [0,1]$ ($D^2 := \{z \in \mathbb{C} \mid |z| \le 1\}$) by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ by the map $(x, 0) \mapsto (x \exp(2p\pi\sqrt{-1}/q), 1)$ for coprime integers p and q. Note that this is homeomorphic to a solid torus $D^2 \times S^1$ ($S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$).

Example 5.4. The complement of the trefoil 3_1 is Seifert fibered. To see this I first note that 3_1 can be put on a standard torus (Figure 24). If we remove the

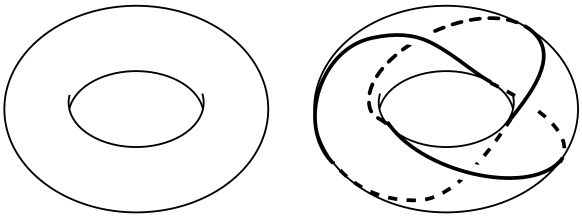


FIGURE 24. Standard torus and the trefoil on it.

standard torus from S^3 we have another standard torus. This is because we first decompose S^3 into two three-balls, dig a hole through a three-ball (remove a solid cylinder) to make a solid torus, and attach the cylinder to the other three-ball to make another solid torus. (See Figure 25.)

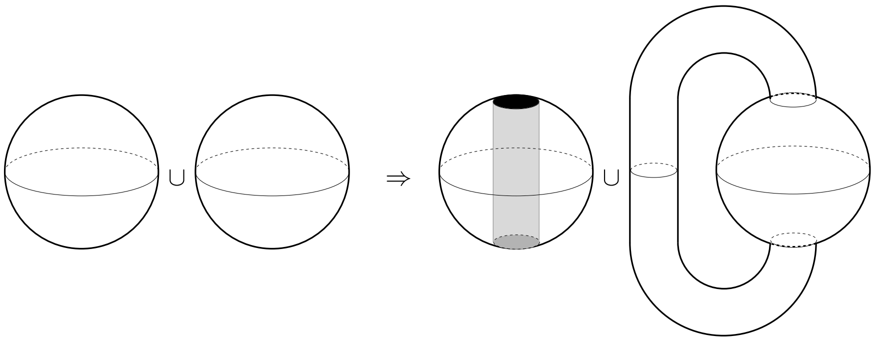


FIGURE 25. $S^3 = \text{ball} \cup \text{ball} = \text{ball without cylinder} \cup \text{ball with cylinder}$.

Then our trefoil is on the common boundary of the two solid tori. See Figure 26. Let us regard the solid torus on the left as $D^2 \times S^1 := D^2 \times [0,1]/(z,0) \sim (z,1)$ (the quotient space obtained from $D^2 \times [0,1]$ by identifying $(z,0)$ and $(z,1)$.) and that on the right as $S^1 \times D^2 := [0,1] \times D^2/(0,w) \sim (1,w)$. Then the trefoil in the left of Figure 26 can be described as

$$\left(\exp\left(\frac{3}{2} \times 2\pi s\sqrt{-1}\right), s\right) \cup \left(-\exp\left(\frac{3}{2} \times 2\pi s\sqrt{-1}\right), s\right)$$

and that on the right as

$$\begin{aligned}
&\left(u, \exp\left(\frac{2}{3} \times 2\pi u\sqrt{-1}\right)\right) \\
\cup &\left(u, \exp\left(\frac{1}{3} \times 2\pi\sqrt{-1}\right)\exp\left(\frac{2}{3} \times 2\pi u\sqrt{-1}\right)\right) \\
\cup &\left(u, \exp\left(\frac{2}{3} \times 2\pi\sqrt{-1}\right)\exp\left(\frac{2}{3} \times 2\pi u\sqrt{-1}\right)\right).
\end{aligned}$$

Now we define maps $p_1 \colon D^2 \times S^1 \to D^2$ and $p_2 \colon S^1 \times D^2$ by $p_1\big((z,s)\big) \mapsto |z|\exp\big(2(\arg z - \frac{3}{2} \times 2\pi s)\sqrt{-1}\big)$ and $p_2\big((u,w)\big) \mapsto |w|\exp\big(3(\arg w - \frac{2}{3} \times 2\pi u)\sqrt{-1}\big)$. These maps are well defined as maps from the solid tori and also agree on their boundaries. It is easily seen that both $p_1\big|_{D^2\setminus\{0\}\times S^1}$ and $p_2\big|_{S^1 \times D^2\setminus\{0\}}$ is a circle bundle. Therefore p_1 and p_2 define Seifert fibered space structure on $D^2 \times S^1$ and $S^2 \times D^2$ with singular fibers $\{0\} \times S^2$ and $S^2 \times \{0\}$ respectively. Note that each trefoil can be regarded as a regular (non-sigular) fiber.

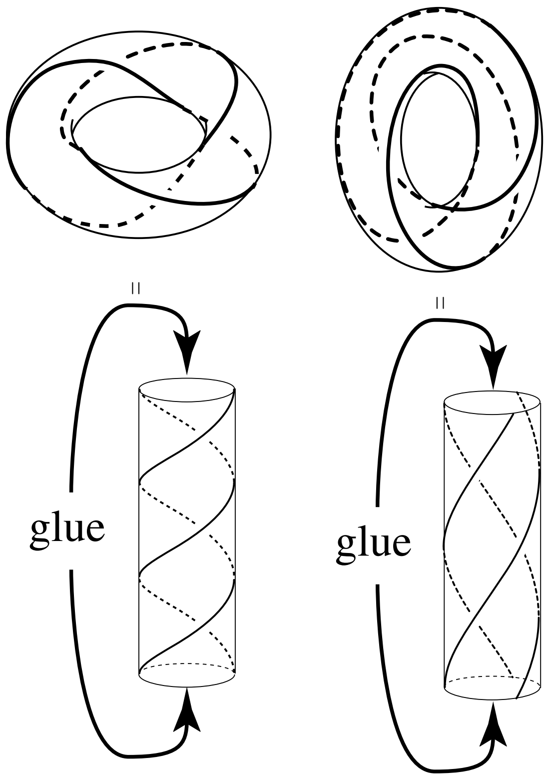


FIGURE 26. S^3 is decomposed into two solid tori and the trefoil is on the common boundary.

Since p_1 and p_2 agree on the boundaries, they define a map from $p\colon S^3 = D^2 \times S^1 \cup S^1 \times D^2 \to D^2 \cup D^2 = S^2$. This also defines a Seifert fibered space structure on S^3 with two singular fibers. Since the trefoil in each solid torus is a regular fiber, we finally see that $S^3 \setminus$ trefoil is a Seifert fibered space with two singular fibers.

Example 5.5. Let us consider the knot depicted in Figure 27, which is called the $(2, 1)$-cable of the figure-eight knot since the knot goes along the figure-eight knot twice and around it once. The complement of this knot can be decomposed as

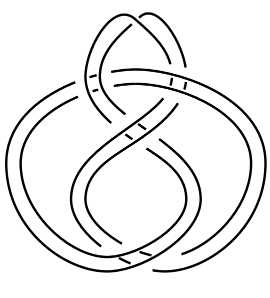


FIGURE 27. $(2, 1)$-cable of the figure-eight knot.

follows: The thin line in the left hand side of Figure 28 shows the decomposing torus. The first figure of the right hand side shows the complement of the figure-eight knot and the second shows the solid torus minus $(2, 1)$-cable, which runs twice along S^1 and once around D^2.

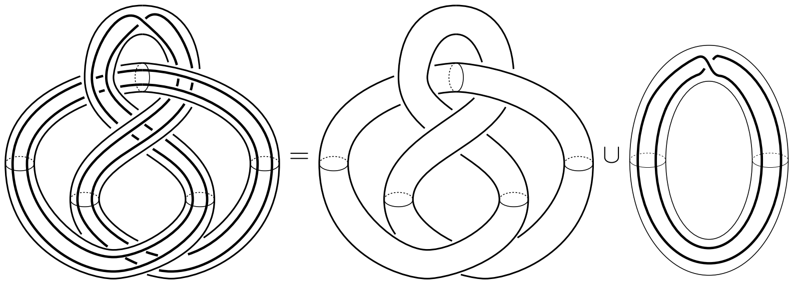


FIGURE 28. The JSJ decomposition of the $(2,1)$-cable of the figure-eight knot.

As noted in the above, the figure-eight knot complement is hyperbolic. The second one $D^2 \times S^1 \setminus (2,1)$-cable is Seifert fibered, which can be proved as before.

Now the simplicial volume of a knot complement is the sum of the hyperbolic volumes of the hyperbolic pieces:

$$\left\|S^3 \setminus K\right\| = \sum_{i=1}^{k} \operatorname{Vol}(H_i).$$

Remark 5.6. The original definition of the simplicial volume $|M|$ for a closed three-manifold is as follows. Let $[M]$ be the fundamental class of the third homology $H_3(M;\mathbb{R})$. (So $[M]$ generates $H_3(M;\mathbb{R}) \cong \mathbb{R}$.) Then the Gromov norm is defined to be the infimum of the norm of three-chains that present $[M]$, where the norm of a chain $\sum_{i=1}^{m} r_i c_i$ is $\sum_{i=1}^{m} |r_i|$.

It is known that the Gromov norm is proportional to the hyperbolic volume; more precisely $\operatorname{Vol}(M) = v_3|M|$ with v_3 is the hyperbolic volume of the regular ideal hyperbolic tetrahedra (see for example [54, Chapter 6]).

In this paper I abuse the notation and call $\|M\| := v_3|M|$ the simplicial volume.

Example 5.7. Since the complement of the trefoil is Seifert fibered, its simplicial volume is zero.

Example 5.8. From Example 5.5, the $(2,1)$-cable of the figure-eight knot can be decomposed into two pieces, one hyperbolic and one Seifert fibered. Therefore its simplicial volume is the same as the figure-eight knot's.

Now the Volume Conjecture can be stated as follows.

Conjecture 5.9 (Volume Conjecture [36])**.** *For any knot K, the asymptotic behavior of $J_N(K;\exp(2\pi\sqrt{-1}))$ determines the simplicial volume of the complement. More precisely the following equality holds.*

$$2\pi \lim_{N\to\infty} \frac{\log\left|J_N\left(K;\exp(2\pi\sqrt{-1}/N)\right)\right|}{N} = \left\|S^3 \setminus K\right\|. \tag{5.3}$$

Remark 5.10. If we drop the absolute value sign from the left hand side in Equation (5.3), we have a complex number whose real part is the volume. In [37] we also conjecture that the imaginary part gives the so called Chern–Simons invariant [5, 31]. It is known that (Volume $+\sqrt{-1}$Chern–Simons) gives in a sense an analytic function [39, 62].

Remark 5.11. The Volume Conjecture was proved so far for

- The figure-eight knot.

- Torus knots (knots on the standardly embedded torus) by Kashaev and O. Tirkkonen [25]. Note that the complement of a torus knot is Seifert fibered and so its simplicial volume is zero.
- The Borromean rings by T. Le. (Interesting topics about the Borromean rings can be found in [7]. Visit also the website: http://www.liv.ac.uk/ spmr02/rings/.) The complement of the Borromean rings is hyperbolic. You can also enjoy a video: [8].

Remark 5.12. Y. Yokota showed (inspired by D. Thurston's observation [53]) that one can obtain a decomposition by ideal tetrahedra corresponding to the limit of Kashaev's invariant for any hyperbolic knot complement whose parameters naturally defines a potential function that gives both volume and Chern–Simons invariant ([59, 57, 58, 60, 61]. See also [34].)

Remark 5.13. For further development of the Volume Conjecture see [33, 35].

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