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Invariants of 3-manifolds via link polynomials and quantum groups

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1. Introduction

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in S^3 the terms of the sequence are equale to the values of the (suitably parametrized) Jones polynomial of the link in the corresponding roots of 1. Thus, for links in S^3 our invariants are essentially equivalent to the Jones polynomial [Jo].

Note that in general we do not know if our invariant of (a framed link in) a closed oriented 3-manifold may be described as the sequence of values of a certain polynomial in the roots of unity.

In the case of manifolds with boundary our invariant is a (sequence of) finite dimensional complex linear operators. This produces from each root of unity q a 3-dimensional topological quantum field theory (see [A1]). In particular, for each q we associate with every closed oriented surface a finite dimensional complex linear space. We construct a projective action of the modular group of the surface in this space.

Our constructions have been partially inspired by the ideas of E. Witten [Wi] who considered quantum field theory defined by the nonabelian Chern-Simons action and applied it to study the topology of 3-manifolds. Using this quantum field theory, Witten has defined (on the physical level of rigor) certain invariants of 3-manifolds and links in 3-manifolds. The constructions of Witten strongly suggested that there may exist a parallel mathematical theory. We believe that our invariants may be viewed as a mathematical realization of the Witten's program.

We use a rather down-to-earth approach to construct our invariants. Namely, we use surgery to reduce the general case to the case of links in S^3 and then apply the classical Jones polynomial and some derived invariants of links in S^3. The reduction of the topology of 3-manifolds to the theory of links in S^3 is very well known. Indeed, each framed link L in S^3 determines a closed, oriented, connected 3-manifold M_L obtained by surgering S^3 along L. Each closed, oriented, connected 3-manifold M is known to be homeomorphic to some M_L by a degree 1 homeomorphism. Two manifolds M_L, $M_{L'}$ are degree 1 homeomorphic if and only if the links L, L' may be related by a series of Kirby moves (see [K]). This suggests that one may construct invariants of 3-manifolds by combining certain link invariants in an expression preserved by Kirby moves. It was the idea to try the Jones polynomial and related invariants which gave the initial impetus to our work.

The Jones polynomial is known to be intricately connected with the quantum enveloping algebra of the Lie algebra $sl_2(\mathbf{C})$, and its fundamental representation. Using other irreducible representations of the algebra, one may produce isotopy invariants of colored links in S^3 where by coloring of a link we mean a function which associates with each component of the link an irreducible representation of $sl_2(\mathbf{C})$. This gives us a large stock of invariants of links in S^3 which are used as groundstones in our constructions concerned with general 3-manifolds. For previous work on link invariants see [Tu], [Re], [ReT] and references therein.

The basic algebraic concept introduced and systematically used in this paper is the one of a modular Hopf algebra. We show that each such algebra gives rise to a topological quantum field theory in dimension 3. In particular, it produces numerical invariants of closed oriented 3-manifolds and links in such manifolds.

We prove that for each root of unity q the quantum q-deformation of sl_2 yields a finite dimensional modular Hopf algebra. This gives the invariants of links and 3-manifolds discussed above.

The paper is organized as follows. In § 2 we recall the results of [ReT] on ribbon Hopf algebras and the associated isotopy invariants of ribbon graphs. In § 3 we introduce modular Hopf algebras and construct associated homeomorphism invariants of closed oriented 3-manifolds and also isotopy invariants of framed links and ribbon graphs in such manifolds. In § 4 we use the results of § 3 to construct operator invariants of 3-dimensional cobordisms. As an application of this construction we get a projective representation of the Teichmüller modular group. §§ 5, 6, 7 are devoted to the proofs of the theorems stated in § 3. In § 8 we briefly consider the modular Hopf algebras associated with the quantum q-deformation of the Lie algebra sl_2, q being a root of unity.

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2. Ribbon Hopf algebras and ribbon graphs

2.1. Ribbon Hopf algebras (see [Dr], [Dr1], [Re1], [Tu1], [ReT])

Let κ be a field of characteristic 0. Let A be a Hopf algebra over κ with the comultiplication $\Delta: A \to A \otimes A$, counit $\varepsilon: A \to \kappa$ and antipode $\gamma: A \to A$. Denote by P the permutation homorphism $a\otimes b\mapsto b\otimes a$: $A\otimes A\to A\otimes A$. Let R be an element of $A\otimes A$. Thus R is a finite sum $\sum_i \alpha_i\otimes\beta_i$ with $\alpha_i, \beta_i\in A$. Put

$$R_{12}=R\otimes 1=\sum_i \alpha_i\otimes\beta_i\otimes 1\in A^{\otimes 3},$$

$$R_{13}=(id\otimes P)(R_{12})=\sum_i \alpha_i\otimes 1\otimes\beta_i\in A^{\otimes 3},$$

$$R_{23}=1\otimes R=\sum_i 1\otimes\alpha_i\otimes\beta_i\in A^{\otimes 3}.$$

The pair (A, R) is called a quasitriangular Hopf algebra if: R is invertible in $A^{\otimes 2}$; for any $a\in A$

$$P(\Delta(a))=R\Delta(a)R^{-1},$$

and

$$(\Delta\otimes id_A)(R)=R_{13}R_{23},$$
$$(id_A\otimes\Delta)(R)=R_{13}R_{12}.$$

The element R is called the universal R-matrix of A. It satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}.$$

Note also the following important equalities

(2.1.1) $$R^{-1}=(\gamma\otimes id)(R)=(id\otimes\gamma^{-1})(R).$$

With the quasitriangular algebra $(A, R=\sum_i \alpha_i\otimes\beta_i)$ one associates the element

$$u=\sum_i \gamma(\beta_i)\,\alpha_i$$

of A. This element is invertible and satisfies the following identities [Dr1, Re1]:

$$\gamma^2(a)=uau^{-1} \quad \text{for all } a\in A;$$
$$u^{-1}=\sum_i \beta_i\gamma^2(\alpha_i); \qquad \varepsilon(u)=1;$$
$$\Delta(u)=(R_{21}R_{12})^{-1}(u\otimes u)=(u\otimes u)(R_{21}R_{12})^{-1},$$

where $R_{21}=P(R_{12})$. Also, $u\gamma(u)$ lies in the center of A.

By a *ribbon Hopf algebra* we shall mean a quasitriangular Hopf algebra (A, R) provided with a central element $v\in A$ such that:

(2.1.2) $$v^2=u\gamma(u), \qquad \gamma(v)=v, \qquad \varepsilon(v)=1,$$
$$\Delta(v)=(R_{21}R_{12})^{-1}(v\otimes v).$$

Since u is invertible, v is also invertible in A.

There exist quasitriangular Hopf algebras which do not contain v as above. If v exists it may be non-unique. For examples of ribbon Hopf algebras see §8.

2.2. *Category Rep A*

With each algebra A over the field κ one associates the category Rep A of its finite dimensional linear representations. The objects of Rep A are left A-modules finitely generated over κ. The morphisms of Rep A are A-linear homomorphisms. The action of A in any A-module V induces an algebra homomorphism $A \to \operatorname{End} V$ denoted by ρ_V.

If A is a Hopf algebra then the comultiplication Δ in A induces a tensor multiplication in Rep A: for objects V, W their tensor product is the vector space $V \otimes_k W$ equipped with the (left) action of A defined by the formula

$$\rho_{V\otimes W}(a) = (\rho_V \otimes \rho_W)(\Delta(a))$$

for $a \in A$. The antipode $\gamma: A \to A$ enables one to define duals for A-modules. Namely, for any A-module V we provide the dual linear space $V^\vee = \operatorname{Hom}_\kappa(V, \kappa)$ with the action of A:

$$\rho_{V^\vee}(a) = (\rho_V(\gamma(a)))^* \in \operatorname{End} V^\vee$$

where $a \in A$ and the asterisque denotes the dual of a linear homomorphism. If A is a quasitriangular Hopf algebra then all objects of Rep A are reflexive up to a canonical isomorphism. Indeed,

$$\rho_{V^{\vee\vee}}(a) = \rho_V(\gamma^2(a)) = u_V \, \rho_V(a) \, u_V^{-1},$$

where $u_V = \rho_V(u)$ and where we have identified $V^{\vee\vee}$ and V via the canonical isomorphism. In other words, the composition of the canonical identification $V^{\vee\vee} \to V$ and the homomorphism $x \mapsto \rho_V(u^{-1})\, x: V \to V$ is an A-linear isomorphism.

If (A, R, v) is a ribbon Hopf algebra and V is an object of Rep A we define the quantum dimension $\dim_q V$ of V to be the trace over κ of the linear operator

$$x \mapsto \rho_V(uv^{-1})\, x: V \to V.$$

For any linear operator $f: V \to V$ we define its quantum trace $\operatorname{tr}_q f$ to be the ordinary trace over κ of the linear operator

$$x \mapsto \rho_V(uv^{-1})\, f(x): V \to V.$$

Clearly, $\dim_q V = \operatorname{tr}_q(id_V)$.

2.3. *Ribbon graphs*

A ribbon in $\mathbf{R}^3$ is the square $[0, 1] \times [0, 1]$ smoothly imbedded in $\mathbf{R}^3$. The images of the segments $[0, 1] \times 0$ and $[0, 1] \times 1$ under the imbeddings are called bases of the ribbon. The image of the segment $(1/2)\times[0,1]$ is called the core of the ribbon. Similarly by an annulus in $\mathbf{R}^3$ we shall mean the cylinder $S^1\times[0,1]$ smoothly imbedded in $\mathbf{R}^3$. The image of the circle $S^1\times(1/2)$ under the imbedding is called the core of the annulus.

Clearly, ribbons and annuli are orientable surfaces in $\mathbf{R}^3$. Note that orientation of such a surface is equivalent to a choice of one of its sides. (We fix the right-hand orientation in $\mathbf{R}^3$).

If the core of a ribbon (or annulus) is oriented then we say that this ribbon (annulus) is directed. Each ribbon and annulus may be directed in 2 ways and may be oriented (as the surface) also in 2 ways. Note that the core of a directed ribbon leads from one base to another base. The former base is called initial, the latter one is called final.

Let k, l be non-negative integers. By a ribbon (k, l)-graph we shall mean an oriented surface S imbedded in $\mathbf{R}^2\times[0,1]$ and decomposed into the union of a finite number of directed ribbons and annuli so that:

(a) annuli do not meet each other and do not meet ribbons;

(b) each ribbon is provided with a "type" 1 or 2;

(c) ribbons of the same type never meet each other;

(d) ribbons of different types may meet only in the points of their bases;

(e) S meets $(\mathbf{R}^2\times 0)\cup(\mathbf{R}^2\times 1)$ precisely in some bases of type 2 ribbons, and the collection of these bases is the collection of segments

$$\{[i-\tfrac{1}{4}, i+\tfrac{1}{4}]\times 0\times 0 \mid i=1, \dots, k\}\cup\{[j-\tfrac{1}{4}, j+\tfrac{1}{4}]\times 0\times 1 \mid j=1, \dots, l\}$$

(f) the other bases of type 2 ribbons are contained in some bases of type 1 ribbons;

(g) in a neighborhood of $S\cap(\mathbf{R}^2\times 0)$ and $S\cap(\mathbf{R}^2\times 1)$ the preferred side of S is turned upwards (i.e. towards the reader).

The surface S is called the surface of the graph. The type 2 ribbons will be simply called ribbons, the type 1 ribbons will be called coupons. Those ribbons which are incident to $S\cap(\mathbf{R}^2\times 0)$ or $S\cap(\mathbf{R}^2\times 1)$ will be called border ribbons.

It is convenient to think of coupons as rigid rectangles, whereas the (type 2) ribbons may be thought of as narrow flexible bands. Some examples of ribbon graphs are given in Fig. 1, 3. Note that instead of drawing annuli it suffices to draw their directed cores and to point out the so-called framings, i.e. the number of right-handed twists of the annuli around their cores. The same remark applies to each ribbon whose ends are both incident to the same coupon or both lie on $S\cap(\mathbf{R}^2\times\{0, 1\})$.

In our pictures the preferred side of S will be depicted white, the opposite side will be shaded. The white side of coupons will be always turned towards the reader.

Ribbon graphs which have no coupons will be called ribbon tangles. The isotopy type of a ribbon tangle is uniquely determined by the cores of ribbons and annuli equipped with the framings.

With each ribbon (k, l)-graph Γ we associate two sequences $\varepsilon_*(\Gamma)=(\varepsilon_1, \dots, \varepsilon_k)$ and $\varepsilon^*(\Gamma)=(\varepsilon^1, \dots, \varepsilon^l)$ consisting of ± 1. Namely, if B is the ribbon of Γ having the segment $[i-(1/4), i+(1/4)]\times 0\times 0$ (resp. $[j-(1/4), j+(1/4)]\times 0\times 1$) as a base then the directed core of B looks either "in" this segment or "out" of it. In

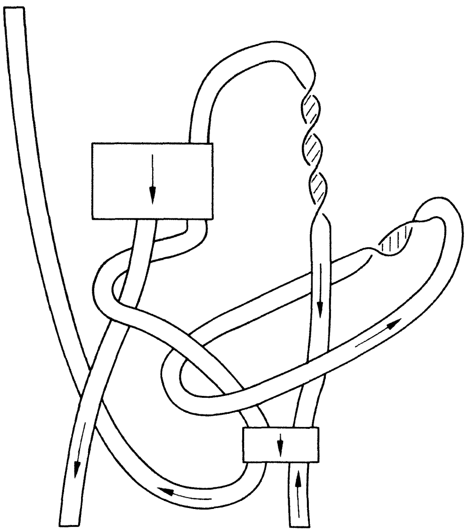


Fig. 1

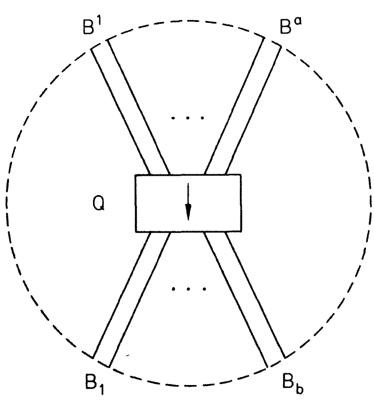


Fig. 2

the first case we put $\varepsilon_i = 1$ (resp. $\varepsilon^j = -1$). In the second case we put $\varepsilon_i = -1$ (resp. $\varepsilon^j = 1$). For example, in Fig. 1 $\varepsilon_1 = 1$, $\varepsilon^1 = \varepsilon_2 = -1$.

Let Q be a coupon of a ribbon graph Γ. Denote by $a = a(Q)$ (resp. by $b = b(Q)$) the number of ribbons of Γ incident to the initial (resp. final) base of Q. A small neighborhood of Q in $\mathbf{R}^2 \times (0, 1)$ is depicted in Fig. 2 where as usual the white side of Q is turned upwards. Denote the ribbons of Γ incident to Q by $B^1(Q), \ldots, B^a(Q)$, and $B_1(Q), \ldots, B_b(Q)$ as shown in Fig. 2. The directions of these ribbons are characterized by numbers $\varepsilon_i(Q)$, $\varepsilon^j(Q) = \pm 1$. Namely, $\varepsilon^j(Q) = -1$ if the base of B^j lying on Q and depicted in Fig. 2 is its initial base, otherwise $\varepsilon^j(Q) = 1$. Similarly, $\varepsilon_i(Q) = -1$ if the base of B_i lying on Q and depicted in Fig. 2 is its final base, and $\varepsilon_i(Q) = 1$ otherwise.

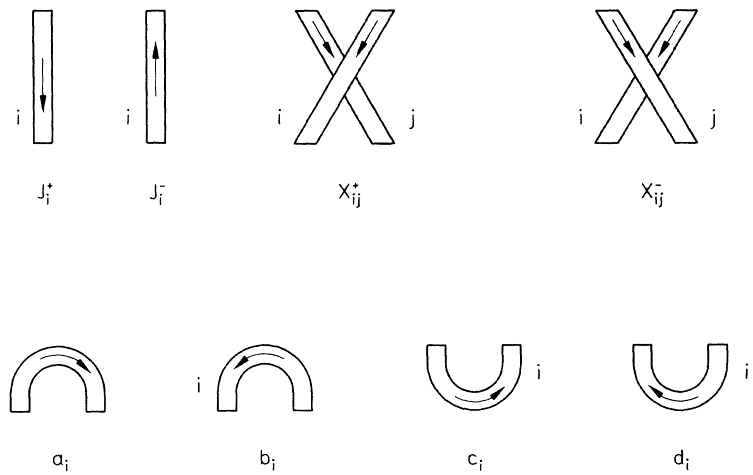


Fig. 3

2.4. Colourings of ribbon graphs

Fix a ribbon Hopf algebra (A, R, v) and set $\{V_i\}_{i\in I}$ of left A-modules of finite dimension over the ground field κ. Denote by N the set of finite sequences $(i_1, \varepsilon_1), \ldots, (i_k, \varepsilon_k)$ where $k \geqq 0$, $i_1, \ldots, i_k \in I$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{1, -1\}$. With such a sequence $\eta = ((i_1, \varepsilon_1), \ldots, (i_k, \varepsilon_k))$ we associate the A-module

$$V(\eta) = V_{i_1}^{\varepsilon_1} \otimes V_{i_2}^{\varepsilon_2} \otimes \ldots \otimes V_{i_k}^{\varepsilon_k}$$

where $V^1 = V$ and $V^{-1} = V^\vee$ (cf. Sect. 2.1). In particular, if $k=0$, i.e. if $\eta = \phi$ then $V(\eta) = \kappa$.

A *colouring* (or an A-colouring) of a ribbons graph Γ is a mapping λ which associates with each ribbon (or annulus) B of Γ its "colour" $\lambda(B) \in I$ and associates with each coupon Q of Γ its "colour" $\lambda(Q) \in \mathrm{Hom}_A(V(\eta), V(\eta'))$ where

$$\eta = ((\lambda(B_1(Q)), \varepsilon_1(Q)), \ldots, (\lambda(B_b(Q)), \varepsilon_b(Q))),$$
$$\eta' = ((\lambda(B^1(Q)), \varepsilon^1(Q)), \ldots, (\lambda(B^a(Q)), \varepsilon^a(Q))),$$

where $a = a(Q)$, $b = b(Q)$.

Two coloured ribbon graphs Γ, Γ' are called isotopic if there exists a smooth isotopy $h_t\colon \mathbf{R}^2 \times [0, 1] \to \mathbf{R}^2 \times [0, 1]$ of the identity $h_0 = \mathrm{id}$ so that each h_t is a self-diffeomorphism of the strip $\mathbf{R}^2 \times [0, 1]$ fixing its boundary $\mathbf{R}^2 \times \{0, 1\}$ pointwise for all $t \in [0, 1]$, and h_1 transforms Γ onto Γ' preserving the decomposition into coupons, ribbons and annuli, preserving the directions of cores and the orientation of the graph surfaces and preserving the colours. Isotopy is, of course, an equivalence relation. By an abuse of language isotopy classes of coloured ribbon graphs will be also called coloured ribbon graphs.

Some examples of coloured ribbon graphs and the notation for these graphs are presented in Fig. 3 (where $i, j \in I$). The coloured ribbon (k, l)-graph with 1

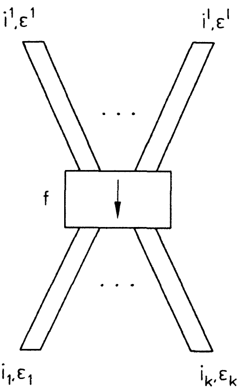


Fig. 4

coupon of colour f presented in Fig. 4 will be denoted by $\Gamma(f, \eta, \eta')$; here $\eta = ((i_1, \varepsilon_1), \ldots, (i_k, \varepsilon_k)) \in N$ and $\eta' = ((i^1, \varepsilon^1), \ldots, (i^l, \varepsilon^l)) \in N$ where i_r, i^s are the colours of the corresponding ribbons and ε_r, ε^s are their directions: 1 means down, -1 means up.

It is convenient to organize from coloured ribbon graphs a category $\mathscr{H}$. Its objects are elements of the set N, i.e. sequences $(i_1, \varepsilon_1), \ldots, (i_k, \varepsilon_k)$ as above. If $\eta, \eta' \in N$ then a morphism $\eta \to \eta'$ is a coloured ribbon graph (considered up to isotopy) such that the sequence of colours and directions of the bottom (resp. top) border ribbons is equal to η (resp. to η'). The composition $\Gamma' \circ \Gamma$ of two such morphisms $\Gamma: \eta \to \eta'$, $\Gamma': \eta' \to \eta''$ is defined as follows: shift Γ' by the vector $(0, 0, 1)$ into $\mathbf{R}^2 \times [1, 2]$; glue the bottom ends of Γ' with the corresponding top ends of Γ; reduce twice the vertical size. This gives a well-defined composition law for morphisms of $\mathscr{H}$. The identity morphisms $\mathrm{id}_\eta: \eta \to \eta$ is the coloured ribbon graph consisting of vertical untwisted ribbons whose directions and colours are determined by η.

We also provide $\mathscr{H}$ with a tensor product $\otimes: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$. The tensor product of objects η, η' is their juxtaposition η, η'. The tensor product of two morphisms (i.e. graphs) Γ, Γ' is obtained by placing Γ' to the right of Γ so that there is no mutual linking or intersection.

2.5. Theorem. *Let $(A, R = \sum_r \alpha_r \otimes \beta_r, v)$ be a ribbon Hopf algebra over the field κ of zero characteristic. Let $\{V_i\}_{i \in I}$ be a set of objects of Rep A. Let $\mathscr{H}$ be the corresponding category of cloured ribbon graphs. There exists a unique covariant functor $F: \mathscr{H} \to$ Rep A which has the following properties:*

(i) *F transforms any object η of $\mathscr{H}$ into the A-module $V(\eta)$;*

(ii) *F preserves the tensor multiplication: for any two coloured ribbon graphs Γ, Γ'*

$$F(\Gamma \otimes \Gamma') = F(\Gamma) \otimes F(\Gamma'); \tag{2.5.1}$$

(iii) *values of F on the ribbon graphs pictured in Fig. 3 are as follows: $F(J_i^+)$ and $F(J_i^-)$ are identity homomorphisms resp. $V_i \to V_i$ and $V_i^\vee \to V_i^\vee$; $F(X_{i,j}^+)$ is the homomorphism*

$$x \otimes y \mapsto \sum_r \rho_{V_j}(\beta_r)\, y \otimes \rho_{V_i}(\alpha_r)\, x\colon V_i \otimes V_j \to V_j \otimes V_i;$$

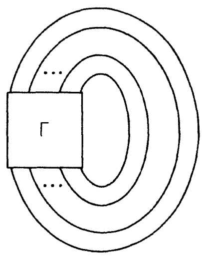


Fig. 5

$F(X^-_{i,j})$ is the homomorphism

$$x \otimes y \mapsto \sum_s \rho_{V_j}(\alpha'_s)\, y \otimes \rho_{V_i}(\beta'_s)\, x \colon V_i \otimes V_j \to V_j \otimes V_i$$

where $R^{-1} = \sum_s \alpha'_s \otimes \beta'_s$; $F(a_i)$ is the canonical pairing

$$x \otimes y \mapsto x(y) \colon V_i^{\vee} \otimes V_i \to \kappa;$$

$F(b_i)$ is the pairing

$$y \otimes x \mapsto x(\rho_{V_i}(uv^{-1})\, y) \colon V_i \otimes V_i^{\vee} \to \kappa;$$

$F(c_i)$ is the homomorphism $\kappa \to V_i \otimes V_i^V$ which transforms 1 into $\sum_m e_m \otimes e^m$ where $\{e_m\}$ is a basis in V_i and $\{e^m\}$ is the dual basis in V_i^V; $F(d_i)$ is the homomorphism $\kappa \to V_i^V \otimes V_i$ which transforms 1 into $\sum_m e^m \otimes \rho_{V_i}(vu^{-1})\, e_m$

(iv) *F transforms each graph $\Gamma(f, \eta, \eta')$ (see Fig. 4) into the homomorphism $f\colon V(\eta) \to V(\eta')$.*

Note that the condition (ii) above is formulated somewhat abusively since the homomorphisms $F(\Gamma \otimes \Gamma')$ and $F(\Gamma) \otimes F(\Gamma')$ act in different modules. However, these modules are canonically isomorphic via rearrangement of brackets. These canonical isomorphisms are tacitly incorporated in the condition (ii).

Uniqueness of the functor F is essentially obvious, since the graphs $J_i^+, J_i^-, X^+_{i,j}, X^-_{i,j}, a_i, b_i, c_i, d_i, \Gamma(f, \eta, \eta')$ generate the category $\mathscr{H}$. This means that an arbitrary coloured ribbon graph (considered up to isotopy) may be obtained from these generators using the tensor product and composition. For a proof of existence and further details see [Tu1, Re1, ReT]. Note that since F is a covariant functor, $F(\Gamma \circ \Gamma') = F(\Gamma) \circ F(\Gamma')$ whenever $\Gamma \circ \Gamma'$ is defined. If Γ is a coloured ribbon (0, 0)-graph then $F(\Gamma)$ is a κ-linear homomorphism $\kappa \to \kappa$, i.e. a multiplication by an element of κ. This element is an isotopy invariant of Γ. It will be also denoted by $F(\Gamma)$. This invariant generalizes the Jones polynomial of links and related polynomials (see [ReT, Sect. 6.1]).

For example, if Γ is the trivial knot with the framing 0 and the colour i then $\Gamma = b_i \circ c_i$ and one may easily compute $F(\Gamma)$ to be $\dim_q V_i$. The following lemma generalizes this computation.

2.6. Lemma. *Let Γ be a coloured ribbon (k, k)-graph which is an endomorphism of a certain sequence $\eta \in N$. Let L be the coloured ribbon* (0, 0)*-graph obtained by closing Γ (see Fig. 5). Then*

$$F(L) = \mathrm{tr}_q F(\Gamma).$$

Proof. Put $f=F(\Gamma)\in\mathrm{End}_A V(\eta)$. Consider first the case $k=1$ so that $\eta=(i,\varepsilon)$ with $i\in I$, $\varepsilon=\pm 1$. Let $\{e_m\}$ be a basis in V_i and $\{e^m\}$ be the dual basis in $V_i^\vee$. If $\varepsilon=1$ then $f\in\mathrm{End}\,V_i$, and

$$L=b_i\circ(\Gamma\otimes J_i^-)\circ c_i.$$

One easily computes

$$F(L)=\sum_m e^m(\rho_{V_i}(uv^{-1})\,f(e_m))=\mathrm{tr}_q f.$$

If $\varepsilon=-1$ then $f\in\mathrm{End}\,V_i^\vee$,

$$L=a_i\circ(\Gamma\otimes J_i^-)\circ d_i,$$

and a similar computation shows that

$$\begin{aligned}F(L)&=\sum_m f(e^m)(\rho_{V_i^\vee}(vu^{-1})\,e_m)\\&=\sum_m(\rho_{V_i^\vee}(\gamma(vu^{-1}))\,fe^m)(e_m)=\sum_m(\rho_{V_i^\vee}(uv^{-1})\,f(e^m))(e_m)\\&=\mathrm{tr}_q f\end{aligned}$$

(the equality $\gamma(vu^{-1})=uv^{-1}$ follows from (2.1.2) since γ is an antiautomorphism of A).

If $k\geqq 2$ then a similar computation shows that $F(L)$ is the (ordinary) trace of the composition of $f=F(\Gamma)$ and the endomorphism

$$x_1\otimes x_2\otimes\ldots\otimes x_k\mapsto\rho_{V_1}(uv^{-1})\,x_1\otimes\rho_{V_2}(uv^{-1})\,x_2\otimes\ldots\otimes\rho_{V_k}(uv^{-1})\,x_k$$

of $V(\eta)$. However the latter endomorphism is just the multiplication by uv^{-1} in the A-module $V(\eta)$. This follows from the equality $\Delta(uv^{-1})=uv^{-1}\otimes uv^{-1}$. Thus

$$F(L)=\mathrm{Tr}\,(uv^{-1}f)=\mathrm{tr}_q f.$$

3. Modular Hopf algebras. Invariants of closed 3-manifolds

3.1. Modular Hopf algebras

Let (A, R, v) be a ribbon Hopf algebra over the ground field κ. Assume that we have fixed the following data: a finite set I with involution $i\mapsto i^*: I\to I$ and with a preferred element $0\in I$ such that $0^*=0$; a set of A-modules $\{V_i\}$ numerated by $i\in I$, where $V_0=\kappa$ with the action of A determined by the counit $A\to\kappa$; a set of A-linear isomorphism

$$\{w_i\colon (V_i)^\vee\to V_{i^*}\,|\,i\in I\}\quad\text{ with } w_0=id_\kappa.$$

The triple (A, R, v) together with this data will be called a *modular Hopf algebra* if the following axioms (3.1.1–6) are satisfied.

(3.1.1) The modules $\{V_i\,|\,i\in I\}$ are mutually non-isomorphic, irreducible (i.e. do not contain proper non-trivial A-submodules), have finite dimension over κ and all have non-zero quantum dimension (see Sect. 2.2).

(3.1.2) For each $i \in I$ the homomorphism

$$w_i^* \circ (w_{i*})^{-1}: V_i \to (V_i)^{\vee\vee} = V_i$$

is the multiplication by uv^{-1}.

(Note that both $(w_i)^*$ and w_{i*} are A-linear, but the canonical identification $V^{\vee\vee} = V$ is not A-linear. This corresponds to the fact that the multiplication by uv^{-1} is not A-linear, cf. Sect. 2.2).

In what follows the direct product $I \times I \times \ldots \times I$ of k copies of I will be denoted by I^k.

(3.1.3) For any $k \geqq 2$ and for any sequence $\theta = (i_1, \ldots, i_k) \in I^k$ there exists an A-linear decomposition

$$V_{i_1} \otimes V_{i_2} \otimes \ldots \otimes V_{i_k} = Z_\theta \oplus \left(\bigoplus_{i \in I} (V_i \otimes \Omega_\theta^i) \right),$$

where Z_θ is a certain A-module satisfying the next axiom (3.1.4) and where $\{\Omega_\theta^i\}_\theta$ are vector spaces over κ.

(3.1.4) For any $k \geqq 2$, $\theta \in I^k$ and any A-linear homomorphism $f: Z_\theta \to Z_\theta$ the quantum trace of f is equal to zero.

The axioms (3.1.1), (3.1.3), (3.1.4) imply that the vector spaces $\{\Omega_\theta^i\}$ and the A-modules $\{Z_\theta\}$ may be uniquely reconstructed from $\{V_i\}$. Indeed, V_i cannot be A-linearly imbedded in Z_θ as a split summand since otherwise the projection $f: Z_\theta \to V_i \subset Z_\theta$ would be an A-linear homomorphism with $\operatorname{tr}_q f = \dim_q V_i \neq 0$. Also $\operatorname{Hom}_A(V_i, V_j) = 0$ for $i \neq j$. Therefore, any split A-linear imbedding

$$V_i \otimes K^m \to V_{i_1} \otimes V_{i_2} \otimes \ldots \otimes V_{i_k}$$

must project monomorphically in $V_i \otimes \Omega_\theta^i$ (where $\theta = (i_1, \ldots, i_k)$) along other summands of decomposition (3.1.3). This implies that the dimensions of κ-modules $\{\Omega_\theta^i\}$ and the isomorphism types of A-modules $\{Z_\theta\}$ are completely determined by $\{V_i \mid i \in I\}$.

(3.1.5) Let $S_{i,j} \in \kappa$ be the quantum trace of the A-linear operator

$$a \mapsto (R_{21} R_{12})\, a: V_i \otimes V_j \to V_i \otimes V_j.$$

(If $R = \sum_r \alpha_r \otimes \beta_r$ then this operator transforms $x \otimes y \in V_i \otimes V_j$ into $\sum_{r,s} \rho_{V_i}(\beta_s \alpha_r)\, x \otimes \rho_{V_j}(\alpha_s \beta_r)\, y$). Then the square matrix $(S_{i,j})_{i,j \in I}$ must be invertible over κ.

To formulate the last axiom we need some more notation. Since $v \in A$ lies in the centre of A the homomorphism

$$x \mapsto \rho_{V_i}(v)\, x: V_i \to V_i$$

is A-linear. Since V_i is irreducible this homomorphism is actually a multiplication by certain element v_i of the ground field κ. (Indeed, if $f: V_i \to V_i$ is an A-linear homomorphism then for any $\lambda \in k$ the set $\{x \in V_i \mid f(x) = \lambda x\}$ is a submodule of V_i. At least one of these submodules is non-zero and therefore coincides with

V_i. Here we assume that either κ is algebraically closed, say $\kappa = \mathbf{C}$, or the modules $\{\bar{\kappa} \otimes_\kappa V_i\}_i$ are irreducible over the algebraic closure $\bar{\kappa}$ of κ).

Invertibility of v in A implies that $v_i \neq 0$. Since the matrix $(S_{i,j})$ introduced above is invertible, there exists a unique solution $(d_i)_{i \in I}$ of the following system of linear equations:

(*) $$\sum_{i \in I} v_i S_{i,j} d_i = v_j^{-1} \dim_q V_j$$

$(j \in I)$. Put

$$C = \sum_{i \in I} v_i^{-1} \dim_q (V_i) d_i \in \kappa.$$

The last axiom says:

(3.1.6) $$C \neq 0.$$

An example of a modular Hopf algebra will be given in § 8.

3.2. *Presentation of closed 3-manifolds via framed links*

A *framed link* in the 3-sphere S^3 is a finite collection L of disjoint smoothly imbedded circles $L_1, \ldots, L_m$ in S^3, each circle L_i being provided with an integer n_i (the framing). The framed link L gives rise to a 4-manifold D_L obtained by adding 2-handle to the 4-ball B along $L \subset S^3 = \partial B$. Here are some details of the construction. Let D^2 be the 2-disc with the center x. Then D_L is obtained from B and m copies of $D^2 \times D^2$ by gluing for all $i = 1, \ldots, m$ the piece $\partial D^2 \times D^2 = S^1 \times D^2$ of the i-th copy of $D^2 \times D^2$ to a tubular neighborhood U_i of L_i such that $S^1 \times x$ is identified with L_i and for any $y \in \partial D$ the loop $S^1 \times y$ is identified with the n_i-times twisted parallel of L_i whose linking coefficient with L_i equals n_i. Clearly D_L is a compact 4-manifold with boundary which is formed by the (induced) gluing of $S^3 \setminus \bigcup_{i=1}^{m} \operatorname{Int} U_i$ with m copies of $D^2 \times S^1$. The orientation of the 4-ball B, inducing the righthanded orientation in $S^3 = \mathbf{R}^3 \cup \{\infty\}$, extends to an orientation in D_L and induces thereby an orientation in ∂D_L. Thus $M_L = \partial D$ is a closed oriented 3-manifold. One says that M_L is obtained by the surgery on S^3 along L. Note that in this construction L is a framed but unoriented link.

It is well known that each closed connected oriented 3-manifold can be obtained by surgery on S^3 along certain framed link. Basically this follows from the classical theorem of V.A. Rokhlin which ensures that each closed oriented 3-manifold bounds a compact oriented 4-manifold (see also [L], [W]).

The bilinear intersection form

$$H_2(D_L; \mathbf{R}) \times H_2(D_L; \mathbf{R}) \to \mathbf{R}$$

may be diagonalized with respect to some basis in $H_2(D_L; \mathbf{R}) = \mathbf{R}^m$ where L is an m-component framed link in S^3. Denote by $\sigma_-(L)$ the number of non-positive squares on the diagonal.

3.3. Invariants of closed 3-manifolds

Fix a modular Hopf algebra as defined in Sect. 3.1. We will use the notation introduced in Sect. 3.1. In particular I is the (finite) set numerating the colours $\{V_i\}$ and $C \in \kappa \setminus 0$.

Let M be a closed connected oriented 3-manifold. Let L be a framed link in S^3 with components $L_1, \ldots, L_m$ and framing $n_1, \ldots, n_m$ such that M is degree 1 diffeomorphic to M_L, the result of surgery of S^3 along L. Denote by $\operatorname{col}(L)$ the set of all mappings $\{1, \ldots, \mathrm{m}\} \to I$. Fix an orientation w of L. If $\lambda \in \operatorname{col}(L)$ then the triple (L, w, λ) in the obvious way determines a coloured ribbon (0, 0)-graph $\Gamma(L, w, \lambda)$ consisting of m annuli. Namely, let T_i be the annulus in S^3 with the ω-oriented core L_i and such that the linking number of two circles making ∂T_i is equal to n_i. Provide the annulus T_i with an arbitrary orientation and compress $\bigcup_i T_i$ into $\mathbf{R}^2 \times (0, 1) \subset S^3$. This gives a ribbon (0, 0)-graph, and the mapping λ determines its colouring. Note that the isotopy type of the resulting coloured ribbon graph $\Gamma(L, \omega, \lambda)$ depends only on L, ω and λ, and does not depend on the choice of orientations of the annuli $\{T_i\}$. Indeed, rotating an annulus around its core to the angle 180° we get the same annulus with the opposite orientation. Thus, up to isotopy both orientations of the annulus are the same. According to the results of Sect. 2.5 we may associate with $\Gamma(L, \omega, \lambda)$ its isotopy invariant $F(\Gamma(L, \omega, \lambda)) \in \kappa$. Put

$$\{L\}_\lambda = \prod_{i=1}^{m} d_{\lambda(L_i)} F(\Gamma(L, \omega, \lambda)) \in k$$

where $\{d_r\}_{r \in I}$ are the solution of equations (*), Sect. 3.1. Put

$$\{L\} = \sum_{\lambda \in \operatorname{col}(L)} \{L\}_\lambda .$$

Clearly the scalar $\{L\}$ does not depend on the order $L_1, \ldots, L_m$ in the set of components of L.

3.3.1. Lemma. *$\{L\}$ does not depend on the choice of orientation ω in L.*

This Lemma will be proved in § 5. Put

$$F(M; L) = C^{-\sigma_-(L)} \{L\} \in \kappa .$$

3.3.2. Theorem. *$F(M; L)$ is a topological invariant of M.*

This Theorem will be proved in § 7 where we use the Kirby moves on links (discussed in § 6) to show that $F(M; L)$ does not depend on the choice of L.

We shall denote the invariant $F(M; L)$ by $F(M)$. This invariant is multiplicative:

$$F(M_1 \# M_2) = F(M_1) F(M_2).$$

Note that $F(S^3) = 1$ and

$$F(S^1 \times S^2) = C^{-1} \sum_{i \in I} d_i \dim_q V_i .$$

(The latter formula follows from the fact that $S^1 \times S^2$ is the result of surgery on S^3 along the unknotted circle with the zero framing).

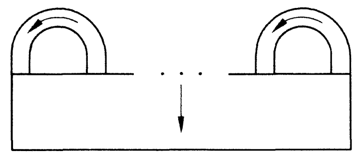


Fig. 6

The definition of the invariant F may be easily extended to the case of closed oriented 3-manifolds with some coloured ribbon graphs sitting inside. By a ribbon graph in an oriented 3-manifold M we mean an oriented surface S imbedded in M and decomposed as the union of a finite collection of directed ribbons and annuli so that conditions (*a*)–(*d*) of Sect. 2.3 hold true and each base of a (type 2) ribbon is contained in a base of a certain coupon (i.e. type 1 ribbon). This definition generalizes the definition of ribbon (0, 0)-graph in $\mathbf{R}^2 \times [0, 1]$. In particular, each framed oriented link in M gives rise to a ribbon graph in M consisting of annuli.

Colourings of ribbon graphs in M and the isotopy relation between coloured ribbon graphs are defined along the lines of Sect. 2.4 with the obvious changes. We cannot organize a category of coloured ribbon graphs in M since we do not consider graphs with border edges. (Such an extension of the theory is possible but we do not pursue this line here.)

Let M be a closed oriented connected 3-manifold and T be a coloured ribbon graph in M. As above, present M as the result of surgery on S^3 along a framed link L with components $L_1, \ldots, L_m$. Applying isotopy to T in M we may push T into $S^3 \backslash U \subset M$ where U is a tubular neighborhood of L in S^3. Thus, assume that $T \subset S^3 \backslash L$. If ω is an orientation of L and $\lambda \in \mathrm{col}(L)$ we may form the ribbon graph $\Gamma = \Gamma(L, \omega, \lambda)$ so that $\Gamma \subset U$. We obtain finally a coloured ribbon (0, 0)-graph $T \cup \Gamma$ in S^3 so that its isotopy invariant $F(T \cup \Gamma) \in K$ may be considered. Put

$$F(M, T; L, \omega) = C^{-\sigma_-(L)} \sum_{\lambda \in \mathrm{col}(L)} \prod_{i=1}^{m} d_{\lambda(L_i)} F(T \cup \Gamma(L, \omega, \lambda))$$

3.3.3. Theorem. *$F(M, T; L, \omega)$ is a topological invariant of the pair $F(M, T)$.*

Put $F(M, T) = F(M, T; L, \omega)$. In particular, (M, T) is an invariant of ambient isotopy of T in M. Clearly, $F(M, \phi) = F(M)$. The invariant $F(M, T)$ has the same multiplicativity property as $F(M)$:

$$F(M_1 \# M_2, T_1 \coprod T_2) = F(M_1, T_1) F(M_2, T_2)$$

where $T_1 \subset M_1$, $T_2 \subset M_2$. If $M = S^3$ then $F(M, T) = F(T)$. Using ribbon graphs consisting of annuli one may specialize Theorem 3.3.3 to obtain an isotopy invariant of framed oriented links in an arbitrary closed oriented 3-manifold.

Theorem 3.3.3 is proven in § 7.

4. Operator invariants of 3-dimensional cobordisms

4.1. Parametrized surfaces

Denote by $\bar{V}_k$ the planar rectangle (the coupon) with k ribbons glued to the top base, as depicted in Fig. 6. Denote by G^k the boundary of the regular neighborhood $U(V_k)$ of V_k in S^3. Provide G^k with the orientation induced by the right-handed orientation in $U(V_k)$. Clearly, G^k is an oriented closed connected surface of genus k. If G is another such surface then by a parametrization of G we shall mean an isotopy class of degree 1 homeomorphisms $G^k \to G$. An oriented closed surface G is said to be *parametrized* if the set of its components $\pi_0(G)$ is totally ordered and each component is provided with a parametrization. By the genus $g(G)$ of a parametrized surface G we mean the sequence of genera of components of G written down in accordance with the order in $\pi_0(G)$.

4.2. Category of 3-dimensional cobordisms

By a *3-dimensional cobordism* we mean a triple (M, G_1, G_2) where: M is an oriented compact 3-manifold; G_1, G_2 are parametrized oriented closed subsurfaces of ∂M with

$$G_1 \cap G_2 = \phi, \qquad G_1 \cup G_2 = \partial M, \qquad \partial[M] = [G_2] - [G_1]$$

(possibly G_1 or G_2 is empty, or even both are empty); the set of components of M is equipped with a total order compatible with the orders in $\pi_0(G_1)$, $\pi_0(G_2)$ (this means that the inclusion in: $\pi_0(G_r) \to \pi_0(M)$ has the property: if in$(a) <$ in(b) then $a < b$; $r = 1, 2$).

Two 3-dimensional cobordisms (M, G_1, G_2) and (M', G'_1, G'_2) are called homeomorphic if there exists a degree 1 homeomorphism

$$h: (M, G_1, G_2) \to (M', G'_1, G'_2)$$

such that for each $r = 1, 2$ the composition of $h|_{G_r}: G_r \to G'_r$ and the parametrization of G_r gives the parametrization of G'_r.

We define the category μ of 3-dimensional cobordisms. Its objects are finite sequences of non-negative integers. If g_1, g_2 are two such sequences then by an μ-morphism $g_1 \to g_2$ we mean a 3-dimensional cobordism (M, G_1, G_2) considered up to homeomorphism and such that $g(G_1) = g_1$, $g(G_2) = g_2$. The composition of two such morphisms (M, G_1, G_2), (M', G'_1, G'_2) with $g(G_2) = g(G'_1)$ is defined by gluing M and M' along $G_2 \approx G'_1$ via the parametrizations of G_2 and G'_1. The identities are presented by cylinders with the same parametrization of both bases. We provide the category μ with the tensor product defined by juxtaposition of sequences and disjoint union of cobordisms.

For each parametrized closed oriented surface G its genus $g(G)$ is an object of μ. By the abuse of language we will call the surface G itself an object of μ. Clearly, two such surfaces determine the same object of μ if and only if they are homeomorphic via a homeomorphism preserving both the order of components and parametrizations.

Remark. For any object $g = (g^1, \ldots, g^n)$ of μ one may consider the group $\mathrm{Aut}_\mu(g)$ of those μ-morphisms $g \to g$ which have two-sided inverses. It follows from standard facts of 3-dimensional topology that such morphisms are represented by cylinders over the surface of genus g with, possibly, different parametrizations of bases. This implies that

$$\mathrm{Aut}_\mu(g) = \mathop{\times}_{i=1}^{n} \mathrm{Aut}_\mu(g^i)$$

and that the group $\mathrm{Aut}_\mu(g^i)$ is canonically isomorphic to the Teichmüller modular group of genus g^i (cf. Sect. 4.6). This explains the role of the modular group in the present frameworks.

4.3. *Functor F*

Fix a modular Hopf algebra $(A, R, v, I, \{V_i\}, \ldots)$ over the field κ. In this section we construct a projective covariant functor $\mu \to \mathrm{Vect}(\kappa)$ where $\mathrm{Vect}(\kappa)$ is the category of finite dimensional κ-modules and κ-linear homomorphisms. The word "projective" means that the functor in question transforms composition of morphisms in μ into the composition of corresponding linear operators multiplied by a scalar. This functor generalizes the invariant of closed 3-manifolds introduced in Sect. 3.3.

For a sequence $\theta=(i_1, \ldots, i_k)\in I^k$ (cf. § 3) put $\theta\theta^*=(i, i_1^*, i_2, i_2^*, \ldots, i_k, i_k^*)$. Thus

$$\Omega^0_{\theta\theta^*}=\Omega^0_{i_1, i_1^*, i_2, i_2^*, \ldots, i_k, i_k^*}$$

where 0 is the preferred element of I (cf. Sect. 3.1).

For $k\geqq 1$ put

$$\Psi_k=\bigoplus_{\theta\in I^k}\Omega^0_{\theta\theta^*}.$$

Thus Ψ_k is a finite dimensional vector space over κ. Put also $\Psi_0=\kappa$.

We assign to a parametrized surface G with components $G_1, \ldots, G_n$ the vector space

$$F(G)=\Psi_{g_1}\otimes\Psi_{g_2}\otimes\ldots\otimes\Psi_{g_n}$$

where $g_i=g(G_i)$, $i=1, \ldots, n$. In particular, F associates with a connected parametrized surface G the vector space $\Psi_{g(G)}$. If $G=\phi$ then $F(G)=\kappa$.

To define operators corresponding to 3-dimensional cobordisms we need some preliminary constructions. Recall the planar surface V_k introduced in Sect. 4.1. Provide V_k with an orientation so that the preferred side of V_k is the one turned to the reader. Provide the coupon and ribbons of V_k with the directions shown in Fig. 6. Thus V_k gains the structure of a ribbon (0, 0)-graph. According to our definitions a colouring of V_k is a sequence $\theta=(i_1, \ldots, i_k)\in I^k$ of colours of the ribbons of V_k and an A-linear homomorphism (the colour of the coupon)

$$\kappa\to V_{i_1}\otimes V_{i_1}^\vee\otimes\ldots\otimes V_{i_k}\otimes V_{i_k}^\vee. \tag{4.3.1}$$

Since $V_0=\kappa$ each element

$$f\in\Omega^0_{\theta\theta^*} \tag{4.3.2}$$

gives rise (via the isomorphisms (3.1.3) and $\{w_i\}$) to an A-linear homomorphism (4.3.1). Thus f together with the sequence θ determines a colouring of V_k. It will be denoted by $\bar{f}$.

Denote by V_l' the mirror image of the surface V_l with respect to the horizontal plane (see Fig. 7, here $l\geqq 0$). Provide V_l' with the orientation so that the preferred side of V_l' is the one turned to the reader. Provide the coupon and ribbons

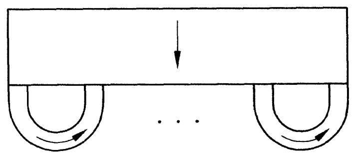


Fig. 7

of $\bar{V}_l'$ with the directions shown in Fig. 7. Thus $\bar{V}_l'$ becomes a ribbon (0, 0)-graph. A colouring of $\bar{V}_l'$ is a sequence $\eta=(j_1, \ldots, j_l) \in I^l$ of colours of the ribbons, and the colour of the coupon, i.e. an A-linear homomorphism

$$V_{j_1} \otimes V_{j_1}^{\vee} \otimes \ldots \otimes V_{j_l} \otimes V_{j_l}^{\vee} \to K. \tag{4.3.3}$$

Since $V_0 = K$ each element

$$h \in (\Omega^0_{\eta\eta^*})^{\vee} \tag{4.3.4}$$

gives rise to a A-linear homomorphism (4.3.3) annihilating

$$Z_{\eta\eta^*} \oplus \bigoplus_{i \in I \setminus \{0\}} (V_i \otimes \Omega^i_{\eta\eta^*}).$$

Thus h together with η determines a colouring of $\bar{V}_l'$. Denote this colouring by $\bar{h}$.

Let (M, G_1, G_2) be a 3-dimensional cobordism with connected M. Assume first that G_1, G_2 are connected surfaces of genus respectively k and l. Since G_1, G_2 are parametrized we have the degree 1 parametrization homeomorphisms, say, $\varphi: G^k \to G_1$ and $\psi: G^l \to G_2$. Glue the regular neighborhood $U(\bar{V}_k)$ of $\bar{V}_k \subset \mathbf{R}^3$ to M along $\varphi: \partial U(\bar{V}_k) = G^k \to G_1$. Glue the regular neighborhood $U(\bar{V}_l')$ of $\bar{V}_l' \subset \mathbf{R}^3$ to M along the homeomorphism $\partial U(\bar{V}_l') \to G_2$ which is the composition of the reflection in a horizontal plane $\partial U(\bar{V}_l') \to \partial U(\bar{V}_l) = G^l$ and ψ. The result of these gluings is a closed 3-manifold, say, $\bar{M}$ consisting of 3 pieces: M, $U(\bar{V}_k)$ and $U(\bar{V}_l')$. In particular, the ribbon graphs $\bar{V}_k$, $\bar{V}_l'$ are imbedded in $\bar{M}$. Provide $\bar{M}$ with the orientation which extends the given orientation in M and the right-handed orientations in $U(\bar{V}_k)$, $U(\bar{V}_l')$.

Each pair of sequences $\theta=(i_1, \ldots, i_k) \in I^k$, $\eta=(j_1, \ldots, j_l) \in I^l$ and each pair f, h as in 4.3.2, 4.3.4 determines a colouring $\bar{f} \coprod \bar{h}$ of the ribbon graph $\bar{V}_k \coprod \bar{V}_l'$ imbedded in $\bar{M}$. According to the results of Sect. 3.3 we may consider an element

$$F(\bar{M}, (\bar{V}_k \coprod \bar{V}_l', \bar{f} \coprod \bar{h}))$$

of K. Denote this element by $\langle f | M | h \rangle$. Theorem 3.3.3 implies that $\langle f | M | h \rangle$ is a topological invariant of the cobordism (M, G_1, G_2) and the pair f, h. It will be convenient to normalize this invariant. Put

$$\langle f | M | h \rangle' = \prod_{r=1}^{l} d_{j_r} \langle f | M | h \rangle \in K. \tag{4.3.5}$$

The formula $(f, h) \to \langle f | M | h \rangle'$ defines a bilinear pairing

$$\Omega^0_{\theta\theta^*} \times (\Omega^0_{\eta\eta^*})^\vee \to K.$$

The associated homomorphism $\Omega^0_{\theta\theta^*} \to \Omega^0_{\eta\eta^*}$ is the "block" of the desired operator. Denote this block by $F(M; \theta, \eta)$. Combined together, these blocks corresponding to arbitrary $\theta \in I^k$, $\eta \in I^l$ define the operator

$$F(M, G_1, G_2)\colon \Psi_k \to \Psi_l.$$

In the case of non-connected G_1, G_2 the definition goes along the same lines. One has to glue in as many handlebodies $U(V_*)$, $U(V'_*)$ as there are components in G_1, G_2. In the multiple $\prod_j d_j$ the index j should vary over the colours of all ribbons of all V'_*.

It follows directly from definitions that the operator $F(M, G_1, G_2)$ is invariant under homeomorphisms of cobordisms. This operator invariant generalizes the invariant of closed manifolds introduced in Sect. 3.3. Indeed, if $\partial M = \phi$, i.e. if $G_1 = G_2 = \phi$, then $F(M, G_1, G_2)$ is just the homomorphism $a \mapsto F(M)\, a\colon K \to K$.

If M is non-connected we define $F(M, G_1, G_2)$ to be the tensor product of operators corresponding to the components of M.

The definition of $F(M, G_1, G_2)$ given above appeals to the constructions of Sect. 3.3. We would like to describe a more direct procedure for computing this operator.

Assume first that G_1, G_2 are connected surfaces of genus resp. k and l. Present the manifold $\bar{M}$ constructed above as the result of surgery on S^3 along a framed link L with components $L_1, \ldots, L_m$. Provide L with an arbitrary orientation. Pushing the handlebodies $U(V_k)$, $U(V_l') \subset \bar{M}$ into $S^3 \backslash U(L) \subset \bar{M}$ we may assume that V_k and V_l' lie in $S^3 \backslash U(L)$. These imbeddings of V_k and V_l' in $S^3 \backslash L$ are possibly knotted. However, applying an isotopy, if necessary, we may assume that the coupons of V_k and V_l' lie respectively below and above the other parts of the picture (see Fig. 8). In the shaded region of Fig. 8 lie without intersections the components of L and the ribbons of V_k, V_l'. (These ribbons may be twisted, knotted, and linked with each other and with L. It is understood that the preferred side of the coupons is turned towards the reader.) Cutting out both coupons we get a ribbon $(2k, 2l)$-tangle Γ. Let $\theta = (i_1, \ldots, i_k)$, $\eta = (j_1, \ldots, j_l)$, as above. Provide the bottom k ribbons of Γ with the colours $i_1, \ldots, i_k$ and the top l ribbons of Γ with the colours $j_1, \ldots, j_l$. This colouring together with an arbitrary colouring λ of L yield a colouring of Γ denoted by $\bar{\lambda}^{\theta,\eta}$ or briefly by $\bar{\lambda}$. According to the results of §2 we have an A-linear operator

$$F(\Gamma, \bar{\lambda})\colon V_{i_1} \otimes V_{i_1}^\vee \otimes \ldots \otimes V_{i_k} \otimes V_{i_k}^\vee \to V_{j_1} \otimes V_{j_1}^\vee \otimes \ldots \otimes V_{j_l} \otimes V_{j_l}^\vee.$$

Let v_θ and ζ_η be respectively the inclusion and the projection

$$\Omega^0_{\theta\theta^*} \to V_{i_1} \otimes V_{i_1}^\vee \otimes \ldots \otimes V_{i_k} \otimes V_{i_k}^\vee,$$

$$V_{j_1} \otimes V_{j_1}^\vee \otimes \ldots \otimes V_{j_l} \otimes V_{j_l}^\vee \to \Omega^0_{\eta\eta^*}$$

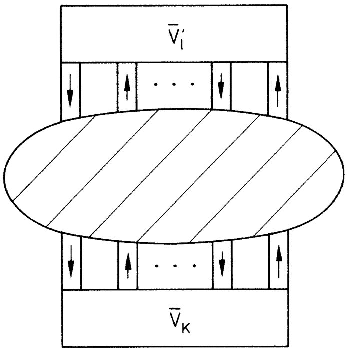


Fig. 8

provided by the axiom (3.1.3) and the isomorphisms $\{w_i\}$. Then the operator

$$(4.3.6) \qquad C^{-\sigma_-(L)} \prod_{r=1}^{l} d_{j_r} \sum_{\lambda \in \operatorname{col}(\Delta)} \left(\prod_{s=1}^{m} d_{\lambda(L_s)} (\zeta_\eta \circ F(\Gamma, \lambda) \circ v_\theta) \right)$$

is precisely the block

$$F(M; \theta, \eta) \colon \Omega^0_{\theta\theta*} \to \Omega^0_{\eta\eta*}$$

of the operator $F(M, G_1, G_2)$. In the case of non-connected G_1 (resp. G_2) there is a similar formula for $F(M, G_1, G_2)$ the only difference is that here we have several bottom (resp. top) coupons which should be drawn in a row (in accordance with the prescribed order in the set of components of G_1, resp. G_2) and then should be cut out to get a ribbon tangle.

It is easy to reconstruct the cobordism (M, G_1, G_2) with connected G_1, G_2 from the ribbon tangle Γ constructed in the preceding paragraph. This gives a convenient way to present 3-dimensional cobordisms with connected bases. More exactly, let us call a $(2k, 2l)$-ribbon tangle Γ *small if* for any ribbon Q of Γ the ends of Q lie either both on the line $\mathbf{R} \times 0 \times 0$ or both on $\mathbf{R} \times 0 \times 1$ and the directions of the ribbons of Γ are the same as in Fig. 8. Each small $(2k, 2l)$-tangle Γ gives rise to a 3-dimensional cobordism (M, G_1, G_2) with genus $G_1 = k$, genus $G_2 = l$. For this we just invert the construction described above: glue in 2 coupons (one on the top of Γ, one on the bottom of Γ) to get some imbeddings $\alpha\colon V_k \hookrightarrow S^3$, $\beta\colon V_l' \hookrightarrow S^3$ identical on the coupons of V_k, V_l'; surger S^3 along the framed link determined by the annulus components of Γ; cut out neighborhoods of $\alpha(V_k)$, $\beta(V_l')$ from the closed 3-manifold obtained by the surgery. The imbedding α and the composition of β with the reflection $V_l \to V_l'$ induce parametrizations of G_1 and G_2.

4.4. Lemma. *Let Γ, Γ' be resp. small ribbon $(2k, 2l)$-tangle and small ribbon $(2l, 2t)$-tangle, determining some morphisms $G_1 \to G_2$, $G_2 \to G_3$ of the category μ. Then $\Gamma' \circ \Gamma$ is a small ribbon $(2k, 2t)$-tangle which determines the composition $G_1 \to G_3$ of these two morphisms.*

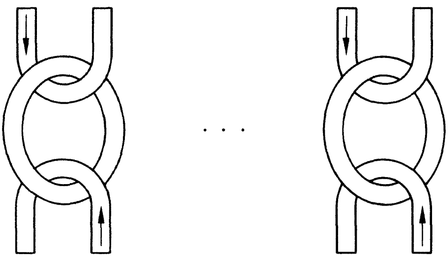


Fig. 9

This Lemma will be proved in Sect. 4.7. It has a natural extension to the case of non-connected G_1, G_2, G_3. We leave the precise formulation of this extension to the reader.

To give an example we consider the small ribbon tangle depicted in Fig. 9. It is easy to see that this tangle presents id_{G^k}, i.e. the cylinder $G^K \times [0, 1]$ with the identity parametrizations of both boundaries. (Remark that the cobordism associated with a small ribbon tangle does not depend on the directions of the annulus components of the tangle.) Unfortunately, we do not know in general if the operator $F(id_{G^k})$ acting in Ψ_k is the identity. Thus we cannot claim that F is a functor in the usual sense. On the other hand F has the following important property.

4.5. Theorem. *If a 3-dimensional cobordism (M, G_1, G_3) is a composition of two cobordisms (M_1, G_1, G_2) and (M_2, G_2, G_3) then for some integer n*

$$F(M, G_1, G_3) = C^n F(M_2, G_2, G_3) \circ F(M_1, G_1, G_2).$$

This Theorem will be proven in Sect. 4.8.

Theorem 4.5 implies in particular that for any parametrized surface G we have

$$(F(id_G))^2 = C^m F(id_G^2) = C^m F(id_G)$$

with $m \in \mathbf{Z}$. In other words up to a scalar multiple $F(id_G)$ is a projector acting in the vector space $F(G)$. Denote the image of this projector by $\bar{F}(G)$. For any μ-morphism $M: G \to G'$ we have

$$F(M) = C^m F(id_{G'}) \circ F(M) = C^n F(M) \circ F(id_G)$$

with $m, n \in \mathbf{Z}$. Therefore $F(M)$ maps $\bar{F}(G)$ into $\bar{F}(G')$. Associating with each parametrized surface G the vector space $\bar{F}(G)$ and with each μ-morphism $M: G \to G'$ the homomorphism $F(M)|_{\bar{F}(G)}$ we get a projective covariant functor $\bar{F}: M \to \mathrm{Vect}(\kappa)$. Note that $\bar{F}(G) \neq 0$ for any G. This follows from Theorem 4.4 and the fact that G is embeddable in S^3 and $F(S^3) = 1$. Note also that $\bar{F}(G_1 \coprod G_2) = \bar{F}(G_1) \otimes \bar{F}(G_2)$ and that for $G = \phi$ $\bar{F}(G) = \kappa$.

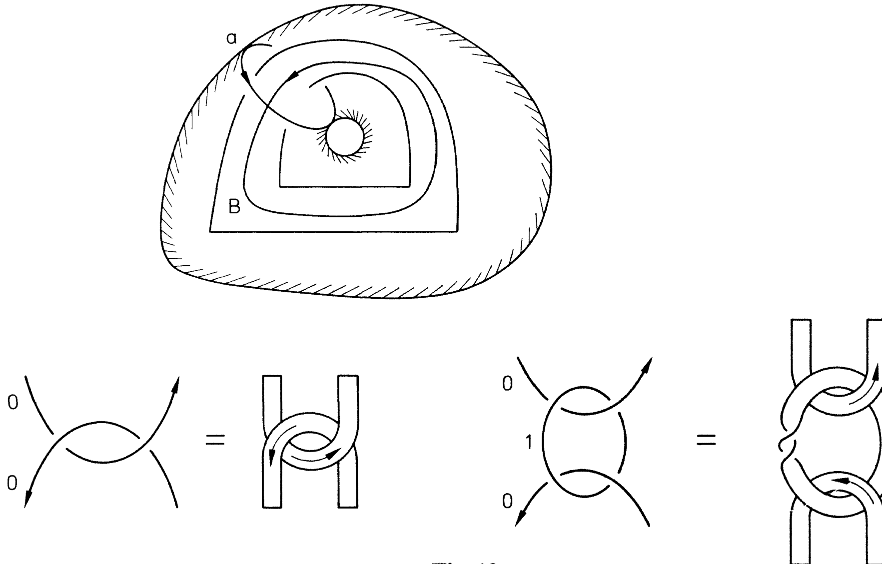


Fig. 10

4.6. *A representation of* Mod_k

Recall that the Teichmüller modular group Mod_k is the group of isotopy classes of orientation preserving homeomorphisms $G^k \to G^k$. The projective functor $\bar{F}$ constructed in Sect. 4.5 gives rise to a projective linear repesentation of Mod_k acting in $\bar{\Psi}_k = \bar{F}(G^k) \subset \Psi_k$. Indeed, each degree 1 homeomorphism $\alpha: G^k \to G^k$ enables us to parametrize both components of the boundary of the cylinder $G^k \times [0, 1]$: we parametrize $G^k \times 0$ via α and we parametrize $G^k \times 1$ via the identity. This yields a morphism, say, M_α of the category μ. Put

$$e(\alpha) = \bar{F}(M_\alpha) \in \mathrm{End}\, \bar{\Psi}_k.$$

If $\alpha, \beta \in \mathrm{Mod}_k$ then the cobordism $M_{\alpha \circ \beta}$ splits into a composition of M_β and a cobordism homeomorphic to M_α. Theorem 4.5 implies that $e(\alpha \circ \beta) = c^n e(\alpha) e(\beta)$ with $n \in \mathbf{Z}$. By the very definition of the space $\bar{\Psi}_k$ the operator $e(1)$ is the multiplication by a non-zero scalar. Thus e is a projective linear representation of Mod_k.

There is a canonical extension

$$0 \to \mathbf{Z} \to \widetilde{\mathrm{Mod}}_k \to \mathrm{Mod}_k \to 1$$

(see [A]) and one may show that e lifts to an honest linear representation of $\widetilde{\mathrm{Mod}}_k$. If C is a root of unity, $C^r = 1$, then e lifts to a linear representation of $\widetilde{\mathrm{Mod}}_k / r\mathbf{Z}$. Actually, one may explicitly compute the 2-cocycle corresponding to e.

The constructions of Sect. 4.3 enable one to present the cylinder M_α associated with $\alpha \in \mathrm{Mod}_k$ by a ribbon $(2k, 2k)$-tangle and to compute $e(\alpha)$ via the operators associated with colourings of this tangle. We give some examples of ribbon $(2, 2)$-tangles associated with the autohomeomorphisms of the torus G^1. We fix the basis a, b in $H_1(G^1) = \mathbf{Z} \oplus \mathbf{Z}$ depicted in Fig. 10. (Recall that $G^1 = \partial U(V_1)$).

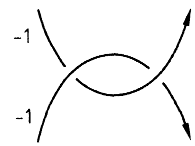


Fig. 11

Now the group Mod_1 may be canonically identified with $sl(2;\mathbf{Z})$. In Fig. 10, we present two (2, 2)-tangles which may be easily shown to give rise to elements of Mod_1 corresponding to the generators

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

of $sl(2;\mathbf{Z})$. (The integers 0, 1 in Fig. 10 are framings, cf. Sect. 2.3). Note that the composition of $(2k, 2k)$-tangles corresponds to the composition of associated elements of Mod_k. This enables one to construct a (2, 2)-tangle corresponding to an arbitrary element of Mod_1. Note that different tangles may present the same homeomorphism. For example, an application of Kirby $(+1)$-move (see § 6) shows that the tangle depicted in Fig. 11 also presents $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Note finally that the majority of small $(2k, 2k)$-tangles do not correspond to any homeomorphism of G^k.

4.7. *Proof of Lemma 4.4*

Consider the l framed circles $L_1, \ldots, L_l \subset S^3$ which are obtained by gluing the (cores of) top boundary ribbons of Γ with bottom boundary ribbons of Γ'. Each of these circles transversally hits the plane $\mathbf{R}^2 \times (1/2) \subset \mathbf{R}^3$ (along which Γ is glued to Γ') in exactly two points. It is convenient to complete this plane to a 2-sphere $S^2 = (\mathbf{R}^2 \times (1/2)) \cup \{\infty\} \subset S^3$. Take a narrow cylinder $S^2 \times [0, \varepsilon] \subset S^3$ over this 2-sphere. We may arrange $L_1, \ldots, L_l$ so that each of these components meet the cylinder in two vertical segments $\{pt\} \times [0, \varepsilon]$. When we form the 3-dimensional cobordism corresponding to $\Gamma' \circ \Gamma$ we surger S^3 along all annulus components of $\Gamma' \circ \Gamma$. In particular, we cut out regular neighborhoods $U_1, \ldots, U_l$ of $L_1, \ldots, L_l$ and glue in l solid tori, say $W_1, \ldots, W_l$. Put

$$N = \left((S^2 \times [0, \varepsilon]) \bigcup_{i=1}^{l} U_i \right) \bigcup_{\text{gluing}} \bigcup_{i=1}^{l} W_i .$$

The manifold N may be easily identified to be the cylinder $G_2 \times [0, 1]$ which is imbedded in the 3-cobordism corresponding to $\Gamma' \circ \Gamma$. When we cut N out of this cobordism there remain two connected pieces which may be easily identified with the cobordisms corresponding to our morphisms $G_1 \to G_2$ and $G_2 \to G_3$. The cylinder structure of N corresponds to gluing the two latter cobordisms along G_2. This proves the lemma.

4.8. Proof of Theorem 4.5

Assume for simplicity that G_1, G_2, G_3 are connected surfaces of genus resp. k, l, t. Present the cobordisms M_1, M_2 by small ribbon tangles resp. Γ_1 and Γ_2. According to Lemma 4.4 the cobordism M is presented by $\Gamma = \Gamma_2 \circ \Gamma_1$. Fix some sequences $\theta = (i_1, \ldots, i_k) \in I^k$, $\eta = (j_1, \ldots, j_t) \in I^t$.

We must prove that

$$F(M; \theta, \eta) = C^n \sum_{\rho \in I^l} F(M_2; \rho, \eta) \circ F(M_1; \theta, \rho) \tag{4.8.1}$$

where n does not depend on the choice of θ, η. To show this we shall use (4.3.5). Let L_1, L_2, L be the framed links formed by the cores of annuli of the tangles resp. $\Gamma_1, \Gamma_2, \Gamma$. Clearly L consists of L_1, L_2 and l circles which may be linked with L_1 and L_2. We fix some colourings λ_1, λ_2 or L_1, L_2 and a sequence $\rho \in I^l$ which determines the colours of these l circles. Let λ be the resulting colouring of L. We claim that

$$\zeta_\eta \circ F(\Gamma, \bar{\lambda}^{\theta, \eta}) \circ v_\theta = (\zeta_\eta \circ F(\Gamma_2, \bar{\lambda}_2^{\rho, \eta}) \circ v_\rho (\zeta_\rho \circ F(\Gamma_1, \bar{\lambda}_1^{\theta, \rho}) \circ v_\theta). \tag{4.8.2}$$

This follows from (3.1.3) and 3 facts: (i) since F is a covariant functor,

$$F(\Gamma, \bar{\lambda}^{\theta, \eta}) = F(\Gamma_2, \bar{\lambda}_2^{\rho, \eta}) \circ F(\Gamma_1, \bar{\lambda}_1^{\theta, \rho});$$

(ii) each composition of A-linear homomorphisms

$$K = V_0 \to V_i \to K$$

equals 0 unless $i = 0$; (iii) each composition of A-linear homomorphisms

$$K \to Z_{\rho\rho^*} \to K$$

equals 0.

Multiplying both parts of (4.8.2) by $\prod_{r=1}^{t} d_{j_r} \times \prod_s d_{\lambda(L^s)}$ where L^s runs over all components of L we will get (4.8.1) with $n = \sigma_-(L_1) + \sigma_-(L_2) - \sigma_-(L)$. (It is here that the multiple $\prod_r d_{j_r}$ which was artificially introduced in (4.3.5) plays its role). This finishes the proof.

4.9. Remarks. 1. The operator invariants of cobordisms introduced in Sect. 4.3 formally depend on the choice of decomposition (3.1.3). Actually it is easy to see that they do not depend on this choice.

2. One may easily extend the definition of the functor F to the case of 3-cobordisms equipped with coloured ribbon graphs sitting inside. Moreover, one may generalize our constructions to graphs with boundary edges whose ends lie on the boundary of the cobordism. To this aim one needs to define the vector space $F(G)$ where G is a connected closed oriented surface provided

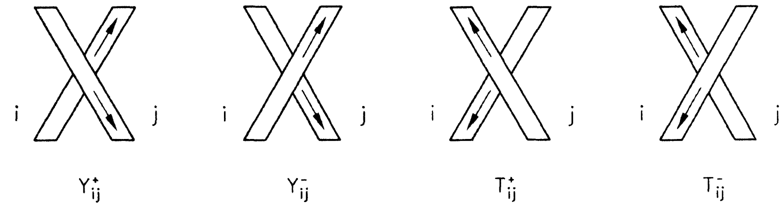


Fig. 12

with a finite set of points $x_1, \ldots, x_n$ each x_r being also provided with a colour $j_r \in I$. We put

$$F(G) = \bigoplus_{i_1, \ldots, i_m \in I} \Omega^0_{j_1, \ldots, j_n, i_1, i_1^*, \ldots, i_m, i_m^*},$$

where $m = \text{genus}\,(G)$. The details are left to the reader.

3. With an arbitrary non-parametrized closed oriented surface G we may associate a projective space. Namely, consider all parametrizations α of G and identify the corresponding vector spaces $\bar{F}(G, \alpha)$ via the action e of the modular group:

$$e(\alpha^{-1} \circ \alpha'): \bar{F}(G, \alpha') \xrightarrow{\approx} \bar{F}(G, \alpha).$$

Since e is a projective action this gives a projective space.

5. Proof of Lemma 3.3.1

5.1. Lemma. *Let Γ be a coloured ribbon graph in S^3. Let p be an annulus of Γ coloured by $i \in I$. Let Γ' be the coloured ribbon graph obtained from Γ by reversing the direction of p and replacing the colour i of p by i^*. Then $F(\Gamma') = F(\Gamma)$.*

Proof. The claim of the Lemma has the following local version:

(5.1.1) $$F(J_{i^*}^-) = w_{i^*}^{-1} F(J_i^+) w_{i^*},$$

(5.1.2) $$F(J_{i^*}^+) = w_i F(J_i^-) w_i^{-1},$$

(5.1.3) $$F(d_{i^*}) = (w_{i^*}^{-1} \otimes w_i) F(c_i),$$

(5.1.4) $$F(b_{i^*}) = F(a_i)(w_i^{-1} \otimes w_{i^*}),$$

(5.1.5) $$F(Y^\varepsilon_{i^*, j}) = (1 \otimes w_{i^*}^{-1}) F(X^{-\varepsilon}_{i,j})(w_{i^*} \otimes 1),$$

(5.1.6) $$F(T^\varepsilon_{j, i^*}) = (w_{i^*}^{-1} \otimes 1) F(X^{-\varepsilon}_{j,i})(1 \otimes w_{i^*})$$

where $\varepsilon = \pm 1$ and X, Y, T are the (2, 2)-tangles shown in Fig. 3 and 12. Since an arbitrary coloured ribbon graph may be obtained from $J^\pm$, a, b, c, d, $X^\pm$ and the graphs $\Gamma(f, \eta, \eta')$ via composition and tensor product, the "local" equalities above imply the claim of the Lemma. (It is, of course, important that p misses the coupons of Γ, so that the coupons of Γ' and Γ are the same).

The equalities (5.1.1, 5.1.2) are obvious since $F(J_i^+)$ and $F(J_i^-)$ are identity endomorphisms. Let us prove (5.1.4). If $x\in V_{i*}$, $y\in V_{i*}^\vee$ then

$$F(a_i)(w_i^{-1}\otimes w_{i*})(x\otimes y)=w_{i*}(y)(w_i^{-1}(x))=y((w_{i*}^*\circ w_i^{-1})(x))$$
$$=y(uv^{-1}x)=F(b_{i*})(x\otimes y).$$

The proof of (5.1.3) is also a direct combination of definitions and the axiom (3.1.2). Let us prove (5.1.6) with $\varepsilon=1$. A direct calculation shows that the R.H.S. of (5.1.6) applied to $a\otimes b\in V_j\otimes V_{i*}^\vee$ yields

$$\sum_s \rho_{V_{i*}^\vee}(\alpha_s')(b)\otimes\rho_{V_j}(\beta_s')(a)\in V_{i*}^\vee\otimes V_j \tag{5.1.7}$$

where $R^{-1}=\sum_s \alpha_s'\otimes\beta_s'$. In particular, the R.H.S. of (5.1.6) actually does not depend on the choice of $\{w_i\}$. Fix a basis $\{e_m\}$ in V_{i*} and the dual basis $\{e^m\}$ in $V_{i*}^\vee$. Clearly

$$T_{j,i*}^+=(J_{i*}^-\otimes J_j^+\otimes b_{i*})\circ(J_{i*}^-\otimes X_{i*,j}^+\otimes J_{i*}^-)\circ(d_{i*}\otimes J_j^+\otimes J_{i*}^-).$$

Then an easy computation shows that

$$F(T_{j,i*}^+)(a\otimes b)=\sum_{m,r} e^m\otimes\rho_{V_j}(\beta_r)\,a\,\langle\rho_{V_{i*}}(u\alpha_r u^{-1})\,e_m,b\rangle$$

where $R=\sum_r \alpha_r\otimes\beta_r$ and $\langle e,b\rangle=b(e)$. (Here we have used that v lies in the centre of A). Now we have (cf. Sect. 2.2)

$$\sum_m e^m\langle\rho_{V_{i*}}(u\alpha_r u^{-1})\,e_m,b\rangle=(\rho_{V_{i*}}(\gamma^2(\alpha_r))^*\,b=\rho_{V_{i*}^\vee}(\gamma(\alpha_r))(b).$$

Thus

$$F(T_{j,i*}^+)(a\otimes b)=\sum_r \rho_{V_{i*}^\vee}(\gamma(\alpha_r))\,b\otimes\rho_{V_j}(\beta_r)\,a. \tag{5.1.8}$$

In view of (2.1.1) we have

$$\sum_r \gamma(\alpha_r)\otimes\beta_r=\sum_s \alpha_s'\otimes\beta_s'.$$

Therefore (5.1.7) and (5.1.8) imply (5.1.6) with $\varepsilon=1$. The case $\varepsilon=-1$ and the formula (5.1.5) are considered along the same lines.

5.2. Proof of Lemma 3.3.1

Lemma 2.6 implies that the invariant F of the trivial knot with framing 0 and with coulour $i\in I$ is equal to $\dim_q V_i$. Now Lemma 5.1 shows that

(5.2.1) $$\dim_q(V_{i*})=\dim_q(V_i).$$ (5.2.1)

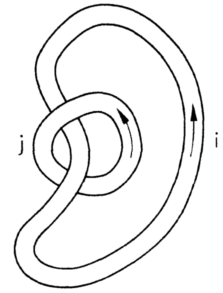


Fig. 13

Let us show that $v_{i*}=v_i$. Since V_{i*} is isomorphic to $V_i^\vee$ over A we see that the endomorphism $x\mapsto\rho_{V_i^\vee}(v)\,x$ of $V_i^\vee$ is actually the multiplication by v_{i*}. One the other hand, for $x\in V_i^\vee$

$$\rho_{V_i^\vee}(v)\,x=(\rho_{V_i}(\gamma(v)))^*(x)$$

(see Sect. 2.2). Since $\gamma(v)=v$ and $\rho_{V_i}(v)$ is the multiplication by v_i we have

(5.2.2) $$v_{i*}=v_i.$$

Lemma 2.6 implies that the scalar $S_{i,j}$ introduced in Sect. 3.1 is equal to $F(H_{i,j})$ where $H_{i,j}$ is the Hopf link depicted in Fig. 13. Inversing directions of both components and trading i,j for i^*,j^* we get H_{i^*,j^*}. Lemma 5.1 implies that

(5.2.3) $$S_{i,j}=S_{i^*,j^*}.$$

The equalities (5.2.1–3) and the equation (*) of Sect. 3.1 imply that $d_{i*}=d_i$ for all $i\in I$. This fact together with Lemma 5.1 imply the claim of the Lemma.

6. The Kirby calculus

6.1. Kirby's moves

It is easy to see that one and the same closed oriented 3-manifold may be obtained from S^3 by surgeries along different links. R. Kirby [K] introduced certain transformations (moves) on framed links and proved that two framed links L, L' determine the same (up to degree 1 homeomorphism) closed 3-manifold if and only if L' may be obtained from L by such transformations. The hard part "only if" of this theorem is based on J. Cerf's theory of critical points of families of Morse functions.

R. Fenn and C. Rourke [FR] introduced another set of transformations on framed links which generate the same equivalence relation as the Kirby transformations but have the advantage of being local, i.e. they proceed entirely inside a small ball which intersects the link in a standard fashion. We will follow here the approach of Fenn-Rourke (see also D. Rolfsen [R]).

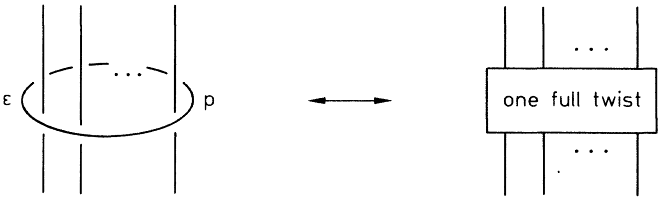


Fig. 14

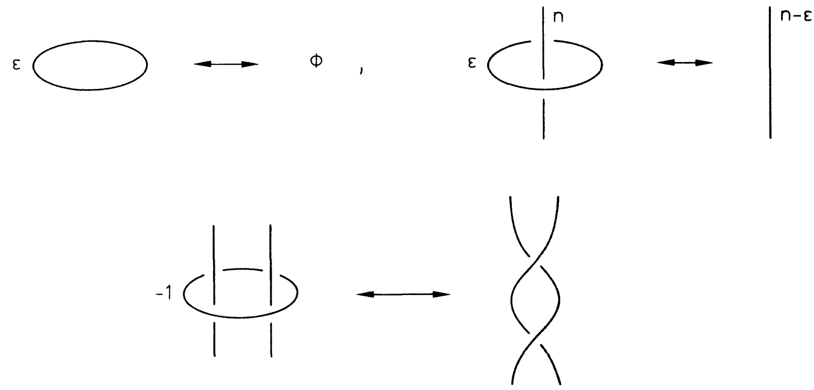


Fig. 15

Let $\varepsilon = \pm 1$. Two framed links L, L' are related by a *Kirby ε-move* if they are identical except for the pieces shown in Fig. 14. Here p is an unknotted component of L with framing ε. This component disappears in L'. The box denotes the full left hand twist if $\varepsilon = 1$ and the full right hand twist if $\varepsilon = -1$ applied to the bunch of vertical strings. If a component L_i of L distinct from p has a framing n then the corresponding component of L' has the framing $n - \varepsilon(lk(L_i, p))^2$ where lk denotes the linking coefficient. Some examples of the local pictures in question are given in Fig. 15. Note that by the Kirby move we mean both the passage from L to L' and the passage from L' to L. In particular, elimination or insertion of an unknotted component with the framing $\varepsilon = \pm 1$ which lies in a ball in the complement of other components is a Kirby ε-move, called special Kirby ε-move.

To understand the action of Kirby moves on the framings it is convenient to use the language of ribbons. Recall that (considered up to isotopy) framed links bijectively correspond to ribbon graphs consisting of annuli. In this language the strings of L piercing the disc bounded by p in Fig. 14 are narrow ribbons. Applying to these ribbons the full twist as above we get a new system of annuli which corresponds to the link L' with the framing described above (cf. Fig. 16).

6.2. Theorem ([K], [FR]). *The closed oriented 3-manifolds obtained from S^3 by surgeries along framed links L_1, L_2 are degree 1 homeomorphic if and only if L_2 may be obtained from L_1 by Kirby $(\pm)$-moves.*

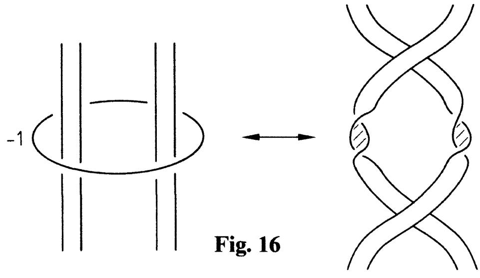


Fig. 16

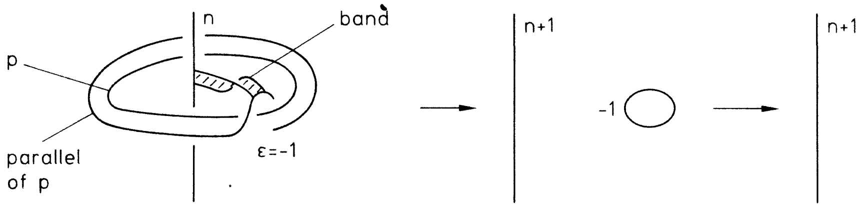


Fig. 17

We will need a slightly improved version of this Theorem.

6.3. Theorem. *The closed oriented 3-manifolds obtained from* S^3 *surgeries along framed links* L_1, L_2 *are degree 1 homeomorphic of and only if* L_2 *may be obtained from* L_1 *by Kirby* (± 1)*-moves and special Kirby* (-1)*-moves.*

Proof. It suffices to present Kirby (-1)-moves as compositions of $(+1)$-moves and special (-1)-moves. Recall first the β-move on framed links, which replaces a component by its band sum with the parallel of another component (see [K]). More exactly, let α, μ be two components of a framed link L with the framings resp. m, n. Let α' be the parallel of α twisted m times around α. Let b: $[0, 1] \times [0, 1] \to S^3$ be a ribbon in S^3 disjoint from L except that one base of b lies on μ and the other base of b lies on α'. Replace μ by the band sum

$$\mu \#_b \alpha' = \mu \cup \alpha' \cup b(\{0, 1\} \times [0, 1]) \backslash b([0, 1] \times \{0, 1\})$$

provided with the framing $m+n+2lk(\alpha, \mu)$. Here to compute lk we take orientations of α, μ compatible with a certain orientation of the ribbon b. This move on L is called β-move. (Its effect on the 4-manifold D_L is to isotope the attaching map of the 2-handle corresponding to μ over the 2-handle corresponding to α. This transformation does not change the homeomorphism types of this 4-manifold and of its boundary. The β-move together with special $(\pm)$-moves were originally used by Kirby to generate the equivalence relation on framed links corresponding to homeomorphisms of 3-manifolds).

The arguments of [FR, page 5] show that if α is an unknotted component with non-positive framing then each β-move as above may be presented as the composition of several Kirby $(+1)$-moves. It is easy to decompose any Kirby (-1)-move in the composition of such type β-moves and one special (-1)-move, see for instance Fig. 17. This implies our claim.

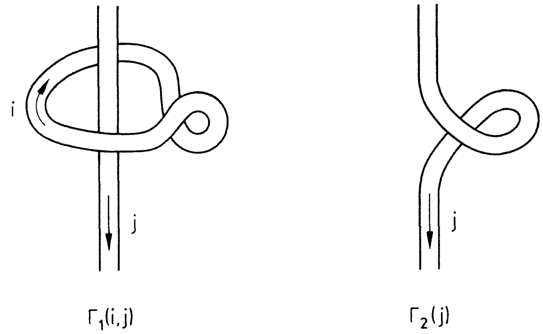


Fig. 18

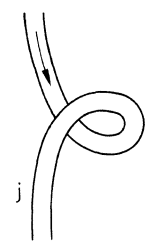


Fig. 19

7. Proof of Theorems 3.3.2 and 3.3.3

7.1. Lemma. *Let* $\Gamma_1(i,j)$ *and* $\Gamma_2(j)$ *be the coloured ribbon* (1, 1)*-tangles pictured in Fig.* 18. *Then*

$$\sum_{i\in I} d_i F(\Gamma_1(i,j)) = F(\Gamma_2(j)) \in \operatorname{End}_A V_j. \tag{7.1.1}$$

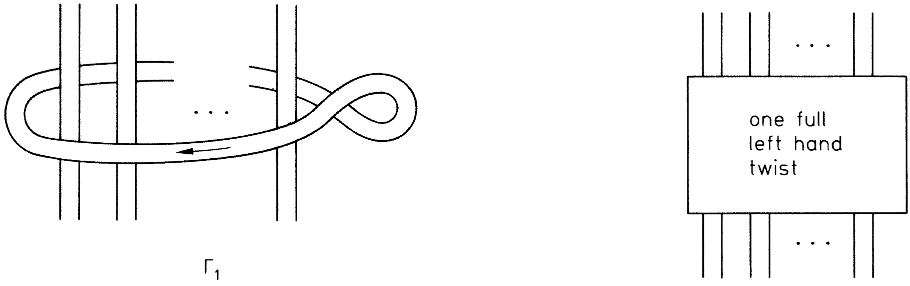
Proof. Since V_j is an irreducible A-module both $F(\Gamma_1(i,j))$, and $F(\Gamma_2(j))$ are multiplications by some scalars resp. $g_{i,j}$, $h_j \in \kappa$. According to computations of [RT, Sect. 5.4], $h_j = v_j^{-1}$, and also the operator $V_j \to V_j$ corresponding to the ribbon (1, 1)-tangle pictured in Fig. 19 is the multiplication by v_j. Recall the ribbon link $H_{i,j}$ introduced in Fig. 13. Lemma 2.6 implies that

$$g_{i,j} \dim_q V_j = v_i F(H_{i,j}) = v_i S_{i,j}.$$

Therefore

$$\sum_{i\in I} d_i g_{i,j} = \sum_{i\in I} d_i v_i S_{i,j} (\dim_q V_j)^{-1} = v_j^{-1}.$$

This implies (7.1.1).



Γ_1

Fig. 20

Γ_2

7.2. Proof of Theorem 3.3.2

In view of Theorem 6.3 it suffices to verify that the Kirby moves on L do not change $F(M; L)$. Assume first that two framed links L and L' in S^3 are related by a special Kirby ε-move. Let p be the unknotted circle of L with the framing ε.

Let p_i be the same circle equipped with the colour $i \in I$. Fix an orientation ω of L and the induced orientation ω' of L'. In view of (2.5.1) for any $\lambda \in \mathrm{col}(L')$ we have

$$F(\Gamma(L', \omega', \lambda)) = F(\Gamma(L, \omega, \lambda|_L)) F(p_i)$$

where $i = \lambda(p)$. Note also that $\sigma_-(L') = \sigma_-(L)$ if $\varepsilon = 1$ and $\sigma_-(L') = \sigma_-(L) + 1$ if $\varepsilon = -1$. Thus we have

$$F(M; L') = F(M; L) C^{(\varepsilon-1)/2} \sum_{i \in I} d_i F(p_i).$$

Lemma 2.6 and the interpretation of v_i mentioned in Sect. 7.1 show that

$$F(p_i) = v_i^{\varepsilon} \dim_q V_i.$$

If $\varepsilon = -1$ then

$$\sum_{i \in I} d_i F(p_i) = \sum_{i \in I} d_i v_i^{-1} \dim_q V_i = C,$$

and therefore $F(M; L') = F(M; L)$. To establish the same equality in the case of $\varepsilon = 1$ we have to show that

$$\sum_{i \in I} d_i v_i \dim_q V_i = 1.$$

The latter formula follows from (*), (Sect. 3.1) with $j = 0$. Indeed, $V_0 = \kappa$ and, as it is easy to see, $f_{i,0} = \dim_q V_i$, $v_0 = 1$, $\dim_q V_0 = 1$.

Let L, L' be two framed links in S^3 related by the Kirby $(+1)$-move such that the small disc in S^3 bounded by the unknotted component p of L is pierced by k branches of L. Provide L' with an arbitrary orientation and provide L with the induced orientation in $L \backslash p$, while p is oriented as in Fig. 20. Present L as the closure of $\Gamma \circ \Gamma_1$ where T_1 is the ribbon (k, k)-tangle presented in Fig. 20 and Γ is a certain ribbon (k, k)-tangle. Present L' as the closure of $\Gamma \circ \Gamma_2$ where Γ_2 is the ribbon (k, k)-tangle drawn in Fig. 20 and consisting of k ribbons.

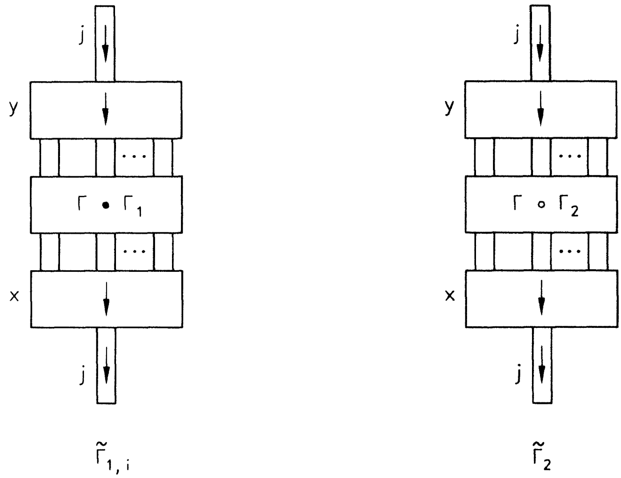


Fig. 21

Fix a colouring λ of L'. For $i \in I$ denote by λ_i the colouring of L which coincides with λ on $L \backslash p$ and associates i with p. We shall prove that

$$\sum_{i \in I} \{L\}_{\lambda_i} = \{L'\}_\lambda. \tag{7.2.1}$$

This will imply that $\{L\} = \{L'\}$, and since $\sigma_-(L) = \sigma_-(L')$ we will get $F(M; L) = F(M; L')$.

Denote by Λ_i and Λ the colourings of the tangles $\Gamma \circ \Gamma_1$ and $\Gamma \circ \Gamma_2$ induced respectively by λ_i and λ. To prove (7.2.1) it suffices to show that

$$\sum_{i \in I} d_i \operatorname{tr}_q (F(\Gamma \circ \Gamma_1, \Lambda_i)) = \operatorname{tr}_q (F(\Gamma \circ \Gamma_2, \Lambda)). \tag{7.2.2}$$

Note that in case $k=1$ the latter equality directly follows from Lemma 7.1, the equality $F(\Gamma \circ \Gamma') = F(\Gamma) \circ F(\Gamma')$ and additivity of tr_q. The following argument works for any $k \geq 1$.

Let $i_1, \ldots, i_k$ be the λ-colours of the bottom border ribbons of Γ_1 (and Γ_2). Let $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$ be the directions of these ribbons (as usual, $+1$ means down and -1 means up). Let x and y be an arbitrary pair of A-linear homomorphisms

$$V_j \xrightarrow{x} V_{i_1}^{\varepsilon_1} \otimes V_{i_2}^{\varepsilon_2} \otimes \ldots \otimes V_{i_k}^{\varepsilon_k} \xrightarrow{y} V_j$$

where $j \in I$ and, as usual, $V^{+1} = V$, $V^{-1} = V^{\vee}$. We first prove the following operator equality:

$$\sum_{i \in I} d_i (y \circ F(\Gamma \circ \Gamma_1, \Lambda_i) \circ x) = y \circ F(\Gamma \circ \Gamma_2, \Lambda) \circ x. \tag{7.2.3}$$

Clearly, $y \circ F(\Gamma \circ \Gamma_1, \Lambda_i) \circ x = F(\tilde{\Gamma}_{1,i})$ and $y \circ F(\Gamma \circ \Gamma_2, \Lambda) \circ x = F(\tilde{\Gamma}_2)$ where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are (1, 1)-tangles pictured in Fig. 21 and coloured respectively via Λ_i and Λ.

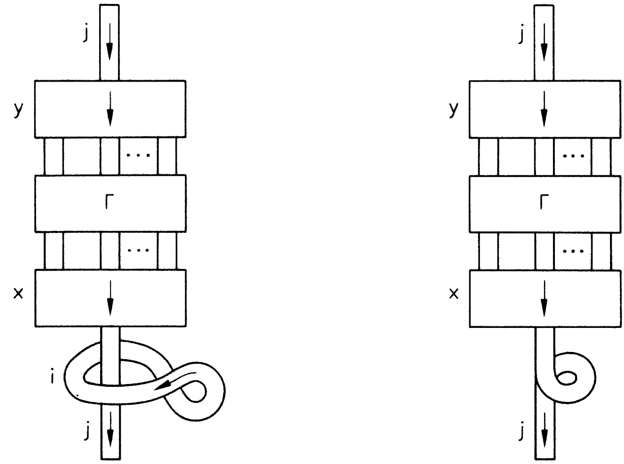


Fig. 22

These tangles are isotopic respectively to tangle $\tilde{\Gamma}'_{1,i}$ and $\tilde{\Gamma}'_2$ pictured in Fig. 22. The equality

$$\sum_{i\in I} d_i F(\tilde{\Gamma}'_{1,i}) = F(\tilde{\Gamma}'_2)$$

directly follows from (7.1.1) and functoriality of F. This implies (7.2.3).

Put

$$V = V_{i_1}^{\varepsilon_1} \otimes V_{i_2}^{\varepsilon_2} \otimes \ldots \otimes V_{i_k}^{\varepsilon_k}.$$

In view of (3.1.3) and existence of A-linear isomorphisms $\{V_i^\vee \to V_{i*}\}$ we have an A-splitting

$$V = Z_\theta \otimes \bigoplus_{j\in I} (V_j \otimes \Omega_\theta^j)$$

with $\theta = (i_1^{\varepsilon_1}, \ldots, i_k^{\varepsilon_k})$ where $i^{+1} = i$ and $i^{-1} = i^*$. We need some generalities on the computation of tr_q for an A-linear homomorphism $\varphi: V \to V$. Clearly

$$\mathrm{tr}_q\, \varphi = \mathrm{tr}_q(r \circ \varphi \circ q) + \sum_{i\in I} \mathrm{tr}_q(r_j \circ \varphi \circ q_j)$$

where q, q_j are the inclusions $Z_\theta \hookrightarrow V$, $V_j \otimes \Omega_\theta^j \hookrightarrow V$ and r, r_j are the projections $V \to Z_\theta$, $V \to V_j \otimes \Omega_\theta^j$. Axiom (3.1.4) implies that $\mathrm{tr}_q(r \circ \varphi \circ q) = 0$. Schur's lemma shows that

$$r_j \circ \varphi \circ q_j\colon V_j \otimes \Omega_\theta^j \to V_j \otimes \Omega_\theta^j.$$

equals $id \otimes \varphi_j$ where φ_j is a certain κ-linear endomorphism of Ω_θ^j. Therefore

$$\mathrm{tr}_q(r_j \circ \varphi \circ q_j) = \dim_q(V_j)\, \mathrm{Tr}(\varphi_j)$$

where Tr is the ordinary trace. Fix a basis $\{e^j_m\}_m$ in Ω^j_θ. With respect to this basis φ_j is presented by a matrix $(\varphi_j^{m,n})$ over κ and $\mathrm{Tr}\,\varphi_j=\sum_m \varphi_j^{m,m}$. We have

$$V_j\otimes\Omega^j_\theta=\bigoplus_m (V_j\otimes\kappa e^j_m),$$

so that each basis vector e^j_m gives rise to an embedding, say, $x^j_m\colon V_j\to V_j\otimes\Omega^j_\theta$ and to a projection, say $y^j_m\colon V_j\otimes\Omega^j_\theta\to V_j$. The composition $y^j_m\circ r_j\circ\varphi\circ q_j\circ x^j_m\colon V_j\to V_j$ is the multiplication by $\varphi_j^{m,m}$. Therefore

$$\dim_q(V_j)\,\mathrm{Tr}(\varphi_j)=\sum_m \mathrm{tr}_q(y^j_m\circ r_j\circ\varphi\circ q_j\circ x^j_m).$$

Finally, we get

$$\mathrm{tr}_q\,\varphi=\sum_{j\in I}\sum_m \mathrm{tr}_q((y^j_m\circ r_j)\circ\varphi\circ(q_j\circ x^j_m)).$$

This formula applied to $\varphi=F(\Gamma\circ\Gamma_1,\Lambda_i)$ and $\varphi=F(\Gamma\circ\Gamma_2,\Lambda)$ and the formula (7.2.3) imply (7.2.2). (Actually we use not the operator equality (7.2.3) but rather the equality of corresponding quantum traces).

7.3. Proof of Theorem 3.3.3

Each coloured ribbon (0, 0)-graph T lying in the exterior of a framed link $L\subset S^3$ survives the surgery of S^3 along L and determines thereby a ribbon graph T_L in the 3-manifold $M_L=\partial D_L$ obtained by the surgery. This construction gives all coloured ribbon graphs in M_L considered up to isotopy (cf. Sect. 3.3). Here are 3 moves on the pair (L, T) which do not change (M_L, T_L) up to degree 1 homeomorphisms: (i) Kirby β-moves on L such that the band of the move does not hit T (cf. Sect. 6.3); (ii) special Kirby (± 1)-moves applied to L far away from T; (iii) the analogues of β-moves which replace a ribbon of T by its band sum with a component of L. (The latter operation does not change (M_L, T_L) since it amounts to an isotopy of T_L in M_L which slides the ribbon over a 2-handle of D_L). Conversely if the pair (M_L, T_L) is degree 1 homeomorphic to another such pair $(M_{L'}, T_{L'})$ then (L', T') may be obtained from (L, T) by the operations (i–iii). This follows from Kirby's argument given in [K, §2]. Indeed the graph may be always isotoped far apart from the band connecting 2 components of L. So the graph does not produce any obstruction to make the handle sliding and to deform thereby the Morse functions as in [K, end of §2].

The same argument as in [FR] shows that the system of moves (i–iii) is equivalent to Kirby (± 1)-moves in which the strings piercing the disc are permitted to be not only the (segments of) components of L but also ribbons of T. These latter moves are purely local and so all the arguments of §§5–7 apply. This finishes the proof of the theorem.

8. Modular Hopf algebra U_t

8.1. Hopf algebra U_t

For a non-zero $q\in\mathbf{C}$ one defines the Hopf algebra $U_q(sl_2)$ which is a q-deformation of the universal enveloping algebra of the Lie algebra $sl_2(\mathbf{C})$ (see [KR],

[Dr], [Ji], [FRT], [KiR]). We will consider a quotient Hopf algebra U_t defined in the case when q is a root of unity. Specifically, let $t=\exp(\pi\sqrt{-1}\,m/2r)$ where m, r are mutually prime integers with odd m and $m\geqq 1$, $r\geqq 2$, $q=t^4$. We define U_t to be the associative algebra over the cyclotomic field $\mathbf{Q}(t)$ with 4 generators K, K^{-1}, X, Y subject to the following relations:

$$XY-YX=\frac{K^2-K^{-2}}{t^2-t^{-2}}, \tag{8.1.1}$$

$$XK=t^{-2}KX, \quad YK=t^2KY, \quad KK^{-1}=K^{-1}K=1 \tag{8.1.2}$$

$$K^{4r}=1, \quad X^r=Y^r=0. \tag{8.1.3}$$

The relations (8.1.1, 8.1.2) define the algebra $U_q(sl_2)$ (see, for instance, [KiR] where X and Y are denoted respectively by X_+ and X_- and $t^4=q$). Thus U_t is a quotient of $U_q(sl_2)$. It is easy to see that U is finite dimensional over $\mathbf{Q}(t)$. Note, however, that U_t is not semisimple.

The known structure of Hopf algebra in $U_q(sl_2)$ induces a structure of Hopf algebra in U_t. The action of comultiplication Δ, counit ε and the antipode γ is given on the generators by the following formulas:

$$\begin{aligned}
&\Delta(X)=X\otimes K+K^{-1}\otimes X,\\
&\Delta(Y)=Y\otimes K+K^{-1}\otimes Y,\\
&\Delta(K)=K\otimes K,\\
&\gamma(K)=K^{-1}, \quad \gamma(X)=-t^2X, \quad \gamma(Y)=-t^{-2}Y,\\
&\varepsilon(K)=1, \quad \varepsilon(X)=\varepsilon(Y)=0.
\end{aligned} \tag{8.1.4}$$

The known structure of (topological) ribbon Hopf algebra in $U_q(sl_2)$ induces a structure of ribbon Hopf algebra in U_t. In particular the universal R-matrix $R\in U_t\otimes U_t$ is given by the following formulas. For $p=0, 1, \ldots, 4r-1$ put

$$h_p=\prod_{\substack{l=0\\ l\neq p}}^{4r-1}\frac{K-t^l}{t^p-t^l}=\frac{1}{4r}\,t^p\prod_{l\neq p}^{4r-1}(K-t^l)$$

Put

$$R=\sum_{p=0}^{4r-1}\sum_{n=0}^{r-1} t^{n(n-1)}\frac{(1-q^{-1})^n}{[n]!}\,h_p(KX)^n\otimes K^p(K^{-1}Y)^n,$$

where $q=t^4$ and

$$[n]=\frac{t^{2n}-t^{-2n}}{t^2-t^{-2}}=\frac{\sin(\pi mn/r)}{\sin(\pi m/r)}$$

and $[n]!=[n][n-1]\ldots[2][1]$.

The element $u\in U_t$ associated with the R-matrix as in Sect. 2.1 is easily computed:

$$u=\sum_{p=0}^{4r-1}\sum_{n=0}^{r-1}\frac{(1-q^{-1})^n}{[n]!}\,t^{-p^2}(-KY)^n h_p(KX)^n t^{n(n-1)}$$

(This follows from the definition of u and the equalities

$$h_p K = K h_p = t^p h_p$$

so that

$$\gamma(K^p) h_p = K^{-p} h_p = t^{-p^2} h_p.$$

The formulas (8.1.4) imply that for any $a \in U_t$

$$\gamma^2(a) = K^2 a K^{-2}.$$

According to the results quoted in Sect. 2.1, $\gamma^2(a) = uau^{-1}$. Therefore, $v = uK^{-2}$ lies in the centre of U_q. Moreover, it is easy to see that

$$\varepsilon(v) = 1 \quad \text{and} \quad \Delta(v) = (R_{21} R_{12})^{-1} (v \otimes v).$$

Lemma.

(8.1.5) $$\gamma(v) = v.$$

For generic q it was proved in [KiR, Dr1]. Because for generic q this formula is equivalent to some algebraic identity we have the same identity in the case when $q^r = 1$.

Lemma 8.1.5 implies that

$$v^2 = v\gamma(v) = uK^{-2} K^{-2} K^2 \gamma(u) = u\gamma(u).$$

Thus the triple (U_t, R, v) is a ribbon Hopf algebra.

8.2. Irreducible modules over U_t

For a complex number $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ and for integer $i \in \{0, 1, \dots, r-2\}$ we define a $(i+1)$-dimensional U_t-module $V^i(\alpha)$ as follows. This module has a basis $e_0^i(\alpha), e_1^i(\alpha), \dots, e_i^i(\alpha)$ and the action $\rho = \rho_i$ of the generators of U_t is given by the following formulas:

$$\rho(K)\, e_n^i(\alpha) = \alpha t^{i-2n} e_n^i(\alpha)$$
$$\rho(X)\, e_n^i(\alpha) = \alpha^2 [n][i+1-n]\, e_{n-1}^i(\alpha),$$
$$\rho(Y)\, e_n^i(\alpha) = e_{n+1}^i(\alpha),$$

where $n = 0, 1, \dots, i$ and $e_{-1}^i(\alpha) = e_{i+1}^i(\alpha) = 0$.

The U_t-modules $V^i(\alpha)$ are irreducible. Moreover, these modules remain irreducible under the ground field extension of $\mathbf{Q}(t)$ to its algebraic closure, the field of complex numbers $\mathbf{C}$.

The quantum dimension of all these modules is non-zero:

$$\dim_q V^i(\alpha) = \operatorname{Tr}_i(uv^{-1}) = \operatorname{Tr}_i(K^2) = \alpha^2 \frac{t^{2i+2} - t^{-2i-2}}{t^2 - t^{-2}} = \alpha^2 [i+1],$$

where for $a \in U_t$ we denote by $\operatorname{Tr}_i(a)$ the (ordinary) trace of the operator $x \mapsto \rho(a)x\colon V^i(\alpha) \to V^i(\alpha)$. In particular, $\dim_q V^i(1) = [i+1]$.

Recall that the dual to $V^i(\alpha)$ module $(V^i(\alpha))^\vee$ is defined to be the vector space of the $\mathbf{Q}(t)$-valued linear functionals on $V^i(\alpha)$ with the action $\rho^\vee$ of U_t given by the formula

$$\langle \rho^\vee(a)\, x, y\rangle = \langle x, \rho(\gamma(a))\, y\rangle$$

where $a \in U_t$, $x \in (V^i(\alpha))^\vee$ and $y \in V^i(\alpha)$. Let $\{f_n^i(\alpha),\ n=0, 1, \ldots, i\}$ be the basis in $V^i(\alpha)$ dual to $\{e_n^i(\alpha)\}_n$:

$$\langle f_n^i(\alpha), e_k^i(\alpha)\rangle = \delta_{n,k}.$$

It is easy to compute that

$$\begin{aligned}
\rho^\vee(K)\, f_n^i(\alpha) &= \alpha^{-1} t^{-(i-2n)} f_n^i(\alpha),\\
\rho^\vee(X)\, f_n^i(\alpha) &= -\alpha^2 t^2 [n+1]\,[i-n]\, f_{n+1}^i(\alpha),\\
\rho^\vee(Y)\, f_n^i(\alpha) &= -t^{-2} f_{n-1}^i(\alpha).
\end{aligned}$$

It is straightforward to verify that the linear mapping

$$w_i(\alpha)\colon (V^i(\alpha))^\vee \to V^i(\alpha^{-1})$$

defined by the rule

$$w_i(\alpha)\,(f_n^i(\alpha)) = (-1)^{n-(i/2)}\, t^{i-2n}\, e_{i-n}^i(\alpha^{-1})$$

is an U_t-isomorphism.

8.3. Modular Hopf algebra 1

Put $I = \{0, 1, \ldots, r-2\}$ and provide I with the identity involution $* = id$. For $i \in I$ put $V^i = V^i(1)$. We have the U_t-isomorphisms

$$\{w_i(1)\colon V_i^\vee \to V_i = V_{i*}\} \tag{8.3.1}$$

constructed in Sect. 8.2.

We claim that the triple U_t, R, v constructed in Sect. 8.1 together with the modules $\{V_i\}_{i\in I}$ and isomorphisms (8.3.1) make a modular Hopf algebra.

The verification of the axioms (3.1.1), (3.1.2) is straightforward. The axioms (3.1.3) and (3.1.4) will be discussed in the next section. Here we check (3.1.5) and (3.1.6). We first state the result of computations: for any $i, j \in I$

$$S_{i,j} = \frac{\sin(\pi(i+1)(j+1)\, m/r)}{\sin(\pi m/r)}. \tag{8.3.2}$$

This matrix is invertible over $\mathbf{Q}(t)$ which is the content of the axiom (3.1.5). Indeed, we have the orthogonality relation

$$\sum_{k=1}^{r-1} \sin\left(\frac{\pi(i+1)\,km}{r}\right) \sin\left(\frac{\pi(j+1)\,km}{r}\right) = \frac{r}{2}\,\delta_{i,j}. \tag{8.3.3}$$

If $m=1$ this is well known; for general odd m with $(m, r)=1$ this equality follows from the fact that the residues $\{\pm km (\mathrm{mod}\, 2r) \mid k=1, \ldots, r-1\}$ fill in the whole

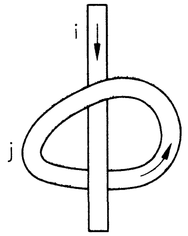


Fig. 23

set 1, 2, ..., r−1, r+2, ..., 2r−1. (Note that for even m the matrix $(S_{i,j})$ is not invertible since $S_{i,j} = -S_{i,r-2-j}$).

Further computations show that for $i \in I$ we have

(8.3.4) $$v_i = t^{-i(i+2)}$$

(8.3.5) $$d_i = \sqrt{\frac{2}{r}} \exp(\sqrt{-1}\,d) \sin(\pi(i+1)\,m/r),$$

where

(8.3.6) $$d = \varphi - \frac{3\pi m}{2} + \frac{\pi}{2}$$

the number φ being determined from the following Gauss sum

(8.3.7) $$\sqrt{2r} \exp(\sqrt{-1}\,\varphi) = \sum_{k=0}^{2r-1} \exp(\sqrt{-1}\,\pi k^2 m/2r)$$

We also have (3.1.6):

(8.3.8) $$C = \exp(2\sqrt{-1}\,d) \neq 0.$$

In particular if $m=1$ then $\varphi = \pi/4$ and

$$C = \exp\left(-\frac{\sqrt{-1}\,\pi}{2}\,\frac{3(2-r)}{r}\right).$$

This value of C shows the accordance of our approach with that of Witten [Wi] who considered the case $m=1$, $q = \exp(2\pi\sqrt{-1}/r)$ from a physical point of view.

Proof of (8.3.2). We will use the following formulas which directly follow from the definition of h_p:

$$\rho(h_p)\,e_n^i = \delta_{p,\,i-2n}\,e_n^j$$

(here $e_n^i = e_n^i(1)$).

Denote by Γ the coloured ribbon (1, 1)-tangle drawn in Fig. 23. The operator $F(\Gamma)\colon V_i \to V_i$ is U_t-linear; therefore $F(\Gamma)$ is the multiplication by certain $b \in K$.

The Hopf link $H_{i,j}$ (see Fig. 13) is clearly isotopic to the closure of Γ. In view of Lemma 2.6 and the definition of $S_{i,j}$ (3.1.5) we have

$$S_{i,j}=F(H_{i,j})=b\dim_q(V_i)=b[i+1]$$

Thus, to prove (8.3.2) it suffices to show that

$$b=\frac{\sin(\pi m(i+1)(j+1)/r)}{\sin(\pi m(i+1)/r)}. \tag{8.3.9}$$

If the R-matrix of U_t is presented as $\sum_k \alpha_k\otimes\beta_k$ then

$$R_{21}R_{12}=\sum_{k,l}\beta_k\alpha_l\otimes\alpha_k\beta_l.$$

The same argument as in the proof of Lemma 2.6 shows that

$$be_n^i=\sum_{k,l}\rho_i(\beta_k\alpha_l)\operatorname{Tr}_j(K^2\alpha_k\beta_l)e_n^i$$

Using the definition of R we get

$$be_n^i=\sum_{p,s=0}^{4r-1}\sum_{n,v=0}^{r-1}t^{n(n-1)+v(v-1)}\frac{(1-q^{-1})^{n+v}}{[n]![v]!}$$
$$K^p(K^{-1}Y)^n h_s(KX)^v\times\operatorname{Tr}_j(K^2h_p(KX)^nK^s(K^{-1}Y)^v)e_m^i.$$

Since the operator $V_i\to V_i$ staying in the R.H.S. is the multiplication by a scalar, we may compute this operator applying it to e_0^i. Clearly, $h_s(KX)^t e_0^i=0$ unless $t=0$ and $s=i$. Thus

$$be_0^i=\sum_{p=0}^{4r-1}\sum_{n=0}^{r-1}\frac{(1-q^{-1})^n}{[n]!}t^{n(n-1)}$$
$$K^p(K^{-1}Y)^n\operatorname{Tr}_j(K^2h_p(KX)^nK^i)e_0^i.$$

The latter trace equals 0 unless $n=0$. Thus,

$$be_n^i=\sum_{p=0}^{4r-1}K^pe_0^i\operatorname{Tr}_j(h_pK^{i+2}).$$

For $p\in\{j,j-2,\ldots,-j\}$ the trace $\operatorname{Tr}_j(h_pK^{i+2})$ equals $t^{p(i+2)}$; for other p this trace equals 0. Therefore

$$b=\sum_{m=j,j-2,\ldots,-j}t^{2m(i+1)}=\frac{t^{2(i+1)(j+1)}-t^{-2(i+1)(j+1)}}{t^{2(i+1)}-t^{-2(i+1)}}.$$

This implies (8.3.9).

Proof of (8.3.4). Applying $\rho_i(u)$ to e_0^i we easily get

$$\rho_i(u)\, e_0^i = \sum_{p=0}^{4r-1} \rho_i(K^{-p} h_p)\, e_0^i = \rho_i(K^{-i} h_i)\, e_0^i = t^{-i^2} e_0^i.$$

Therefore,

$$\rho_i(v)\, e_0^i = \rho_i(uK^{-2})\, e_0^i = t^{-i(i+2)} e_0^i.$$

This implies (8.3.4).

Proof of (8.3.5). To prove (8.3.5) we need some simple number-theoretic computations. For integer n put

$$\begin{aligned} S_n &= \sum_{j=1}^{r-1} t^{j^2+2nj} \\ &= \sum_{j=1}^{r-1} \exp\left(\frac{\sqrt{-1}\pi m}{2r}(j^2+2nj)\right). \end{aligned}$$

Clearly

$$\begin{aligned} S_n &= \sum_{j=1}^{r-1} \exp\left(\frac{\sqrt{-1}\pi m}{2r}((2r-j)^2-2n(2r-j))\right) \\ &= \sum_{j=r+1}^{2r-1} \exp\left(\frac{\sqrt{-1}\pi m}{2r}(j^2-2nj)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} S_n + S_{-n} &= \sum_{j=1}^{2r-1} \exp\left(\frac{\sqrt{-1}\pi m}{2r}(j^2+2nj)\right) \\ &\quad - \exp\left(\sqrt{-1}\pi m\left(n+\frac{r}{2}\right)\right) \\ &= -\exp\left(\sqrt{-1}\pi m\left(n+\frac{r}{2}\right)\right) - 1 + \\ &\quad + \exp\left(-\frac{\sqrt{-1}\pi m}{2r} n^2\right) \sum_{j=1}^{2r-1} \exp\left(\frac{\sqrt{-1}\pi m}{2r} j^2\right). \end{aligned}$$

(Here we have made substitution $j \mapsto j-n$). From (8.3.7) we get

$$S_n + S_{-n} = -\exp\left(\sqrt{-1}\pi m\left(n+\frac{r}{2}\right)\right) + \sqrt{2r}\exp\left(\sqrt{-1}\left(\varphi - \frac{\pi m\, n^2}{2r}\right)\right) - 1$$

Finally we get

$$\begin{aligned} (S_n + S_{-n}) - (S_{n+2} + S_{-n-2}) &= 2\sqrt{2r}\exp(\sqrt{-1}\varphi)\sin\frac{\pi m(n+1)}{r} \\ &\quad \cdot \exp\left(\frac{-\sqrt{-1}\pi m(n^2+2n+2)}{2r}\right). \end{aligned}$$

Now we may compute $\{d_i\}$. The equations (*) of Sect. 3.1 are equivalent to the following equations

$$\sum_{i=0}^{r-2} t^{-i(i+2)} \sin\left(\frac{\pi(i+1)(j+1)m}{r}\right) d_i = t^{j(j+2)} \sin\frac{\pi(j+1)m}{r}$$

for $j=0, 1, \ldots, r-2$. Using (8.3.3) we get

$$\begin{aligned} d_k &= t^{k(k+2)}(2r)^{-1}) \sum_{j=0}^{r-2} t^{j(j+2)} \sin\frac{\pi(j+1)m}{r} \\ &\cdot \sin\left(\frac{\pi(j+1)(k+1)m}{r}\right) = t^{k(k+2)-1} \frac{(2r)^{-1}}{4\sqrt{-1}} \\ &\cdot \sum_{j=1}^{r-1} t^{j^2}(t^{2j(k+2)} + t^{-2j(k+2)} - t^{2jk} - t^{-2jk}) \\ &= (\sqrt{-1})^{k^2+2k-1}(2r)^{-1}(-S_{k+2} - S_{-k-2} + S_k + S_{-k}) \\ &= \sqrt{-\frac{2}{r}}\, t^{-3} \sin\frac{\pi m(k+1)}{r} \exp(\sqrt{-1}\,\varphi) \end{aligned}$$

This implies (8.3.5).

Proof of (8.3.8). Substituting $J=0$ in the equality (*) of Sect. 3.1 we get

$$\sum_{i\in I} v_i \dim_q(V_i)\, d_i = 1.$$

In the present setting, $\dim_q(V_i)$ is a real number $d_i = |d_i| \exp(\sqrt{-1}\,d)$ and $v_i^{-1} = \overline{v_i}$ where the overbar denotes the complex conjugation. Therefore

$$\begin{aligned} C &= \sum_i v_i^{-1} \dim_q(V_i)\, d_i \\ &= \sum_i \overline{v_i \dim_q(V_i)\, d_i} \times \exp(2\sqrt{-1}\,d) \\ &= \exp(2\sqrt{-1}\,d). \end{aligned}$$

8.4. Modular Hopf algebra 2

Now, we are going to prove the property (3.1.4) of representations V_i. To do this we should consider in more details indecomposable representations of the algebra U_t (see [Lu]).

Let U_t^+ and U_t^- be the subalgebras of U_t generated by K, X and K, Y respectively. The simplest nontrivial extensions of the irreducible representation $V^j(\alpha)$ are of the Verma modules $W^j(\alpha)$ and $\tilde{W}^j(\alpha)$ and $\tilde{W}^j(\alpha)$ which are free over U_t^- and U_t^+ respectively. They are labeled by integers $j=0, 1, \ldots, r-1$,

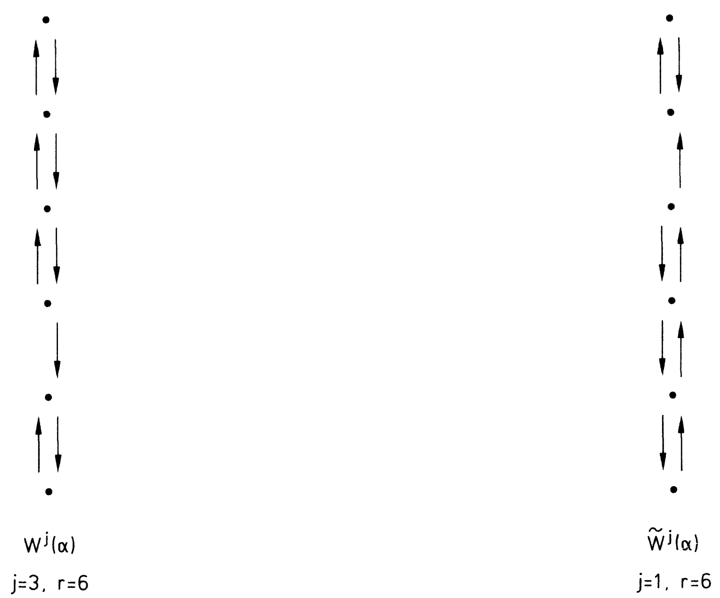


Fig. 24

and by $\alpha \in \mathbf{C}$ such that $\alpha^4 = 1$. In the weight basis $e_n^j(\alpha) \in W^j(\alpha)$ and $\tilde{e}_n^j(\alpha) \in \tilde{W}(\alpha)$ the action of the generators has the following form:

$$K e_n^j \alpha = \alpha t^{j-2n} e_n^j(\alpha), \qquad K \tilde{e}_n^j(\alpha) = \alpha t^{-j+2n} \tilde{e}_n^j(\alpha),$$

$$Y e_n^j(\alpha) = \alpha^2 [n][j+1-n] e_{n1}^j(\alpha), \; X \tilde{e}_n^j(\alpha) = \tilde{e}_{n+1}^j(\alpha),$$

$$X e_n^j(\alpha) = e_{n+1}^j(\alpha), \qquad Y \tilde{e}_n^j(\alpha) = \alpha^2 [n][j+1-n] \tilde{e}_{n-1}^j(\alpha).$$

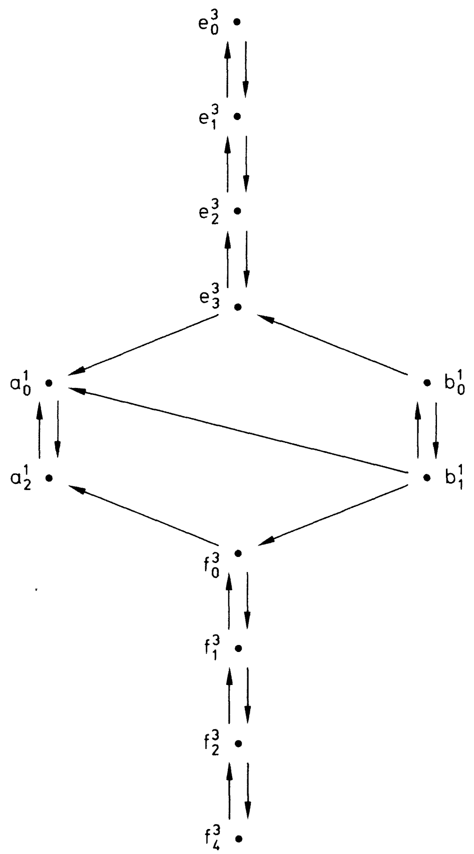
Here $n = 0, 1, \ldots, r-1$. It is obvious that

$$W^{r-1}(\alpha) \simeq \tilde{W}^{r-1}(\alpha).$$

Studying the action of X, Y, K on the vectors e_0^j, e_j^j one easily shows that $W^{r-1}(\alpha)$ may be imbedded in another U_t-module W only as a split summand of W. Also, if $W^{r-1}(\alpha)$ is a quotient of W then it is a split summand of W. This gives the following lemma.

8.4.1 Lemma. *The module $W^{r-1}(\alpha)$ has no extensions.*

It is convenient to use a graphical representation for the structure of U_t-modules. Let us represent vectors from the module by verteces ordered vertically according to the values of the weights. Arrows pointing down show the action of Y. Arrows pointing up show the action of X. The absence of arrows coming out of a vertex means that the corresponding vector is annihilated by one of the generators X or Y. Examples of such representations are given in Fig. 24.



$X^j(\alpha)$, $j=3$, $r=6$

Fig. 25

The following sequences are exact:

$$0 \to V^{r-2-j}(\alpha t^{-r}) \to W^j(\alpha) \to V^j(\alpha) \to 0$$
$$0 \to V^j(\alpha) \to \tilde{W}^j(\alpha) \to V^{r-2-j}(\alpha t^r) \to 0$$

Dual Verma modules have the same structure and the following isomorphisms hold:

$$(W^j(\alpha))^\vee \simeq W^{r-2-j}(\alpha^{-1} t^r)$$
$$(\tilde{W}^j(\alpha))^\vee \simeq W^{r-2-j}(\alpha^{-1} t^{-r}).$$

The Verma modules $W^j(\alpha)$ for $j \neq r-1$ have important extensions $X^j(\alpha)$ of dimension $2r$. The structure of these representations is given on Fig. 25.

Let $e_n^j(\alpha)$, $n=0, \ldots, j$ and $a_n^j(\alpha)$, $b_n^j(\alpha)$, $n=0, \ldots, r-2-j$, $f_n^j(\alpha)$, $n=0, \ldots, j$ be the elements of the weight basis in $X^j(\alpha)$ (see Fig. 25). The action of the generators K, X, and Y has the following form in the basis:

$$\left.\begin{aligned} Ke_n^j(\alpha)&=\alpha t^{j-2n} e_n^j(\alpha),\\ Xe_n^j(\alpha)&=\alpha^2[n][j+1-n]\, e_{n-1}^j(\alpha),\\ Ye_n^j(\alpha)&=e_{n+1}^j(\alpha)\end{aligned}\right\} \quad n=0, \ldots, j$$

$$\left.\begin{aligned} Ka_n^j(\alpha)&=\alpha t^{-j-2-2n} \alpha_n^j(\alpha)\\ Xa_n^j(\alpha)&=\alpha^2 t^{2r}[n][r-j-1-n]\, a_{n-1}^j(\alpha)\\ Ya_n^j(\alpha)&=a_{n+1}^j(\alpha)\end{aligned}\right\} \quad n=0, \ldots, r-2-j$$

$$\left.\begin{aligned} Kb_n^j(\alpha)&=\alpha t^{-j-2n-2} b_n^j(\alpha)\\ Xb_n^j(\alpha)&=\alpha^2 t^{2r}[n][r-j-1-n]\, b_{n-1}^j(\alpha)+a_{n-1}^j(\alpha)\\ Yb_n^j(\alpha)&=b_{n+1}^j(\alpha)\end{aligned}\right\} \quad n=0, \ldots, r-j-2$$

$$\left.\begin{aligned} Kf_n^j(\alpha)&=\alpha t^{-2r} t_0^{j-2n} f_n^j(\alpha)\\ Xf_n^j(\alpha)&=\alpha^2[n][j+1-n]\, f_{n-1}^j(\alpha)+\delta_{n,0}\, a_{r-2-j}^j(\alpha)\\ Yf_n^j(\alpha)&=f_{n+1}^j(\alpha).\end{aligned}\right\} \quad n=0, \ldots, j$$

Here $e_{j+1}^j(\alpha)\equiv a_0^j(\alpha)$, $a_{-1}^j(\alpha)\equiv e_j^j(\alpha)$, $b_{r-1-j}^j(\alpha)\equiv f_0^j(\alpha)$.

Let us denote by V_1, V_2, V_3 and V_4 the subspaces in X^j generated by vectors $\{e_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{f_n\}$ respectively. Also denote by W_{12}, W_{34}, W_{24}, and W_{13} the subspaces generated by double sets $\{e_n, a_m\}$, $\{b_n, f_m\}$, $\{a_n, f_m\}$ and $\{e_n, f_m\}$ respectively. The modules $X^j(\alpha)$ have the following submodules:

$$V_2 \simeq V^{r-j-2}(\alpha), \qquad W_{12}\simeq W^j(\alpha t^r), \qquad W_{24}\simeq \tilde{W}^j(\alpha t^{-r})$$

and the following structure of factormodules:

$$\begin{aligned} &0\to W_{12} &&\to X^j(\alpha) &&\to W_{34}\to 0\\ &0\to W_{24} &&\to X^j(\alpha) &&\to W_{13}\to 0\\ &0\to V_1\otimes V_4 &&\to X^j(\alpha)/V_2 &&\to V_3 \to 0\end{aligned} \tag{8.4.1}$$

where

$$W_{34}\simeq W^{r-2-j}(\alpha), \qquad W_{13}\simeq \tilde{W}^{r-2-j}(\alpha t^r)$$

$$V_1\simeq V^j(\alpha t^r), \qquad V_3\simeq V^{r-j-2}(\alpha), \qquad V_4\simeq V^j(\alpha t^{-r})$$

The next lemma is analogous to Lemma 8.4.1.

Lemma 8.4.2. *The Modules $X^j(\alpha)$ have no extensions.*

8.4.3. Theorem. *Let $j_1, j_2 \leqq r-2$, then*

1. If $j_1+j_2 \leqq r-2$:

$$V^{j_1}(\alpha_1)\otimes V^{j_2}(\alpha_2)=\bigoplus_{\substack{|j_1-j_2|\leqq j\leqq j_1+j_2\\ j=|j_1-j_2| \bmod 2}} V^j(\alpha_1\alpha_2),$$

2. If $j_1+j_2>r-2$:

$$V^{j_1}(\alpha_1)\otimes V^{j_2}(\alpha_1)=\Big(\bigoplus_{\substack{|j_1-j_2|\leqq j\leqq 2r-4-j_1-j_2\\ j=|j_1-j_2| \bmod 2}} V^j(\alpha_1\alpha_2)\Big)$$

$$\otimes\Big(\bigoplus_{\substack{0\leqq j\leqq j_1+j_2-r\\ j=j_1+j_2 \bmod 2}} X^j(\alpha,\alpha_2)\Big)$$

$$\oplus\begin{cases}\phi, & \text{if } j_n+j_2=r \ (\mathrm{mod}\ 2)\\ V^{r-1}(\alpha,\alpha_2), & \text{if } j_1+j_2=r-1 \ (\mathrm{mod}\ 2)\end{cases}$$

Proof. If $j_1+j_2\leqq r-1$ the decomposition 1 follows from the decomposition for generic q [Ji, KiR]. Suppose $j_1+j_2\leqq r$. Solving explicitly the equations

$$\Delta X\cdot x=0$$

where

$$X=\sum_{n_1 n_2} x_{n_1 n_2}\, e^{j}_{n_1}(\alpha)\otimes e^{j_2}_{n_2}(\alpha_2)$$

we find the highest weight vectors in $V^{j_1}(\alpha_1)\otimes V^{j_2}(\alpha_2)$ (for generic q see [Ji]). There are two types of h.w.v. The h.w.v. from the first one are parameterized by integers j: $|j_1-j_2|\leqq j\leqq r-1$, $j=|j_1-j_2| \bmod 2$ and have the form

$$x^j=\sum_{n_1+n_2=\frac{1}{2}(j_1-j_2-j)} C^{j_2 j}_{j_1 n_1 n_2}\, e^{j_1}_{n_1}(\alpha_1)\otimes e^{j_2}_{n_2}(\alpha_2)$$

where

$$C^{j_1 j_2 j}_{n_1 n_2}=\text{const.}\,(\alpha_1\alpha_2)^{n_2}\, t^{n_1(j-2)}\frac{[\frac{1}{2}(j_2-j_1+j)+n_1]!\,[j_1-n_1]!}{[\frac{1}{2}(j_1+j_2-j_2-j)-n_1]!\,[n_1]!}$$

This vector has weight $\alpha_1\alpha_2 t^j$:

$$\Delta K x^j=\alpha_1\alpha_2 t^j x^j.$$

The h.w.v. of the second type are parametrized by integers j:

$$0\leqq j\leqq j_1+j_2-r,\qquad j=j_1+j_2 \ (\mathrm{mod}\ 2)$$

$$\Delta K\tilde{x}^j=(\alpha_1\alpha_2 t^r)\cdot t^j\tilde{x}^j$$

$$\tilde{x}^j=\sum_{n_1+n_2=\frac{1}{2}(j_1+j_2-j-r)}\tilde{C}^{j_1 j_2 j}_{n_1 n_2}\, e^{j_1}_{n_1}(\alpha_1)\otimes e^{j_2}_{n_2}(\alpha_2)$$

where

$$\tilde{C}^{j_1 j_2 j}_{n_1 n_2}=C^{j_1 j_2 j+r}_{n_1 n_2}$$

So, for each weight $\alpha_1\alpha_2 t^j$, $|j_1-j_2|\leqq j\leqq j_1+j_2$ there is a unique h.w.v. in $V^{j_1}(\alpha_1)\otimes V^{j_2}(\alpha_2)$. Acting by the operator ΔY on the vectors x^j and $\tilde{x}^j$, for each $|j_1-j_2|\leqq j\leqq r-1$, $j=|j_1-j_2| \bmod 2$, we obtain an irreducible submodule $V^j(\alpha_1\alpha_2)$ (we put $V^{r-1}(\alpha_1\alpha_2)\equiv W^{r-1}(\alpha_1\alpha_2)$). For each $0\leqq j\leqq j_1+j_2-r$, $j=j_1+j_2$ (mod 2) we obtain a Verma submodule $W^j(\alpha_1\alpha_2 t^r)$. The total number of highest weight vectors is equal to $\min(j_1+1, j_2+1)$ and is the same as the number of h.w.v. in the decomposition of the tensor product of the two corresponding irreducible representations of $SU(2)$.

The analysis of the lowest weight vectors y^j gives us the irreducible submodules $V^j(\alpha_1\alpha_2)$ mentioned above and the submodules which are isomorphic to the Verma modules $\tilde{W}^j(\alpha_1\alpha_2 t^{-r})$ with $0 \leqq j \leqq j_1+j_2-r$, $j=j_1+j_2 \pmod 2$. The total number of l.w.v. is the same as the total number of h.w.v., and for each weight $\alpha_1\alpha_2 t^{-j}$, $|j_1-j_2| \leqq j \leqq j_1+j_2$, $j=j_1+j_2 \pmod 2$ there is a unique l.w.v. in $V^{j_1}(\alpha_1)\otimes v^{j_2}(\alpha_2)$.

From the uniqueness of h.w.v. with weight $\alpha_1\alpha_2 t^{r-j-2}$ follows identity

$$(\Delta Y)^{j+1}\tilde{x}^j = C\cdot x^{r-j-2}, \qquad 0 \leqq j \leqq j_1+j_2-r$$

for some nonzero constant C.

From the uniqueness of l.w.v. with weight $\alpha_1\alpha_2 t^{-r+j+2}$ follows a similar identity for l.w.v's:

$$(\Delta X)^{j+1}\tilde{y}^j = C\cdot y^{r-j-2}$$

Therefore the subrepresentations $V^{r-j-2}(\alpha_1\alpha_2)$, $W^j(\alpha_1\alpha_2 t^r)$, $\tilde{W}^j(\alpha_1\alpha_2 t^r)$, in $V^{j_1}(\alpha_1)\otimes Vj_2(\alpha_2)$ are the subrepresentations of indecomposable reducible subrepresentations $M^j(\alpha_1\alpha_2)$ formed by vectors

$$\{(\Delta Y)^n x^j\}_{n=0}^j = \{(\Delta X)^n y^j\}_{n=0}^j$$

for

$$|j_1-j_2| \leqq j \leqq 2r-4-j_1-j_2, \qquad j=|j_1-j_2| \pmod 2$$

$$\{(\Delta Y)^n \tilde{x}^j\}_{n=0}^j, \qquad \{(\Delta X)^n \tilde{y}^j\}_{n=0}^j$$

for

$$0 \leqq j \leqq j_1+j_2-r, \qquad j=j_1+j_2 \pmod 2.$$

This representation $M^j(\alpha)$ has the following structure:

$$M^j(\alpha)/W^j(\alpha t^r) \simeq V^{r-j-2}(\alpha),$$
$$M^j(\alpha)/\tilde{W}^j(\alpha t^{-r}) \simeq V^{r-j-2}(\alpha),$$
$$M^j(\alpha)/V^{r-j-2}(\alpha) \simeq V^j(\alpha t^r)\otimes V^j(\alpha t^{-r})$$

Let us define vectors

$$x_n^j = \sum_{n_1+n_2=\frac{1}{2}(j_1+j_2-j)+n} C_{n_1 n_2 n}^{j_1 j_2 j}\, e_{n_1}^{j_1}(\alpha_1)\otimes e_{n_2}^{j_2}(\alpha_2)$$

where

$$C_{n_1 n_2 n}^{j_1 j_2 j} = a\cdot\alpha_1^{-n}(\alpha_1\alpha_2)^{n_1} t^{-nj_1} \sum_{p=0}^{\frac{1}{2}(j_1+j_2-j)} (-1)^p t^{2p(n-j+1)}$$
$$\cdot\frac{[\frac{1}{2}(j_2-j_1+j)-p]!\,[j_1-p]!}{[n_1-p]!\,[n-n_1+p]!\,[\frac{1}{2}(j_1+j_2-j)-p]!\,[p]!}$$

for generic t. Let us fix the constant a by the condition

$$x_n^j = (\Delta Y)^n x^j$$

for

$$|j_1-j_2| \leqq j \leqq r-1\,(j=|j_1-j_2| \bmod 2),\ t^{4r}=1.$$

Then, consider vectors

$$z^{r-j-2}(t) = -\frac{1}{[j][r]}(x_{j+1}^{j+r} - c \cdot x^{r-j-2})$$

for generic t. Acting by the element ΔX on these vectors we have:

$$\Delta X z^{r-j-2}(t) = x_{j+1}^{j+r}. \tag{8.4.2}$$

In the limit $t^{4r} \to 1$, we have $x_{j+1}^{j+r} \to (\Delta Y)^{j+1} \tilde{x}^j = c x^{r-j-2}$. From (8.4.2) and from the structure of l.w.v. it follows that $z^{r-j-2}(t)$ has a finite limit when $t^{4r} \to 1$. Vectors $(\Delta Y)^n z^{r-j-2}(t)|_{t^{4r}=1}$ for $n=0, \ldots, j-r-2$ generate a linear space isomorphic to $V^{r-j-2}(\alpha)$. By direct computation one can prove that

$$(\Delta Y)^{r-j-1} z^{r-j-2}(t)|_{t^{4r}=1} = C \cdot (\Delta X)^j \tilde{y}^j$$

for some nonzero C. So, we obtain an extension of the representation $M^j(\alpha_1 \alpha_2)$ which is isomorphic to $X^j(\alpha_1 \alpha_2)$:

$$0 \to M^j(\alpha_1 \alpha_2 \to X^j(\alpha_1 \alpha_2) \to V^{r-j-2}(\alpha_1 \alpha_2) \to 0$$

The subrepresentations $X^j(\alpha_1 \alpha_2)$ for different values of j are nonisomorphic. They are nonisomorphic to irreducible subrepresentations $W^{r-1}(\alpha_1 \alpha_2)$, $V^j(\alpha_1 \alpha_2)$ either. Therefore we have a decomposition 2. Q.E.D.

8.4.4. Lemma. *Tensor products of Verma modules and irreducible representations of U_t have the following structure:*

1. If $j_1 + j_2 \leqq r-2, j_1 \leqq j_2$

$$V^{j_1}(\alpha_1) \otimes W^{j_2}(\alpha_2) = \bigoplus_{j_2 - j_1 \leqq j \leqq j_1 + j_2} W^j(\alpha_1 \alpha_2),$$

2. If $j_1 + j_2 \leqq r-2, j_1 > j_2$

$$V^{j_1}(\alpha_1) \otimes W^{j_2}(\alpha_2) = \Big(\bigoplus_{j_1 - j_2 \leqq j \leqq j_1 + j_2} W^j(\alpha_1 \alpha_2)\Big) \otimes \Big(\bigoplus_{0 \leqq j \leqq j_1 - j_2 - 2} X^j(\alpha_1 \alpha_2 t^{-r})\Big)$$

$$\otimes \begin{cases} \phi, j_1 - j_2 = 0 & (\text{mod } 2) \\ V^{r-1}(\alpha_1 \alpha_2 t^{-r}, j_1 - j_2 = 1 & (\text{mod } 2) \end{cases}$$

3. If $j_1 + j_2 \leqq r-1, j_1 \leqq j_2$

$$V^{j_1}(\alpha_1) \otimes W^{j_2}(\alpha_2) = \Big(\bigoplus_{j_2 - j_1 \leqq j \leqq 2r-4-j_1-j_2} W^j(\alpha_1 \alpha_2)\Big) \otimes \Big(\bigoplus_{0 \leqq j \leqq j_1 + j_2 - r} X^j(\alpha_1 \alpha_2)\Big)$$

$$\otimes \begin{cases} \phi, j_1 + j_2 = r & (\text{mod } 2) \\ V^{r-1}(\alpha_1 \alpha_2), j_1 + j_2 = r-1 & (\text{mod } 2) \end{cases}$$

4. If $j_1+j_2\leqq r-1, j_1>j_2$

$$V^{j_1}(\alpha_1)\otimes W^{j_2}(\alpha_2)=\Big(\bigoplus_{j_1-j_2\leqq j\leqq 2r-4-j_1-j_2} W^j(\alpha_1\alpha_2)\Big)\otimes\Big(\bigoplus_{0\leqq j\leqq j_1+j_2-r} X^j(\alpha_1\alpha_2)\Big)$$

$$\otimes(\otimes X^j(\alpha_1\alpha_2 t^{-r}))\otimes\begin{cases}\phi, & \text{if } j_1-j_2=0 \pmod 2\text{: } r \text{ is even.}\\ V^{r-1}(\alpha_1\alpha_2)\otimes V^{r-1}(\alpha_1\alpha_2 t^{-r}), & \text{if } j_1-j_2=1 \pmod 1\text{: } r \text{ is even.}\\ V^{r-1}(\alpha_1\alpha_2 t^{-r}), & \text{if } j_1-j_2=0 \pmod 2\text{: } r \text{ is odd.}\\ V^{r-1}(\alpha_1\alpha_2), & \text{if } j_1-j_2=1 \bmod 2)\text{: } r \text{ is odd.}\end{cases}$$

Here $W^j(\alpha)\equiv\phi$, $X^j(\alpha)=\phi$ if $j<0$.

Proof. Let us prove 2. Recall that the sequence

$$0\to V^{r-j-2}(\alpha t^{-r})\to W^j(\alpha)\to V^j(\alpha)\to 0$$

is exact. Consider the decompositions of the tensor products of $V^{j_1}(\alpha_1)$ with sub and factor modules of $W^{j_2}(\alpha_2)$. For the case 2 we have:

$$W^{j_1}(\alpha_1)\otimes V^{j_2}(\alpha_2)=\bigoplus_{j_1-j_2\leqq j\leqq j_1+j_2} V^j(\alpha_1\alpha_2).$$

$$V^{j_1}(\alpha_1)\otimes V^{r-2-j_2}(\alpha_2 t^{-r})=\Big(\bigoplus_{j_1-j_2\leqq j\leqq j_1+j_2} V^{r-2-j}(\alpha_1\alpha_2 t^{-r})\Big)$$
$$\otimes\Big(\bigoplus_{0\leqq j\leqq j_1-j_2-2} X^j(\alpha_1\alpha_2 t^{-r})\Big)$$
$$\otimes\begin{cases}\phi, j_1-j_2=0 \pmod 2\\ V^{r-1}(\alpha_1\alpha_2 t^{-r}), j_1-j_2=1 \pmod 2).\end{cases}$$

Now from the fact that $X^j(\alpha)$ and $V^{r-1}(\alpha)$ have no extensions and from the exactness of the sequence (8.4.1) claim 2 follows immediately. For other cases the proof is similar.

8.4.5. Theorem. *The tensor products of irreducible representations V^j with representations X^j have the following form:*

1. If $j_1+j_2\leqq r-2, j_1\leqq j_2$,

$$V^{j_1}(\alpha_1)\otimes X^{j_2}(\alpha_2)=\bigoplus_{j_2-j_1\leqq j\leqq j_1+j_2} X^j(\alpha_1\alpha_2).$$

2. If $j_1+j_2\leqq r-2, j_1>j_2$

$$V^{j_1}(\alpha_1)\otimes X^{j_2}(\alpha_2)=\Big(\bigoplus_{j_1-j_2\leqq j\leqq j_1+j_2} X^j(\alpha_1\alpha_2)\Big)$$

$$\otimes\Big(\bigoplus_{0\leqq j\leqq j_1-j_2-2}(\mathbf{C}^2\otimes X^j(\alpha_1\alpha_2))\Big)\otimes\begin{cases}\phi, j_1-j_2=0 \pmod 2\\ \mathbf{C}^2\otimes V^{r-1}(\alpha_1\alpha_2), j_1-j_2=1 \pmod 2)\end{cases}$$

3. If $j_1+j_2>r-2, j_1\leqq j_2$

$$V^{j_1}(\alpha_1\otimes X^{j_2}(\alpha_2)=(\bigoplus_{j_2-j_1\leqq j\leqq 2r-4-j_1-j_2} X^j(\alpha_1\alpha_2))$$

$$\otimes(\bigoplus_{0\leqq j\leqq j_1+j_2-r}(X^j(\alpha_1\alpha_2 t^r)\otimes X^j(\alpha_1\alpha_2 t^{-r}))$$

$$\otimes\begin{cases}\phi, j_1-j_2=r \pmod 2\\ V^{r-1}(\alpha_1\alpha_2 t^r), j_1-j_2=r-1 \pmod 2).\end{cases}$$

4. If $j_1+j_2>r-2, j_1\leqq j_2$

$$V^{j_1}(\alpha_1)\otimes X^{j_2}(\alpha_2)=(\bigoplus_{j_1-j_2\leqq j\leqq 2r-4-j_1-j_2} X^j(\alpha_1\alpha_2))\otimes(\bigoplus_{0\leqq j\leqq j_1-j_2-2}(\mathbf{C}^2\otimes X^j(\alpha_1\alpha_2)))$$

$$\otimes(\bigoplus_{0\leqq j\leqq j_1+j_2-r}(X^j(\alpha_1\alpha_2 t^r)\otimes X^j(\alpha_1\alpha_2 t^{-r})))$$

$$\otimes\begin{cases}\phi, j_1-j_2=0 \pmod 2)\text{: } \textit{even } r\\ \mathbf{C}^2\otimes V^{r-1}(\alpha_1\alpha_2)\otimes V^{r-1}(\alpha_1\alpha_2 t^r)\oplus V^{r-1}(\alpha_1\alpha_2 t^{-r}), j_1-j_2=1 \pmod 2)\text{: } \textit{even } r\\ \mathbf{C}^2\otimes V^{r-1}(\alpha_1\alpha_2), j_1-j_2=0 \pmod 2)\text{: } \textit{odd } r\\ V^{r-1}(\alpha_1\alpha_2 t^r)\otimes V^{r-1}(\alpha_1\alpha_2 t^{-r}), j_1-j_2=1 \pmod 2)\text{: } \textit{odd } r.\end{cases}$$

Proof. To obtain these decompositions it is sufficient to use exact sequences (8.4.1) and decompositions from the Lemma. As in the proof of the Lemma the main point is the absence of extensions of the representations $X^j(\alpha)$ and $V^{r-1}(\alpha)$.

Now we are going to prove the property (3.1.4) for the algebra U_t. As follows from Theorem 8.4.3

$$V^{j_1}\otimes\ldots\otimes V^{j_N}=\bigoplus_{0\leqq j\leqq r-2}(\Omega_j^{j_1\ldots j_N}\otimes V^j)\oplus Z(j_1\ldots j_N),$$

where

$$Z(j_1\ldots j_N)=\bigoplus_{\alpha^4=1}(\Omega_{r-1}^{j_1\ldots j_N}(\alpha)\otimes W^{r-1}(\alpha)))$$

$$\oplus(\bigoplus_{\substack{a^4=1\\0\leqq j\leqq r-2}}\tilde{\Omega}_j^{j_1}\ldots j_N(\alpha)\otimes X^j(\alpha)).$$

Therefore to prove that the quantum trace of each U_t-linear operator acting on $Z(j_1,\ldots,j_N)$ is equal to zero is sufficient to prove the same property of U_t-linear operators as acting on $W^{r-1}(\alpha)$ and $X^j(\alpha)$. Because the representation $W^{r-1}(\alpha)$ is irreducible each U_t-linear operator β is proportional to the unit operator $\beta=b\cdot I$ (Schur lemma), and we have

$$\operatorname{tr}(K^2\beta)=b\operatorname{tr}_{W^{r-1}(\alpha)}(K^2)=b\sum_{n=0}^{r-1}\alpha^2 t^{2r-2-4n}$$

$$=b\alpha^2 t^{2r-2}\frac{1-t^{-4r}}{1-t^{-4}}=0$$

8.4.6. Lemma. *Each U_t-linear operator in $W^j(\alpha)$ or $\tilde{W}^j(\alpha)$ or $\tilde{W}^j(\alpha)$ is proportional to the unit operator.*

This lemma also follows from Schur's lemma. As a consequence, for each U_t-linear operator we have

$$\mathrm{tr}_{W^j(\alpha)}(K^2\beta)=b\alpha^2 t^{2j}\sum_{n=0}^{r-1} t^{-4n}=0$$

$$\mathrm{tr}_{\tilde{W}^j(\alpha)}(K^2\beta)=b\alpha^2 t^{-2j}\sum_{n=0}^{r-1} t^{-4n}=0$$

Combining the Lemma and the structure (8.4.1) of $X^j(\alpha)$ we obtain the following block structure for U_t-linear operators acting on $X^j(\alpha)$:

$$\beta=\left|\begin{array}{c|c} b_1 I_{W^{r-j-2}(\alpha)} & * \\ \hline 0 & \mathrm{b}_2 I_{W^j(\alpha t^r)}\end{array}\right| .$$

And therefore

$$\mathrm{tr}_{X^j(\alpha)}(K^2\beta)=b_1\,\mathrm{tr}_{W^{r-j-2}(\alpha)}(1)+b_2\,\mathrm{tr}_{W^j(\alpha t^r)}(1)=0.$$

So, we proved the theorem:

8.4.7. Theorem. *The quantum trace of each U_t-linear operator acting in $Z(j_1\dots j_n)$ is equal to zero.*

9. Concluding remarks

1. One may compute the operator $F(\mathrm{id}_{G^1})$ to be the identity. So

$$\bar{F}(S^1\times S^1)=\bar{F}(G^1)=\Psi_1$$

and we get essentially the Verlinde representation of Mod_1 in the $(r-1)$-dimensional vector space Ψ_1 [MS].

Conjecture 1. *For U_t the operator $F(\mathrm{id}_{G^K})$ is the unit operator.*

2. With each simple Lie algebra $\mathscr{G}$ one can associate the Hopf algebra $U_t(\mathscr{G})$ [Dr, J] which is the deformation of the universal enveloping algebra of $\mathscr{G}$. This algebra has a finite dimensional factor algebra $U_t[\mathscr{G}]$ [Lu]. One may show that $U_t[\mathscr{G}]$ is a modular Hopf algebra. Details will be presented in a separate publication. In the general case we obtain the following representation of Mod_1:

$$S_{\lambda\mu}=\chi_\lambda(t^{2(\mu+\rho)})\,\chi_\mu(t^{2\rho}),$$

$$T_{\lambda\mu}=\delta_{\lambda,\mu^*}\,t^{(\lambda,\lambda+2\rho)}.$$

Here λ, μ are truncated dominant highest weights of the algebra $\mathscr{G}$: $(\lambda+\rho, \alpha_{\max}) < r$, $(X_\lambda(x) = \mathrm{tr}_{V^\lambda}(x)$ is a character of the representation V^λ of the algebra $\mathscr{G}$ and $\mu^* = -w_0\,\mu$ where w_0 is the element of the Weyl group with maximal length.

$$(ST)^3 = C\cdot\alpha$$
$$S^2 = C.$$

Here $C_{\lambda\mu} = \delta_{\lambda,\mu^*}$, $\alpha \neq 0$. It is remarkable that this representation for $t = \exp\left(\frac{i\pi}{2r}\right)$ was found in [KaW] from the representation theory of Kac-Moody algebras [Ka]. In this case $\alpha = \exp\left(i\frac{\pi}{4}\,\frac{(r-g)\dim\mathscr{G}}{r}\right)$, where g is the dual Coxeter number.

3. The structure of tensor products of representations $V^j(\alpha)$, $V^{r-1}(\alpha)$, $X^j(\alpha)$ reflects the fact that they form a closed quasitensorial category.

4. When this work was finished we were informed that the similar results about the structure of tensor products in U_t had been independently obtained by A. Wasserman (private communication).

5. Our invariant depends on the orientation of the manifold M. Consider M as a result of surgery of S^3 along some framed link L. The manifold M with opposite orientation is homeomorphic to the manifold M' obtained from S^3 by the surgery along the mirror image L' of the link L with opposite framing numbers for each component. In our example related to $U_q(sl_2)$ $F_q(L, w, \lambda) = F_{q^{-1}}(L', -w, \lambda)$ and we have the following relations between invariants

$$F_q(M) = F_{q^{-1}}(M')$$

where the bar is the complex conjugation. If M has the boundary, then $F(M')$ is the conjugated operator to $F(M)$ with respect to standard bilinear form on $r^0_{\theta\theta^*}$. This follows from the fact that $F(M, T)$ is expressed in terms of $F(S^3, \Gamma)$ where $\{\Gamma\}$ are certain coloured ribbon graphs in S^3 dependent on T and M (see sections 3, 4, 6).

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