THE COLORED JONES POLYNOMIAL AND THE A-POLYNOMIAL FOR TWIST KNOTS

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ABSTRACT. @We show that for a twist knot, the A-polynomial can be obtained from recurrences for the summand in Masbaum's formula of the colored Jones polynomial. Our result supports the *AJ conjecture* due to S.Garoufalidis.

1. INTRODUCTION

The colored Jones polynomial is a one variable polynomial invariant of a knot colored with irreducible finite dimensional representation of $sl(2,\mathbb{C})$ [14]. The original Jones polynomial [6] corresponds to the colored Jones polynomial of a knot colored with the irreducible 2-dimensional representation. On the other hand, in [1], D.Cooper, M.Culler, H.Gillet, D.D.Long, and P.R.Shalen introduced a two variable polynomial invariant $A_K(l,m)$ of a knot K, called the A-polynomial, by using the representations of the fundamental group of the complement of the knot into $SL_2(\mathbb{C})$. In this paper, we prove that for a twist knot, the A-polynomial can be obtained from the colored Jones polynomial.

Recently, C.Frohman, R.Gelca, W.Lofaro [2] introduced the peripheral ideal of a knot, and the noncommutative A-ideal of a knot as a generalization of the A-polynomial, via Kauffman bracket skein theory. A nontrivial element in the peripheral ideal of a knot induces a recursive relation of the Kauffman bracket polynomial of the knot, which essentially equals to the colored Jones polynomial of the knot. It is not yet proved that for a knot except for the $(2,2p+1)$-torus knot and the figure eight knot, the peripheral ideal is nontrivial.

Afterwards, in [4], S.Garoufalidis and T.T.Q.Le proved that the colored Jones polynomial of a knot satisfies a nontrivial recursive relation. Moreover, in [3], Garoufalidis defined the recursion ideal, which is identified with the set of recursive relations of the colored Jones polynomial, and defined the noncommutative A-polynomial $A_q(K)$ of a knot K called *A_q-polynomial*, as a generator of the ideal. Let us denote by $J_K(n)$ the n-colored Jones polynomial associated with irreducible n-dimensional representation of $sl(2,\mathbb{C})$. We consider two operators E and Q acting on $J_K(n)$ by $EJ_K(n) = J_K(n+1)$ and $QJ_K(n) = q^n J_K(n)$. The element $A_q(K)$ is of the form $A_q(K)(E,Q) = \sum_k a_k E^k$ with a_k in $\mathbb{Z}[q,Q]$. Then, Garoufalidis conjectured

Conjecture 1.1 (*The AJ conjecture*)**.** *For every knot K in S^3, $A_K(l,m) = \varepsilon A_q(K)(l,m^2)$, where ε is the evaluation map at $q = 1$.*

Furthermore, he showed that the conjecture holds for the trefoil knot and the figure eight knot, by using the mathematica package `qZeil.m` develped by Paul and Riese [12] [11], to find a recursive relation of the colored Jones polynomial.

We focus on twist knots, for which we have the formula of the colored Jones polynomial obtained by G.Masbaum in [8] and the formula of the A-polynomial obtained by J.Hoste and P.D.Shanahan in [5]. The purpose of this paper is to prove that for a twist knot, the A-polynomial can be obtained from recursive relations of the summand in Masbaum's formula of the colored Jones polynomial. Our result supports the AJ conjecture for a twist knot.

Moreover, using the mathematica package `qMultisum.m` developed by Reise [13], we observe that the A-polynomial for the knots 5_2 and 6_1 can be obtained from a recursive relation of the colored Jones polynomial. We will also discuss a relation between our result and 'Volume conjecture' due to R.M.Kashaev, H. Murakami and J. Murakami ([7],[10]).

This paper is organized as follows. In Section 2, we recall Masbaum's formula of the colored Jones polynomial for a twist knot and state Main theorem (Theorem 2.2). In Section 3, we prove Theorem 2.2. In Section 4, we relate our result to the AJ conjecture and give a result about recursive relations of the Jones polynomials for the knots 5_2 and 6_1.

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2. Main Theorem

We start with the review of the definition of the A-polynomial of a knot K in S^3 (see [1]). Let M_K be the complement of K and $R(M_K)$ the set of all representations of $\pi_1(M_K)$ into $SL(2,\mathbb{C})$. The set $R(M_K)$ is an affine algebraic variety. Noting that $SL(2,\mathbb{C})$ acts on representations by conjugation, we restrict our attention to the subset R_U of $R(M_K)$ consisting of a represetation ρ satisfying that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matirices for the meridian μ and the preferred longitude λ of K. We can define a projection ξ from R_U to $\mathbb{C}^2$ by $\xi(\rho) = (l,m)$ for $\rho \in R(M_K)$ with

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix},\ \rho(\lambda) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix}.$$

The Zariski closure of $\xi(R_U)$ has a structure of an algebraic variety in $\mathbb{C}^2$ and each of its irreducible complex-dimension-one components is a curve, which is defined by a polynomial with integer coefficients in l and m. The A-polynomial is the product of those defining polynomials. We note that the A-polynomial of K has a factor $l-1$, which correponds to abelian representations. So, we denote by $A_K(l,m)$ the A-polynomial divided by $l-1$.

Nextly, we recall Masbaum's formula of the colored Jones polynomial for a twist knot. Some notations are fixed.

$$\begin{aligned} &\{n\} = s^n - s^{-n},\ s^2 = q, \\ &\{n\}! = \{n\}\{n-1\}\cdots\{1\},\ (q)_n = (1-q)(1-q^2)\cdots(1-q^n). \end{aligned}$$

Let $J_K(n)$ be the colored Jones polynomial of a 0-framing knot K colored with the n-dimensional irreducible representation of $sl_2(\mathbb{C})$, where $J_K(n)$ is normalized by $J_{\bigcirc}(n) = 1$ for the trivial knot $\bigcirc$. ($J'_K(n)$ in [8] is equal to $J_K(n)$ in this paper.)

Let K_p be the twist knot drawn in figure 1. In [8], Masbaum obtained the following formula for the colored Jones polynomial of the twist knot K_p.

Theorem 2.1. [8] *The colored Jones polynomial of the twist knot K_p is given by*

$$J_{K_p}(n) = \sum_{k=0}^{\infty} \mathcal{C}_{K_p}(k) \frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}, \tag{1}$$

where

$$\mathcal{C}_{K_p}(k) = (-1)^{k+1} s^{k(k+3)/2} \sum_{l=0}^{k} (-1)^l q^{l(l+1)p} \{2l+1\} \frac{\{k\}!}{\{k+l+1\}!\{k-l\}!}. \tag{2}$$

We make use of the rearranged Masbaum's formula ([4])

$$J_{K_p}(n) = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^{k+1}q^{k(k+3)/2}q^{nk}\frac{(q^{-1})_{n+k}(q^{-1})_{n-1}}{(q^{-1})_n(q^{-1})_{n-k-1}} \times(-1)^l q^{l(l+1)p+l(l-1)/2}(q^{2l+1}-1)\frac{(q)_k}{(q)_{k+l+1}(q)_{k-l}}.$$

Let us put

$$F(n,k,l) = (-1)^{k+1}q^{k(k+3)/2}q^{nk}\frac{(q^{-1})_{n+k}(q^{-1})_{n-1}}{(q^{-1})_n(q^{-1})_{n-k-1}}(-1)^l q^{l(l+1)p+l(l-1)/2}(q^{2l+1}-1)\frac{(q)_k}{(q)_{k+l+1}(q)_{k-l}}.$$

Then, it easily follows that

$$(3)\quad f_0(q,q^n,q^k,q^l) := \frac{F(n+1,k,l)}{F(n,k,l)} = q^k\frac{(1-q^{-n-k-1})(1-q^{-n})}{(1-q^{-n-1})(1-q^{-n+k})},$$

$$(4)\quad f_1(q,q^n,q^k,q^l) := \frac{F(n,k+1,l)}{F(n,k,l)} = -q^{k+n+2}\frac{(1-q^{-n-k-1})(1-q^{-n+k+1})(1-q^k)}{(1-q^{k+l+2})(1-q^{k-l+1})},$$

$$(5)\quad f_2(q,q^n,q^k,q^l) := \frac{F(n,k,l+1)}{F(n,k,l)} = -q^{(2p+1)l+2p}\frac{(q^{2(l+1)+1}-1)(1-q^{k-l})}{(q^{2l+1}-1)(1-q^{k+l+2})}.$$

We obtain the following result.

Theorem 2.2. *From the three equations*

$$f_0(1,m^2,x,y)=l,\ f_1(1,m^2,x,y)=1,\ f_2(1,m^2,x,y)=1,$$

we can obtain

$$x=\frac{lm^2+1}{m^2+l},\quad (y+1)(1-x)y^p h_p(m,x)=0,$$

where $h_p(m,x)$ *is an element in* $\mathbb{Z}[m^{\pm1},x]$ *and satisfies that if we put*

$$B_{K_p}(l,m) := \begin{cases} (m^2+l)^{2p-1}(m^2)^p h_p(m,\frac{lm^2+1}{m^2+l}), & p>0,\\ (m^2+l)^{2|p|}(m^2)^{|p|}h_p(m,\frac{lm^2+1}{m^2+l}), & p\le 0,\end{cases}$$

then, $B_{K_p}(l,m)=A_{K_p}(l,m)$.

3. Proof of Theorem 2.2

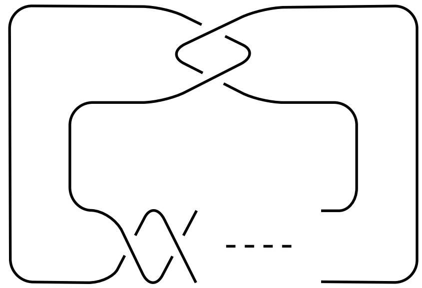
Let K_p be the twist knot pictured in Figure 1. To prove Theorem 2.2, we use the following formula for the A-polynomial of the twist knot K_p due to Hoste and Shanahan in [5].

Theorem 3.1. *(Hoste-Shanahan* [5]*) For* $p\neq -1,0,1,2$, $A_{K_p}(l,m)$ *is given recusively by*

$$(6)\quad A_{K_p}(l,m) = \begin{cases} cA_{K_{p-1}}(l,m)-dA_{K_{p-2}}(l,m), & p>0,\\ cA_{K_{p+1}}(l,m)-dA_{K_{p+2}}(l,m), & p<0,\end{cases}$$

where

$$c = -l+l^2+2lm^2+m^4+2lm^4+l^2m^4+2lm^6+m^8-lm^8,$$
$$d = m^4(l+m^2)^4,$$



p full twists
Figure 1

and with initial conditions

$$\begin{aligned} A_{K_2}(l,m) &= -l^2+l^3+2l^2m^2+lm^4+2l^2m^4-lm^6-l^2m^8 \\ &\quad +2lm^{10}+l^2m^{10}+2lm^{12}+m^{14}-lm^{14}, \\ A_{K_1}(l,m) &= l+m^6, \\ A_{K_0}(l,m) &= 1, \\ A_{K_{-1}}(l,m) &= -l+lm^2+m^4+2lm^4+l^2m^4+lm^6-lm^8. \end{aligned}$$

Now, we give a proof of Theorem 2.2.

Proof of Theorem 2.2 From the first equation

$$f_0(1,m^2,x,y) = x\frac{(1-\frac{1}{xm^2})}{(1-\frac{x}{m^2})} = l,$$

we get

$$x = \frac{lm^2+1}{m^2+l}.$$

Moreover, the second equation

$$f_1(1,m^2,x,y) = -xm^2\frac{(1-\frac{1}{xm^2})(1-\frac{x}{m^2})(1-x)}{(1-xy)(1-\frac{x}{y})} = 1$$

gives

(7) $$y^2+1 = \frac{y}{m^2}(m^4-xm^4+x^2m^2+m^2+1-x).$$

It is clear that the claim for $p=0$ holds. We consider the case $p>0$. The third equation

$$f_2(1,m^2,x,y) = -y^{2p+1}\frac{(1-\frac{x}{y})}{(1-xy)} = 1$$

implies that

(8) $$y^{2p+1}+1-y^{2p}x-xy = 0,$$

which can be changed to

$$\begin{aligned} 0 &= (y+1)(y^{2p}-y^{2p-1}+y^{2p-2}-\cdots+1-xy(y^{2p-2}-y^{2p-3}+\cdots+1)) \\ &= (y+1)\{(y^2+1-(1+x)y)(y^{2(p-1)}+y^{2(p-2)}+\cdots+y^2+1) \end{aligned}$$

$$-(1-x)(y^{2(p-1)}+y^{2(p-2)}+\cdots+y^2)\}.$$

From the equation (7), we can get

$$(y+1)(1-x)H_p(m,x,y)=0, \tag{9}$$

where $H_0(m,x,y)=1$ and for $p\geq 1$,

$$\begin{aligned}&H_p(m,x,y)\\&=\frac{y}{m^2}(m^4-xm^2+1)(y^{2(p-1)}+y^{2(p-2)}+\cdots+y^2+1)-(y^{2(p-1)}+y^{2(p-2)}+\cdots+y^2).\end{aligned}$$

To prove the theorem for $p>0$, we need

Lemma 3.2. *We have $H_p(m,x,y)=y^p h_p(m,x)$, where for $p\geq 2$, $h_p(m,x)$ is given recursively by*

$$h_p(m,x)=\frac{a}{m^2}h_{p-1}(m,x)-h_{p-2}(m,x) \tag{10}$$

with $a=m^4-xm^4+x^2m^2+m^2+1-x$, $h_0(m,x)=1$, $h_1(m,x)=\frac{b}{m^2}$, and $b=m^4-xm^2+1$.

Proof of lemma 3.2 We show by induction on p. When $p=1$, it follows that $H_1(m,x,y)=\frac{y}{m^2}(m^4-xm^2+1)=yh_1(m,x)$. In the case $p=2$, we obtain that

$$\begin{aligned}H_2(m,x,y)&=H_1(m,x,y)+\frac{yb}{m^2}y^2-y^2\\&=yh_1(m,x)+y^3h_1(m,x)-y^2\\&=y(h_1(m,x)(y^2+1)-y)\\&=y(\frac{ay}{m^2}h_1(m,x)-y)\\&=y^2(\frac{a}{m^2}h_1(m,x)-1)\\&=y^2(\frac{a}{m^2}h_1(m,x)-h_0(m,x)).\end{aligned}$$

Here we used the equation (7) in the fourth equality. Suppose that it holds for $p\leq n$. By hypothesis of induction and the definition of $H_p(m,x,y)$, we can obtain

$$\begin{aligned}H_{n+1}(m,x,y)&=H_n(m,x,y)+\frac{yb}{m^2}y^{2n}-y^{2n}\\&=y^nh_n(m,x,y)+y^2(\frac{yb}{m^2}y^{2(n-1)}-y^{2(n-1)})\\&=y^nh_n(m,x,y)+y^2(H_n(m,x,y)-H_{n-1}(m,x,y))\\&=y^nh_n(m,x,y)+y^2(y^n(h_n(m,x,y)-y^{n-1}h_{n-1}(m,x,y))\\&=y^n((y^2+1)h_n(m,x,y)-yh_{n-1}(m,x,y))\\&=y^n(\frac{a}{m^2}yh_n(m,x,y)-yh_{n-1}(m,x,y))\\&=y^{n+1}(\frac{a}{m^2}h_n(m,x,y)-h_{n-1}(m,x,y)).\end{aligned}$$

This completes the proof. □

Let us go back to the proof of the theorem. From Lemma 3.2, it is clear that $h_p(m,x)$ is in $\mathbb{Z}[m^{\pm 1}, x]$. We put

$$B_{K_p}(l,m) := (m^2+l)^{2p-1}(m^2)^p h_p(m, \frac{lm^2+1}{m^2+l}), \tag{11}$$

and then, it is easy to see that $B_{K_p}(l,m)$ satisfies the recursive relation

$$B_{K_p}(l,m) = a(m^2+l)^2 B_{K_{p-1}}(l,m) - m^4(m^2+l)^4 B_{K_{p-2}}(l,m). \tag{12}$$

By comparing it with the recursive relation (6) of $A_{K_p}(l,m)$ by Hoste and Shanahan, and by noting that $a(m^2+l)^2 = -l + l^2 + 2lm^2 + m^4 + 2lm^4 + l^2m^4 + 2lm^6 + m^8 - lm^8 = c$ and $d = m^4(m^2+l)^4$, it is shown that $B_{K_p}(l,m) = A_{K_p}(l,m)$ for $p \geq 0$. In a similar way to the case $p \geq 0$, we can prove the theorem in the case $p < 0$. □

4. Observations

We start with reviewing Wilf-Zeilberger's algorithm. For more details, we refer to [4] and [3]. A disctrete function $F(n, \mathbf{k})$ is called proper q-hypergeometric if it is one of the form

$$F(n,\mathbf{k}) = \frac{\prod_s (A_s;q)_{a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s}}{\prod_t (B_t;q)_{u_t n + \mathbf{v}_t \cdot \mathbf{k} + w_t}} q^{A(n,\mathbf{k})} \xi^{\mathbf{k}},$$

where $\mathbf{k} = (k_1, \ldots, k_r)$, $A_s, B_t \in \mathbb{Q}(q)$, a_s, u_t are integers, $\mathbf{b}_s$, $\mathbf{k}_s$ are vectors of r integers, $A(n,\mathbf{k})$ is a quadratic form, c_s, w_t are variables and ξ is an r vector of elements in $\mathbb{Q}(q)$, and for $k \geq 0$,

$$(A,q)_k := \begin{cases} (1-A)(1-Aq)\cdots(1-Aq^{k-1}) & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

In [4], Garoufalidis and Le proved that the colored Jones polynomial $J_K(n)$ can be written as a multisum

$$J_K(n) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} F(n, k_1, \ldots, k_r)$$

of a proper q-hypergeometric function $F(n, k_1, \ldots, k_r)$, where only finitely many terms in the right hand side are nonzero. Consider the operators E, E_i, Q and Q_j, $(1 \leq i,j \leq r)$ acting on a discrete function $f : \mathbb{N}^r \to \mathbb{Z}[q^{\pm}]$ by

$$\begin{aligned}
(Ef)(n,k_1,\ldots,k_r) &= f(n+1,k_1,\ldots,k_r), \\
(E_i f)(n,k_1,\ldots,k_r) &= f(n,k_1,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots,k_r), \\
(Qf)(n,k_1,\ldots,k_r) &= q^n f(n,k_1,\ldots,k_r), \\
(Q_j f)(n,k_1,\ldots,k_r) &= q^{k_j} f(n,k_1,\ldots,k_r).
\end{aligned}$$

Then, they satisfy the relations

$$\begin{aligned}
&QQ_i = Q_iQ, Q_iQ_j = Q_jQ_i, EE_i = E_iE, E_iE_j = E_jE_i, \\
&Q_iE = EQ_i, Q_iE_j = E_jQ_i \text{ for } i \neq j, EQ = qQE, E_iQ_i = qQ_iE_i,
\end{aligned}$$

which are denoted by (Rel_q). The q-Weyl algebra $\mathcal{A}$ is defined to be a noncomutative algebra with presentation

$$\mathcal{A} = \frac{\mathbb{Z}[q^{\pm}]\langle Q, Q_1, \ldots, Q_r, E, E_1, \ldots, E_r\rangle}{(Rel_q)}.$$

Wilf and Zeilberger proved that

Theorem 4.1. [15] *Every proper q-hypergeometric function $F(n,\mathbf{k})$ satisfies a $\mathbf{k}$-free recurrence*

$$\sum_{i,\mathbf{j}\in S}\sigma_{i,\mathbf{j}}(q^n)F(n+i,\mathbf{k}+\mathbf{j})=0, \tag{13}$$

where S is a finite set, $\mathbf{k}=(k_1,\dots,k_r)$, $\mathbf{j}=(j_1,\dots,j_r)$, and $\sigma_{i,\mathbf{j}}(q^n)$ are polynomials in q^n with coefficients in $\mathbb{Q}(q)$.

Putting

$$P=P(E,Q,E_1,\dots,E_r)=\sum_{i,\mathbf{j}\in S}\sigma_{i,\mathbf{j}}(Q)E^iE^{\mathbf{j}},$$

where $E^{\mathbf{j}}={E_1}^{j_1}\cdots {E_r}^{j_r}$, the recurrence (13) can be written in operator notation as $PF=0$.

Moreover, they showed that any $\mathbf{k}$-free recurrence can be transformed to

$$P=P(E,Q,1,\dots,1)+\sum_{i=1}^{r}(E_i-1)R_i(E,Q,E_1,\dots,E_r).$$

Applying it to $F(n,k_1,\dots,k_r)$ and summing over $\mathbf{k}\geq\mathbf{0}$ give us

$$P(E,Q,1,\dots,1)J_K(n)=error(n),$$

where $error(n)$ is a sum of a multisum of proper q-hypergeometric function with $r-1$ variables. So, repeating the process, we arrive at a homogeneous recursive relation of $J_K(n)$

$$P_1(E,Q)P(E,Q,1,\dots,1)J_K(n)=0,$$

where $P_1(E,Q)$ is a polynomial in E with coefficients in $\mathbb{Q}(q)[Q]$.

On the other hand, from the fact that $F(n,k_1,\dots,k_r)$ is proper q-hypergeometric, we can write $EF/F=A/B$ and $E_iF/F=A_i/B_i$ for polynomials $A,B,A_i,B_i\in\mathbb{Q}(q)[q^n,q^{k_1},\dots,q^{k_r}]$. Then, putting $Q=q^n$ and $Q_i=q^{k_i}$, it follows that $(BE-A)F=0$, $(B_iE_i-A_i)F=0$ and that $BE-A$, $B_iE_i-A_i$ generate the annihilation ideal of F in $\mathcal{A}$. From these generators, the Wilf-Zeilberger's algorithm finds a $\mathbf{k}$-free recurrence of $F(n,\mathbf{k})$ and then a recursive relation of the sum of $\sum_{\mathbf{k}}F(n,\mathbf{k})$.

Now, we apply the above algorithm to $J_{K_p}(n)$. From the equations (3), (4) and (5), setting

$$\begin{aligned}
B&=(1-q^{-n-1})(1-q^{-n+k}), & A&=q^k(1-q^{-n-k-1})(1-q^{-n}),\\
B_1&=(1-q^{k+l+2})(1-q^{k-l+1}), & A_1&=-q^{k+n+2}(1-q^{-n-k-1})(1-q^{-n+k+1})(1-q^k),\\
B_2&=(q^{2l+1}-1)(1-q^{k+l+2}), & A_2&=q^{(2p+1)l+2p}(q^{2(l+1)+1}-1)(1-q^{k-l}),
\end{aligned}$$

we have $(BE-A)F=0$, $(B_iE_i-A_i)F=0$ for $i=1,2$, with $Q=q^n$, $Q_1=q^k$ and $Q_2=q^l$. Identifying Q with m^2, E with l, Q_1 with x, Q_2 with y, from the definition of f_i for $i=0,1,2$ in Section 2, we observe that $\varepsilon(BE-A)$ corresponds to $f_0(1,m^2,x,y)-l$ and that $\varepsilon(B_iE_i-A_i)$ with $E_i=1$ corresponds to $f_i(1,m^2,x,y)-1$ for $i=1,2$, where ε is the evaluation map at $q=1$ defined in [3]. So, our result may support the AJ conjecture.

Nextly, we will present the usage of the mathematica package `qMultiSum.m` with the above theorical background, which is developed by A. Riese [13], to obtain inhomogeneous recurrences of the colored Jones polynomial of the knots 5_2 and 6_1. We will also relate them to the A-polynomial.

The formula in [8] allows us to write

$$J_{5_2}(n)=\sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^{k+1}q^{(3k^2+5k)/2}q^{nk}\frac{(q^{-1})_{n+k}(q^{-1})_{n-1}}{(q^{-1})_n(q^{-1})_{n-k-1}}q^{-l(k+1)}\frac{(q^{-1})_k}{(q^{-1})_l(q^{-1})_{k-l}}. \tag{14}$$

We denote the summand by $F(n,k,l)$. Then, the mathematica program soft `qMultiSum.m` computes the **k**-free recurrence of $F(n,k,l)$

$$
\begin{aligned}
&-q^{4+7n}\left(q-q^n\right)\left(q^2-q^n\right)\left(q^5-q^n\right)\left(q+q^n\right)\left(q^2+q^n\right)\left(q-q^{2n}\right)\left(q^3-q^{2n}\right)F(-5+n,k,l)\\
&+q^{1+3n}\left(q-q^n\right)\left(q^2-q^n\right)\left(q^4-q^n\right)^2\left(q+q^n\right)\left(q^2+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^3-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)\left(q^4-q^n-q^{1+n}\right)F(-4+n,-1+k,-1+l)\\
&-q^{9+2n}\left(q-q^n\right)\left(q^2-q^n\right)\left(q^4-q^n\right)^2\left(q+q^n\right)\left(q^2+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^3-q^{2n}\right)\left(q^9-q^{2n}\right)F(-4+n,-1+k,l)\\
&-q^{5+6n}\left(1+q+q^2+q^3+q^4\right)\left(q-q^n\right)\left(q^4-q^n\right)^2\left(q+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^3-q^{2n}\right)F(-4+n,k,l)\\
&+\left(q-q^n\right)\left(q^2-q^n\right)\left(q^3-q^n\right)^3\left(q^4-q^n\right)\left(q+q^n\right)\left(q^2+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^3-q^{2n}\right)\left(q^7-q^{2n}\right)\left(q^9-q^{2n}\right)F(-3+n,-2+k,-2+l)\\
&+q^{2+2n}\left(q-q^n\right)\left(q^3-q^n\right)^2\left(q^4-q^n\right)\left(q+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^3-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)\left(q^5+q^6+q^{2n}-q^{1+n}-2q^{2+n}-2q^{3+n}-q^{4+n}\right)F(-3+n,-1+k,-1+l)\\
&-q^{5+n}\left(q-q^n\right)\left(q^3-q^n\right)^2\left(q^4-q^n\right)\left(q+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^3-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)\left(q^6+q^{2n}+q^{1+2n}\right)F(-3+n,-1+k,l)\\
&-q^{6+5n}\left(1+q^2\right)\left(1+q+q^2+q^3+q^4\right)\left(q-q^n\right)\left(q^3-q^n\right)^2\left(q+q^n\right)\left(q^3+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)F(-3+n,k,l)\\
&-q\left(q-q^n\right)\left(q^2-q^n\right)^3\left(q^3-q^n\right)\left(q^4-q^n\right)\left(q+q^n\right)\left(q^2+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\\
&\times\left(q-q^{2n}\right)\left(q^3-q^{2n}\right)\left(q^7-q^{2n}\right)\left(q^9-q^{2n}\right)F(-2+n,-2+k,-2+l)\\
&+q^{4+n}\left(q-q^n\right)\left(q^2-q^n\right)^2\left(q^4-q^n\right)\left(q+q^n\right)\left(q^2+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^7-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)\left(q^5+q^{2n}-q^{1+n}-2q^{2+n}-2q^{3+n}-q^{4+n}+q^{1+2n}\right)F(-2+n,-1+k,-1+l)\\
&-q^{3+2n}\left(q-q^n\right)\left(q^2-q^n\right)^2\left(q^4-q^n\right)\left(q+q^n\right)\left(q^2+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^7-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)\left(q^4+q^5+q^{2n}\right)F(-2+n,-1+k,l)\\
&-q^{7+4n}\left(1+q^2\right)\left(1+q+q^2+q^3+q^4\right)\left(q^2-q^n\right)^2\left(q^4-q^n\right)\left(q^2+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)F(-2+n,k,l)\\
&-q^{6+n}(q-q^n)^2\left(q+q^2-q^n\right)\left(q^3-q^n\right)\left(q^4-q^n\right)\left(q+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\\
&\times\left(q^7-q^{2n}\right)\left(q^9-q^{2n}\right)F(-1+n,-1+k,-1+l)\\
&-q^{5+3n}(q-q^n)^2\left(q^3-q^n\right)\left(q^4-q^n\right)\left(q+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q-q^{2n}\right)\left(q^7-q^{2n}\right)\\
&\times\left(q^9-q^{2n}\right)F(-1+n,-1+k,l)\\
&-q^{8+3n}\left(1+q+q^2+q^3+q^4\right)(q-q^n)^2\left(q^4-q^n\right)\left(q+q^n\right)\left(q^4+q^n\right)\left(q^7-q^{2n}\right)
\end{aligned}
$$

$$\times \left(q^9-q^{2n}\right) F(-1+n,k,l)$$
$$+q^{9+2n}\left(q^3-q^n\right)\left(q^4-q^n\right)\left(-1+q^n\right)\left(q^3+q^n\right)\left(q^4+q^n\right)\left(q^7-q^{2n}\right)$$
$$\times \left(q^9-q^{2n}\right) F(n,k,l)=0.$$

Moreover, it can be converted into the inhomogeneous recursive relation of the colored Jones polynomial of the knot 5_2

$$\begin{aligned}
&q^{9+7n}J_{5_2}(n)\\
&+\frac{\left(-1+q^{1+n}\right)^2\left(1+q^{1+n}\right)}{\left(-1+q^n\right)\left(-1+q^{3+n}\right)\left(1+q^{3+n}\right)}\times\\
&q^{5+2n}\left(1-q^{1+n}-q^{1+2n}+q^{2+2n}+q^{3+2n}-q^{6+2n}+q^{2+3n}+q^{7+3n}+q^{5+4n}+q^{6+4n}+2q^{7+4n}\right.\\
&\left.-q^{8+4n}-q^{9+4n}-q^{8+5n}+q^{9+6n}+q^{10+6n}\right)J_{5_2}(1+n)\\
&+\frac{\left(-1+q^{2+n}\right)^2\left(1+q^{2+n}\right)\left(-1+q^{1+2n}\right)}{\left(-1+q^n\right)\left(-1+q^{3+n}\right)\left(1+q^{3+n}\right)\left(-1+q^{7+2n}\right)}\times\\
&q\left(-1+2q^{2+n}+q^{2+2n}-q^{4+2n}+q^{6+2n}+q^{7+2n}-q^{4+3n}+q^{5+3n}+3q^{6+3n}+2q^{7+3n}-q^{8+3n}\right.\\
&-2q^{9+3n}+q^{6+4n}-2q^{8+4n}-2q^{9+4n}+q^{11+4n}-q^{13+4n}-q^{8+5n}+3q^{10+5n}+3q^{11+5n}\\
&-3q^{13+5n}-2q^{14+5n}+q^{15+5n}+q^{10+6n}+q^{11+6n}-q^{13+6n}+2q^{15+6n}+q^{16+6n}+q^{14+7n}\\
&\left.+3q^{15+7n}+2q^{16+7n}-q^{17+7n}-q^{18+7n}-q^{17+8n}-q^{18+8n}+q^{20+9n}\right)J_{5_2}(2+n)\\
&-\frac{\left(-1+q^{1+n}\right)\left(1+q^{1+n}\right)\left(-1+q^{3+n}\right)\left(-1+q^{1+2n}\right)}{\left(1+q^n\right)\left(-1+q^{4+n}\right)\left(1+q^{4+n}\right)\left(-1+q^{7+2n}\right)}\times\\
&\left(-1+q^{2+n}+q^{3+n}-q^{4+2n}-3q^{5+2n}-2q^{6+2n}+q^{7+2n}+q^{8+2n}-q^{5+3n}-q^{6+3n}+q^{8+3n}\right.\\
&-2q^{10+3n}-q^{11+3n}+q^{8+4n}-3q^{10+4n}-3q^{11+4n}+3q^{13+4n}+2q^{14+4n}-q^{15+4n}-q^{11+5n}\\
&+2q^{13+5n}+2q^{14+5n}-q^{16+5n}+q^{18+5n}+q^{14+6n}-q^{15+6n}-3q^{16+6n}-2q^{17+6n}+q^{18+6n}\\
&\left.+2q^{19+6n}-q^{17+7n}+q^{19+7n}-q^{21+7n}-q^{22+7n}-2q^{22+8n}+q^{25+9n}\right)J_{5_2}(3+n)\\
&+\frac{\left(-1+q^{1+n}\right)\left(1+q^{1+n}\right)\left(-1+q^{4+n}\right)\left(-1+q^{1+2n}\right)\left(-1+q^{3+2n}\right)}{\left(-1+q^n\right)\left(-1+q^{3+n}\right)\left(1+q^{3+n}\right)\left(-1+q^{7+2n}\right)\left(-1+q^{9+2n}\right)}\times\\
&q^{1+n}\left(1+q-q^{4+n}+q^{6+2n}+q^{7+2n}+2q^{8+2n}-q^{9+2n}-q^{10+2n}+q^{8+3n}+q^{13+3n}\right.\\
&\left.-q^{12+4n}+q^{13+4n}+q^{14+4n}-q^{17+4n}-q^{17+5n}+q^{21+6n}\right)J_{5_2}(4+n)\\
&+\frac{\left(-1+q^{1+n}\right)\left(1+q^{1+n}\right)\left(-1+q^{2+n}\right)\left(1+q^{2+n}\right)\left(-1+q^{5+n}\right)\left(-1+q^{1+2n}\right)\left(-1+q^{3+2n}\right)}{\left(-1+q^n\right)\left(-1+q^{3+n}\right)\left(1+q^{3+n}\right)\left(-1+q^{4+n}\right)\left(1+q^{4+n}\right)\left(-1+q^{7+2n}\right)\left(-1+q^{9+2n}\right)}\times\\
&q^{4+2n}J_{5_2}(5+n)=G(n),
\end{aligned}$$

where $G(n)$ is a sum of sums of proper q-hypergeometric function. Setting $q=1$, and replacing $J_{5_2}(i+n)$ by l^i and q^n by m^2, the denominators cancell, the left hand side is changed to

$$\begin{aligned}
&(1+m^2 l)^2\times\\
&\{m^{14}+l(m^4-m^6+2m^{10}+2m^{12}-m^{14})-l^2(-1+2m^2+2m^4-m^8+m^{10})+l^3\},
\end{aligned}$$

and the second factor is equal to the A-polynomial of the knot 5_2. So, this supports the AJ conjecture for the knot 5_2.

Furthermore, the colored Jones polynomial of the knot 6_1 can be written

$$J_{6_1}(n) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} q^{-k^2-k}q^{nk}\frac{(q^{-1})_{n+k}(q^{-1})_{n-1}}{(q^{-1})_n(q^{-1})_{n-k-1}}q^{l(k+1)}\frac{(q)_k}{(q)_l(q)_{k-l}}. \tag{15}$$

Using `qMultiSum.m` again, we can obtain the inhomogeneous recursive relation of $J_{6_1}(n)$

$$\begin{aligned}
&q^{8+4n}J_{6_1}(n)\\
&-\frac{(-1+q^{1+n})^2(1+q^{1+n})}{(-1+q^n)(-1+q^{3+n})(1+q^{3+n})}\times\\
&q^{5+2n}\big(1+q-2q^{2+n}-q^{3+n}-q^{1+2n}-q^{2+2n}+q^{3+2n}-q^{6+2n}-q^{7+2n}+q^{3+3n}-q^{5+3n}\\
&-q^{6+3n}-q^{7+3n}+q^{8+3n}-q^{4+4n}+q^{7+4n}+q^{8+4n}-q^{9+4n}-q^{9+5n}+q^{10+6n}\big)J_{6_1}(1+n)\\
&-\frac{(-1+q^{2+n})^2(1+q^{2+n})(-1+q^{1+2n})}{(-1+q^n)(-1+q^{3+n})(1+q^{3+n})(-1+q^{7+2n})}\times\\
&q\big(-1+2q^{2+n}+q^{3+n}+q^{2+2n}+q^{3+2n}-q^{4+2n}-2q^{5+2n}+q^{6+2n}+q^{7+2n}-2q^{4+3n}-q^{5+3n}\\
&+2q^{6+3n}+3q^{7+3n}-2q^{9+3n}-q^{10+3n}-q^{5+4n}+q^{6+4n}+q^{7+4n}-2q^{8+4n}-5q^{9+4n}-3q^{10+4n}\\
&+q^{12+4n}-q^{13+4n}-2q^{8+5n}-3q^{9+5n}+3q^{11+5n}+q^{12+5n}-2q^{13+5n}-3q^{14+5n}+2q^{11+6n}\\
&+q^{12+6n}-2q^{13+6n}-2q^{14+6n}+2q^{16+6n}-q^{13+7n}+2q^{15+7n}+3q^{16+7n}-q^{18+7n}\\
&-2q^{18+8n}+q^{20+9n}\big)J_{6_1}(2+n)\\
&+\frac{(-1+q^{1+n})(1+q^{1+n})(-1+q^{3+n})(-1+q^{1+2n})}{(-1+q^n)(-1+q^{4+n})(1+q^{4+n})(-1+q^{7+2n})}\times\\
&\big(-1+2q^{3+n}+q^{3+2n}-2q^{5+2n}-3q^{6+2n}+q^{8+2n}-2q^{6+3n}-q^{7+3n}+2q^{8+3n}+2q^{9+3n}\\
&-2q^{11+3n}+2q^{8+4n}+3q^{9+4n}-3q^{11+4n}-q^{12+4n}+2q^{13+4n}+3q^{14+4n}+q^{10+5n}-q^{11+5n}\\
&-q^{12+5n}+2q^{13+5n}+5q^{14+5n}+3q^{15+5n}-q^{17+5n}+q^{18+5n}+2q^{14+6n}+q^{15+6n}-2q^{16+6n}\\
&-3q^{17+6n}+2q^{19+6n}+q^{20+6n}-q^{17+7n}-q^{18+7n}+q^{19+7n}+2q^{20+7n}-q^{21+7n}-q^{22+7n}\\
&-2q^{22+8n}-q^{23+8n}+q^{25+9n}\big)J_{6_1}(3+n)\\
&-\frac{(-1+q^{1+n})(1+q^{1+n})(-1+q^{4+n})(-1+q^{1+2n})(-1+q^{3+2n})}{(-1+q^n)(-1+q^{3+n})(1+q^{3+n})(-1+q^{7+2n})(-1+q^{9+2n})}\times\\
&q^{2+n}\big(1-q^{4+n}-q^{4+2n}+q^{7+2n}+q^{8+2n}-q^{9+2n}+q^{8+3n}-q^{10+3n}-q^{11+3n}-q^{12+3n}\\
&+q^{13+3n}-q^{11+4n}-q^{12+4n}+q^{13+4n}-q^{16+4n}-q^{17+4n}-2q^{17+5n}-q^{18+5n}\\
&+q^{20+6n}+q^{21+6n}\big)J_{6_1}(4+n)\\
&+\frac{(-1+q^{1+n})(1+q^{1+n})(-1+q^{2+n})(1+q^{2+n})(-1+q^{5+n})(-1+q^{1+2n})(-1+q^{3+2n})}{(-1+q^n)(-1+q^{3+n})(1+q^{3+n})(-1+q^{4+n})(1+q^{4+n})(-1+q^{7+2n})(-1+q^{9+2n})}\times\\
&q^{18+5n}J_{6_1}(5+n)=G'(n),
\end{aligned}$$

where $G'(n)$ is a sum of sums of proper q-hypergeometric function. Setting $q=1$, and replacing $J_{6_1}(i+n)$ by l^i and q^n by m^2, the left hand side is changed to

$$\begin{aligned}
&(1+m^2l)\{m^8+l(-2m^4+3m^6+3m^8+2m^{14}-m^{16})\\
&+l^2(1-3m^2-m^4+3m^6+6m^8+3m^{10}-m^{12}-3m^{14}+m^{16})\\
&+l^3(-1+m^2+m^4+3m^{10}+3m^{12}-2m^{14})+m^8l^4\},
\end{aligned}$$

and the second factor coincides with the A-polynomial of the knot 6_1.

Finally, we discuss our result in terms of *Volume conjecture* due to Kashaev, H.Murakami and J.Murakami. (see [7],[10]). Let K be a hyperbolic knot in S^3 and M_K the complement of K. We write $\hat{J}_K(n)$ for $J_K(n)$ evaluated at $q = \exp\frac{2\pi\sqrt{-1}}{n}$. Then, *Volume conjecture* is

Conjecture 4.2.

$$2\pi \lim_{n\to\infty} \frac{\log|\hat{J}_K(n)|}{n} = Vol(M_K),$$

where $Vol(M_K)$ is the hyperbolic volume of M_K.

From the formula of $J_{K_p}(n)$ given by Masbaum, we have

$$\hat{J}_{K_p}(n) = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^{k+l}q^{k(k+3)/2}(q)_k(q^{-1})_k q^{l(l+1)p+l(l-1)/2}\frac{(q)_{2l+1}}{(q)_{2l}}\frac{(q)_k}{(q)_{k+l+1}(q)_{k-l}}. \tag{16}$$

Computing the optimistic limit [9] of $\hat{J}_K(n)$, it is conjectured that there exists a solution (x_0, y_0) to the two equations

$$f_1(1,1,x,y) = 1,\ f_2(1,1,x,y) = 1 \tag{17}$$

such that

$$Vol(M_K) = 3D(x_0) - D(x_0y_0) - D(\frac{x_0}{y_0}),$$

where f_i is the same notation as in Section 2, and $D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z)$ with $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-w)}{w}dw$. By using Mathematica 4 and Maple V, we can numerically see that for $p = -1, 2, -2$, this conjecture is true.

In [16], Y.Yokota introduced a way to construct an ideal triangulation of the complement M_K of K associated to Kashaev's R-matrices, and gave a sketch proof of *Volume conjecture*. Moreover, in [17], he explained a method to calculate the A-polynomial of K from the triangulation of ∂M_K in [16]. We hope to find a geometrical interpretation of our method to obtain the A-polynomial, comparing with Yokota's method.

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