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The Yang-Baxter equation and invariants of links

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§ 1. Introduction

The Yang-Baxter equation first appeared in the independent papers of C.N. Yang and R.J. Baxter in the late 1960's – early 1970's. This equation and its solutions play fundamental role in the theory of completely integrable quantum systems and in the theory of exactly solved models of statistical mechanics (see [1, 9]). A relationship between the Yang-Baxter equation and the new polynomial invariants of links was implicit already in the pioneer paper of Jones [5]. In that paper Jones introduced his famous polynomial of links via a study of certain finite dimensional von Neumann algebras. A remark of D. Evans mentioned in [5] points out that these algebras were earlier discovered by physicists who used them to study the Potts model and the ice-type model of statistical mechanics.

After appearance of [5] several authors introduced two new isotopy invariants of links P and F which are (up to reparametrization) Laurent polynomials of 2 variables (see [3, 7]). Both P and F contain the Jones polynomial but can not be deduced from it. Known constructions of P appeal either to von Neumann algebras, or to Hecke algebras, or to a geometric iterative procedure based on a Conway-type relation. The only known construction of F, due to Kauffman [7], appeal to an analogous geometric procedure.

Recently, Jones [6] has shown that P can be constructed using explicit matrix representations of Hecke algebras, introduced in works on the quantum inverse scattering method and related to the Yang-Baxter equation. It is stressed in [6] that "a consistent general picture of the polynomials is starting to emerge, the relevant mathematical formalism being quantum inverse scattering method and quantum statistical mechanics".

The key observation which underlies the present paper is to the effect that one can directly construct P and F using some solutions of the Yang-Baxter equation. This leads to a general scheme which enables one to introduce these (and other) invariants of links. The resources of this scheme are far from being exhausted.

Note, however, that this approach does not shed light on the conceptual problem of understanding the polynomials from the viewpoint of algebraic topology. In particular, it is by no means clear how to extend the definition of the polynomials P, F given below to links in homology 3-spheres. (It is curious to note that a related invariant – the multivariable Conway polynomial can be defined for links in homology spheres, see [10].)

Though I do not consider von Neumann and Hecke algebras in this paper, it would be of great importance to comprehend the algebraic nature of the invariants.

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Organization of the paper. In § 2 the Yang-Baxter equation is recalled and the so-called EYB-operators are introduced. In § 3 with each EYB-operator S I associate an isotopy invariant of links T_S. In § 4 some special EYB-operators are considered, and the corresponding link invariants are studied. These invariants are shown to be equivalent to the polynomials P, F mentioned above. In § 5 under some restrictions on the EYB-operator S a state model for the invariant T_S is presented. In § 6 a nonoriented version of this state model is discussed; this model is used to prove Theorem 4.3.4 formulated in § 4.

Notation and agreements. In the whole paper the symbol K denotes a fixed commutative ring with 1 and V denotes a fixed finitely generated free K-module of rank $m \geqq 1$. For a natural n the n-times tensor product $V \otimes_K V \otimes \dots \otimes_K V$ is denoted by $V^{\otimes n}$. In particular, $V^{\otimes 1} = V$ and $V^{\otimes 2} = V \otimes_K V$. Each basis $v_1, \dots, v_m$ in V gives rise to a basis in $V^{\otimes n}$ which consists of vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$ with $i_1, \dots, i_n \in \{1, 2, \dots, m\}$. Having this basis, each (K-linear) endomorphism f of $V^{\otimes n}$ determines the multiindexed matrix $(f_{i_1, \dots, i_n}^{j_1, \dots, j_n})$, $1 \leqq i_1, j_1, \dots, i_n, j_n \leqq m$ defined by the equation

$$f(v_{i_1} \otimes \dots \otimes v_{i_n}) = \sum_{1 \leqq j_1, \dots, j_n \leqq m} f_{i_1, \dots, i_n}^{j_1, \dots, j_n} v_{j_1} \otimes \dots \otimes v_{j_n}.$$

The symbol K^* will denote the set of invertible elements of K.

By a link we shall mean a tame link in $\mathbf{R}^3$.

§ 2. The Yang-Baxter operators

2.1. Let $R: V^{\otimes 2} \to V^{\otimes 2}$ be a (K-linear) isomorphism. For natural n, i with $n-1 \geqq i \geqq 1$ denote by $R_i(n)$ the isomorphism

$$\mathrm{Id}_V^{\otimes (i-1)} \otimes R \otimes \mathrm{Id}_V^{\otimes (n-i-1)}: V^{\otimes n} \to V^{\otimes n}.$$

Thus for any $v_1, \dots, v_n \in V$

$$R_i(n)(v_1 \otimes \dots \otimes v_n) = v_1 \otimes \dots \otimes v_{i-1} \otimes R(v_i, v_{i+1}) \otimes v_{i+2} \otimes \dots \otimes v_n.$$

The isomorphism R: $V^{\otimes 2} \to V^{\otimes 2}$ is called a *Yang-Baxter operator* (or, briefly, a YB-operator) if the automorphisms $R_1 = R_1(3)$ and $R_2 = R_2(3)$ of $V^{\otimes 3}$ satisfy the equality

$$R_1 \circ R_2 \circ R_1 = R_2 \circ R_1 \circ R_2. \tag{1}$$

This is the Yang-Baxter equality (with zero spectral parameter). For examples of YB-operators and further information the reader is referred to [2, 4, 8]; see also § 4.

2.2. Recall that for each homomorphism $f: V^{\otimes n} \to V^{\otimes n}$ one can define its "operator trace" $\mathrm{Sp}_n(f)$ which is a homomorphism $V^{\otimes(n-1)} \to V^{\otimes(n-1)}$. If $v_1, \ldots, v_m$ is a basis in V then for any $i_1, \ldots, i_{n-1} \in \{1, 2, \ldots, m\}$

$$\mathrm{Sp}_n(f)(v_{i_1} \otimes \ldots \otimes v_{i_{n-1}}) = \sum_{1 \leqq j_1, \ldots, j_{n-1}, j \leqq m} f_{i_1, \ldots, i_{n-1}, j}^{j_1, \ldots, j_{n-1}, j} v_{j_1} \otimes \ldots \otimes v_{j_{n-1}}.$$

($\mathrm{Sp}_n(f)$ does not depend on the choice of basis of V). It is clear that $\mathrm{Sp}(\mathrm{Sp}_n(f)) = \mathrm{Sp}(f) \in K$ where Sp is the ordinary trace of a homomorphism.

2.3 By an *enhanced Yang-Baxter operator* (briefly, EYB-operator) I will understand a collection {a Yang-Baxter operator $R: V^{\otimes 2} \to V^{\otimes 2}$; a K-homomorphism $\mu: V \to V$; invertible elements α, β of K} which satisfy the following two conditions:

(i) The homomorphism $\mu \otimes \mu: V^{\otimes 2} \to V^{\otimes 2}$ commutes with R;

(ii) $\mathrm{Sp}_2(R \circ (\mu \otimes \mu)) = \alpha \beta \mu$; $\mathrm{Sp}_2(R^{-1} \circ (\mu \otimes \mu)) = \alpha^{-1} \beta \mu$.

Note that if μ is an isomorphism then Condition (ii) is equivalent to the following:

$$\mathrm{Sp}_2(R^{\pm 1} \circ (\mathrm{Id}_V \otimes \mu)) = \alpha^{\pm 1} \beta \, \mathrm{Id}_V.$$

We shall mainly consider the case when μ is an isomorphism presented by a diagonal matrix with respect to some basis of V. The following theorem restates Conditions (i), (ii) in this case.

2.3.1. **Theorem.** *Let $R: V^{\otimes 2} \to V^{\otimes 2}$ be a* YB-*operator. Let $v_1, \ldots, v_m$ be a basis of V and μ be an isomorphism $V \to V$ which transforms v_i into $\mu_i v_i$ for $i = 1, \ldots, m$ with $\mu_1, \ldots, \mu_m \in K^*$. The collection $(R, \mu, \alpha \in K^*, \beta \in K^*)$ is a* EYB-*operator if and only if the following two conditions are satisfied:*

(i)′ *For any $i, j, k, l \in \{1, 2, \ldots, m\}$*

$$(\mu_i \mu_j - \mu_k \mu_l) R_{i,j}^{k,l} = 0. \tag{2}$$

(ii)′ *For any $i, k \in \{1, 2, \ldots, m\}$*

$$\sum_{j=1}^{m} R_{i,j}^{k,j} \mu_j = \alpha \beta \delta_i^k; \qquad \sum_{j=1}^{m} (R^{-1})_{i,j}^{k,j} \mu_j = \alpha^{-1} \beta \delta_i^k$$

(here δ_i^k is the Kronecker symbol: $\delta_i^i = 1$, $\delta_i^k = 0$ for $k \neq i$).

Proof. Obvious.

2.4. *Remarks.* Clearly, $\mu \otimes \mu$ commutes with R iff $\mu \otimes \mu$ commutes with R^{-1}. Therefore, any of the conditions (i), (i)′ implies that for arbitrary i, j, k, l

$$(\mu_i \mu_j - \mu_k \mu_l)(R^{-1})_{i,j}^{k,l} = 0. \tag{3}$$

The condition (ii)′ of Theorem 2.3.1 implies that the product of the square $m\times m$-matrix $[R_{i,j}^{i,j}]$ with the column

$$\begin{bmatrix}\mu_1\\ \vdots\\ \mu_m\end{bmatrix}$$

is equal to the constant column

$$\begin{bmatrix}\alpha\beta\\ \vdots\\ \alpha\beta\end{bmatrix}.$$

The same is true for the matrix $[(R^{-1})_{i,j}^{i,j}]$ if we replace α by α^{-1}. Therefore, if at least one of these two square matrices is invertible over K then there exists at most one sequence $\mu_1, \dots, \mu_m$ which satisfy (ii)′ for given α, β.

In the general case $\mu_1, \dots, \mu_m$ (if exist) are not uniquely determined by R, α, β. For example, for any homomorphism $\mu: V\to V$ the collection $(\mathrm{Id}_{V^{\otimes 2}}, \mu, \alpha=1, \beta=\mathrm{Sp}\,\mu)$ is a EYB-operator.

§ 3. Invariants of braids and links

3.1 *Invariants of braids.* Every YB-operator $R: V^{\otimes 2}\to V^{\otimes 2}$ gives rise to a finite-dimensional representation of the Artin n-string braid group

$$\begin{aligned} B_n=\langle \sigma_1, \dots, \sigma_{n-1}: \sigma_i\sigma_j=\sigma_j\sigma_i &\quad \text{for } |i-j|\geqq 2;\\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} &\quad \text{for } i=1, \dots, n-1\rangle \end{aligned}$$

(here $n\geqq 1$). Namely, put $R_i=R_i(n): V^{\otimes n}\to V^{\otimes n}$ and notice that

$$R_i R_j=R_j R_i \quad \text{for } |i-j|\geqq 2$$

and (in view of the Yang-Baxter equality)

$$R_i R_{i+1} R_i=R_{i+1} R_i R_{i+1} \quad \text{for } i=1, \dots, n-1.$$

Therefore, there is a unique homomorphism $B_n\to \mathrm{Aut}(V^{\otimes n})$ which transforms σ_i into R_i for all i. Denote this homomorphism by b_R. We shall also use the homomorphism w from B_n to the additive group of integers which sends $\sigma_1, \dots, \sigma_{n-1}$ into 1.

Every EYB-operator $S=(R, \mu, \alpha, \beta)$ determines a mapping $T_S: \coprod_{n\geqq 1} B_n\to K$ as follows. For $n\geqq 1$ denote the homomorphism $\mu\otimes\mu\otimes\dots\otimes\mu: V^{\otimes n}\to V^{\otimes n}$ by $\mu^{\otimes n}$. For a braid $\xi\in B_n$ but

$$T_S(\xi)=\alpha^{-w(\xi)}\beta^{-n}\,\mathrm{Sp}(b_R(\xi)\circ\mu^{\otimes n}: V^{\otimes n}\to V^{\otimes n}).$$

The most important properties of T_S are given by the following theorem.

3.1.2. **Theorem.** *For any* $\xi, \eta \in B_n$

$$T_S(\eta^{-1}\xi\eta) = T_S(\xi\sigma_n) = T_S(\xi\sigma_n^{-1}) = T_S(\xi).$$

To prove this theorem we need the following lemma which is a direct consequence of definitions.

3.1.3. **Lemma.** *If f, g, h are endomorphisms respectively of* $V^{\otimes(n+1)}$, $V^{\otimes n}$, $V^{\otimes 2}$ *then*

$$\mathrm{Sp}_{n+1}(f\circ(g\otimes \mathrm{Id}_V)) = \mathrm{Sp}_{n+1}(f)\circ g;$$
$$\mathrm{Sp}_{n+1}((g\otimes \mathrm{Id}_V)\circ f)) = g\circ \mathrm{Sp}_{n+1}(f);$$
$$\mathrm{Sp}_{n+1}(\mathrm{Id}_V^{\otimes(n-1)}\otimes h) = \mathrm{Id}_V^{\otimes(n-1)}\otimes \mathrm{Sp}_2(h).$$

3.1.4. *Proof of Theorem* 3.1.2. It follows from the definition of EYB-operator that $\mu^{\otimes n}$ commutes with $b(\eta)$ for any $\eta\in B_n$ where $b=b_R\colon B_n\to \mathrm{Aut}(V^{\otimes n})$. Thus

$$\mathrm{Sp}(b(\eta^{-1}\xi\eta)\circ\mu^{\otimes n}) = \mathrm{Sp}(b(\eta^{-1})\circ b(\xi)\circ\mu^{\otimes n}\circ b(\eta)) = \mathrm{Sp}(b(\xi)\circ\mu^{\otimes n}).$$

Also, $w(\eta^{-1}\xi\eta) = w(\xi)$. Therefore $T_S(\eta^{-1}\xi\eta) = T_S(\xi)$.

Let us prove that $T_S(\xi\sigma_n) = T_S(\xi)$. Clearly,

$$b(\xi\sigma_n) = (b(\xi)\otimes \mathrm{Id}_V)\circ R_n\colon\ V^{\otimes(n+1)}\to V^{\otimes(n+1)}.$$

Thus,

$$\begin{aligned}&\mathrm{Sp}(b(\xi\sigma_n)\circ\mu^{\otimes(n+1)})\\ &\quad=\mathrm{Sp}[(b(\xi)\otimes \mathrm{Id}_V)\circ R_n\circ(\mathrm{Id}_V^{\otimes(n-1)}\otimes\mu\otimes\mu)\circ(\mu^{\otimes(n-1)}\otimes \mathrm{Id}_V^{\otimes 2})]\\ &\quad=\mathrm{Sp}\{\mathrm{Sp}_{n+1}[(b(\xi)\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_V^{\otimes(n-1)}\otimes[R\circ(\mu\otimes\mu)]\circ(\mu^{\otimes(n-1)}\otimes \mathrm{Id}_V^{\otimes 2})]\}.\end{aligned}$$

Lemma 3.1.3 implies that the expression in the figured brackets is equal to

$$b(\xi)\circ[\mathrm{Id}_V^{\otimes(n-1)}\otimes \mathrm{Sp}_2(R\circ(\mu\otimes\mu))]\circ(\mu^{\otimes(n-1)}\otimes \mathrm{Id}_V).$$

In view of the definition of EYB-operator, this is equal to $\alpha\beta(b(\xi)\circ\mu^{\otimes n})$. Hence

$$\mathrm{Sp}(b(\xi\sigma_n)\circ\mu^{\otimes(n+1)}) = \alpha\beta\,\mathrm{Sp}(b(\xi)\circ\mu^{\otimes n}).$$

Clearly, $w(\xi\sigma_n) = w(\xi)+1$. These equalities imply that $T_S(\xi\sigma_n) = T_S(\xi)$. The equality $T_S(\xi\sigma_n^{-1}) = T_S(\xi)$ is proved similarly.

3.2. *Invariants of links.* Recall briefly the well known relationship between braids and links. Each braid gives rise to an oriented link via closing (see Fig. 1). A theorem of J. Alexander asserts that any oriented link is isotopic to the closure of some braid. A theorem of A. Markov asserts that the closures of two braids are isotopic (in the category of oriented links) if and only if these braids are equivalent with respect to the equivalence relation in $\coprod_n B_n$ generated by Markov moves $\xi\to\eta^{-1}\xi\eta$, $\xi\mapsto\xi\sigma_n^{\pm 1}$ where $\xi,\eta\in B_n$.

Theorem 3.2 shows that for any EYB-operator $S=(R,\mu,\alpha,\beta)$ the mapping $T_S\colon \coprod_n B_n\to K$ induces a mapping of the set of oriented isotopy classes of links into K. This latter mapping is also denoted by T_S.

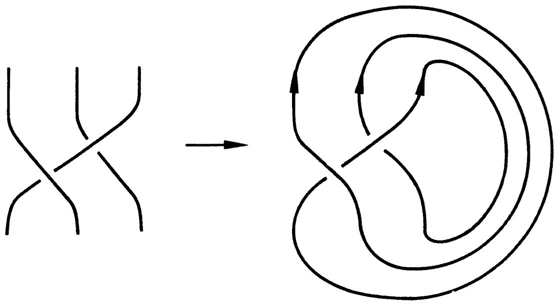

Fig. 1

For the trivial knot O we have

$$T_S(\mathrm{O}) = \beta^{-1}\,\mathrm{Sp}(\mu). \tag{4}$$

It is easy to show (using the evident equality $\mathrm{Sp}(f \otimes g) = \mathrm{Sp}(f)\,\mathrm{Sp}(g)$) that T_S is multiplicative: If a link L is the disjoint union of two links L_1 and L_2 then $T_S(L) = T_S(L_1) \cdot T_S(L_2)$. In particular, if L is the trivial n-component link then $T_S(L) = [\beta^{-1}\,\mathrm{Sp}(\mu)]^n$.

To formulate the next property of T_S we will need the following terminology. Let τ be a mapping of the set of oriented isotopy link types into K. Let $f(t) = \sum_{i=p}^{q} k_i t^i$ be a Laurent polynomial over K (i.e. $f(t) \in K[t, t^{-1}]$). Let us say that $f(t)$ *annihilates* τ and write $f(t) * \tau = 0$ if for any oriented links $L_p, L_{p+1}, \ldots, L_q$, which have diagrams coinciding outside some disk and looking as in Fig. 2 inside this disk, we have $\sum_{i=p}^{q} k_i \tau(L_i) = 0$. In particular, when $f(t) = k_- t^{-1} + k_0 + k_+ t$ the equation $f(t) * \tau = 0$ is a Conway-type relation between the invariants of links $L_- = L_{-1}$, L_0, $L_+ = L_1$ (see Fig. 3).

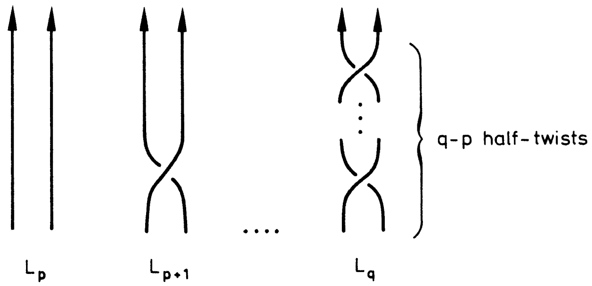

Fig. 2

3.2.1. **Theorem.** *Let $S = (R, \mu, \alpha, \beta)$ be a* EYB-*operator. If the automorphism R of $V^{\otimes 2}$ satisfies the equation $\sum_{i=p}^{q} k_i R^i = 0$ with $k_p, \ldots, k_q \in K$ then the polynomial $\sum_{i=p}^{q} k_i \alpha^i t^i$ annihilates T_S.*

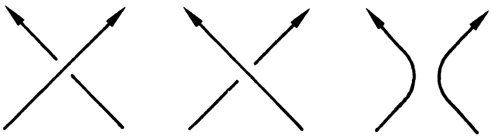

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Fig. 3

Proof. Let $L_p, \ldots, L_q$ be oriented links which have diagrams as above. Then for some braid η these links are isotopic to the closures of the braids η, $\sigma_1 \eta, \ldots, \sigma_1^{q-p} \eta$. Let n be the number of strings of η. Then

$$T_S(L_i) = T_S(\sigma_1^i \eta) = \alpha^{-i-w(\eta)} \beta^{-n} \operatorname{Sp}[(R_1)^i \circ b_R(\eta) \circ \mu^{\otimes n}].$$

Hence,

$$\sum_{i=p}^{q} k_i \alpha^i T_S(L_i) = \alpha^{-w(\eta)} \beta^{-n} \operatorname{Sp}\left[\sum_{i=p}^{q} k_i R_1^i \circ b_R(\eta) \circ \mu^{\otimes n}\right] = 0.$$

3.2.2. **Corollary.** *For any* EYB-*operator S in V the isotopy invariant T_S is annihilated by a polynomial of degree $\leqq m^2$ (where $m = \mathrm{rk}_K V$).*

Proof. Every endomorphism of $V^{\otimes 2}$ is annihilated by its characteristic polynomial.

3.3. *Remarks.* (i) Without loss of generality we can confine ourselves to EYB-operators (R, μ, α, β) with $\alpha = \beta = 1$. Indeed, if $S = (R, \mu, \alpha, \beta)$ is a EYB-operator then $S' = (\alpha^{-1} R, \beta^{-1} \mu, 1, 1)$ also is a EYB-operator and $T_S = T_{S'}$. However, sometimes it may be convenient to have non-trivial α, β.

(ii) If $S = (R, \mu, \alpha, \beta)$ is a EYB-operator then $S_1 = (-R, -\mu, \alpha, \beta)$ and $S_2 = (R, \mu, -\alpha, -\beta)$ are EYB-operators and for any n-component link L

$$T_{S_1}(L) = T_{S_2}(L) = (-1)^n T_S(L).$$

(iii) It is easy to verify that (in the notation used above) $f(t) * \tau = 0$ implies $t^i f(t) * \tau = 0$ for any integer i. If two polynomials annihilate τ then their sum also annihilates τ. Therefore, the set of polynomials annihilating τ is an ideal of the ring $K[t, t^{-1}]$. Let us call this ideal the annulator of τ. Theorem 3.2.1 shows that for any EYB-operator $S = (\mathrm{R}, \mu, \alpha, \beta)$ the annulator of T_S is contained in the annulator of the endomorphism $\alpha^{-1} R$ of $V^{\otimes 2}$. I do not know if this inclusion can be proper.

§4. Examples and applications

4.1. At present there is a general method which in principle enables one to construct EYB-operators from representations of simple (complex) Lie algebras (see [2, 4]). Each pair (a simple Lie algebra X, an automorphism of the Dynkin diagram of X) determines a "universal" YB-operator acting in an infinite dimensional vector space; with any representation of X in a vector space W one associates an induced YB-operator $W^{\otimes 2} \to W^{\otimes 2}$. These induced operators have been explicitly described for the fundamental representations of the Lie algebras of series A_n^1, B_n^1, C_n^1, D_n^1, A_n^2 and D_n^2 (see [4]; here the upper index denotes the order of the automorphism of the Dynkin diagram: 1 corresponds to the identity and 2 corresponds to the non-trivial involution). The YB-operators which correspond to series A^1, B^1, C^1, D^1 and A^2 can be enhanced to EYB-operators, see Sect. 4.2 and 4.3. The case of D^2 has remained unclear. (In this case one definitely can not enhance the YB-operator in the diagonal way with respect to the natural basis).

Up to the end of § 4, K is the Laurent polynomial ring $\mathbf{Z}[q,q^{-1}]$; V is the free K-module with a fixed basis $v_1, \ldots, v_m$. The symbol $E_{i,k}$ denotes the homomorphism $V \to V$ which transforms v_i in v_k and transforms v_r, with $r \neq i$, into 0. The homomorphism $E_{i,k} \otimes E_{j,l}$: $V^{\otimes 2} \to V^{\otimes 2}$ clearly transforms $v_i \otimes v_j$ into $v_k \otimes v_l$ and transforms other basis vectors of type $v_r \otimes v_s$ into 0.

4.2. *The series* A^1. The fundamental vector representation of the simple Lie algebra A_{m-1}^1 gives rise to the following YB-operator $V^{\otimes 2} \to V^{\otimes 2}$ (see [4] and references therein):

$$R = -q \sum_i E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} E_{i,j} \otimes E_{j,i} + (q^{-1} - q) \sum_{i<j} E_{i,i} \otimes E_{j,j}.$$

(Here $i, j = 1, 2, \ldots, m$.) Note that our notation differ from that of [4]; in particular, our operator R corresponds to $(k\xi)^{-1} \check{R}(0)$ in [4]. The equality (1) for R can be rather easily checked directly. From [4] one can also extract a formula for R^{-1}:

$$R^{-1} = -q^{-1} \sum_i E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} E_{i,j} \otimes E_{j,i} + (q - q^{-1}) \sum_{i>j} E_{i,i} \otimes E_{j,j}.$$

It is clear that

$$R - R^{-1} = (q^{-1} - q) \operatorname{Id}_V^{\otimes 2}. \tag{5}$$

4.2.1. **Theorem.** *Put* $\mu_i = q^{2i-m-1}$ *for* $i = 1, \ldots, m$. *Put* $\alpha = -q^m$, $\beta = 1$. *Then* $S = (R, \mu = \operatorname{diag}(\mu_1, \ldots, \mu_m), \alpha, \beta)$ *is a* EYB-*operator such that for any triple of links* (L_+, L_-, L_0) *as in Fig. 3*

$$q^m T_S(L_+) - q^{-m} T_S(L_-) = (q - q^{-1}) T_S(L_0) \tag{6}$$

and $T_S(\mathrm{O}) = (q^m - q^{-m})/(q - q^{-1})$.

Proof. The matrix of R with respect to the basis $\{v_i \otimes v_j \mid i, j = 1, \ldots, m\}$ in $V^{\otimes 2}$ looks as follows:

$$R_{i,j}^{k,l} = \begin{cases} -q & \text{if} \quad i=j=k=l \\ 1 & \text{if} \quad i=l \neq k=j \\ q^{-1} - q & \text{if} \quad i=k<l=j \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $R_{i,j}^{k,l} \neq 0$ then the non-ordered pairs i, j and k, l coincide. The same property holds for the matrix of R^{-1}. Thus, the condition (i)′ of Theorem 2.3.1 is satisfied and the condition (ii)′ is equivalent to the equalities

$$\sum_{j=1}^{m} R_{i,j}^{i,j}\,\mu_j=\alpha\beta; \qquad \sum_{j=1}^{m} (R^{-1})_{i,j}^{i,j}\,\mu_j=\alpha^{-1}\beta. \tag{7}$$

We have

$$\sum_{j=1}^{m} R_{i,j}^{i,j}\,\mu_j=-q\,\mu_i+\sum_{j=i+1}^{m}(q^{-1}-q)\,\mu_j=-q^{2i-m}+(q^{-1}-q)$$
$$\cdot\,[q^{2i-m+1}+q^{2i-m+3}+\ldots+q^{m-1}]=-q^m=\alpha\beta.$$

The second from the formulas (7) is verified analogously. Hence, (R,μ,α,β) is a EYB-operator. Other statements of the theorem follow directly fom the results of Sect. 3.1 and the formula (5).

4.2.2. For an oriented link L *by* $P_m(L)$ I will denote the invariant $T_S(L)$ produced by Theorem 4.2.1 and the constructions of § 3. Using $P_2, P_3, \ldots$ I will give a new proof of the following theorem.

4.2.3. **Theorem** (see [3]). *There exists a unique mapping P from the set of isotopy types of oriented links into the ring $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$ such that $P(\mathrm{O})=1$ and for any triple (L_+, L_-, L_0) as above*

$$x\,P(L_+)+x^{-1}\,P(L_-)=y\,P(L_0).$$

4.2.4. **Lemma.** *Let D be a diagram of an oriented n-component link L. Let u be the number of crossing points of D. If $m\geqq 4u+2n+1$ then the Laurent polynomial $(q-q^{-1})^{u+n}\,P_m(L)$ can be uniquely expressed as a (finite) sum*

$$\sum_{a,b\in\mathbf{Z}} r_{a,b}\,q^{a+mb}, \qquad (r_{a,b}\in\mathbf{Z}), \tag{8}$$

so that $r_{a,b}=0$ for $|a|>2u+n$. The coefficients $\{r_{a,b}\}$ do not depend on the choice of $m\geqq 4u+2n+1$.

Proof. The inequality $2u+n<m/2$ implies that if the desirable decomposition (8) exists then it is unique. Let us proof existence. It is well known that trading overcrossings for undercrossings one can transform any link diagram into a diagram of a trivial link. Therefore, applying the formula (6) in the iterative fashion we obtain that $P_m(L)$ is a finite sum of polynomials of type $\pm q^{me}(q-q^{-1})^f$ $P_m(G_d)$ where $e, f\in\mathbf{Z}$; $0\leqq f\leqq u$; G_d is the trivial d-component link, and $d\leqq u+n$. Clearly,

$$(q-q^{-1})^{u+n}[q^{me}(q-q^{-1})^f\,P_m(G_d)]=q^{me}(q^m-q^{-m})^d(q-q^{-1})^{f+u+n-d}.$$

Note that $f+u+n-d\leqq f+u+n\leqq 2u+n$. This immediately implies existence of the decomposition (8).

The last statement of Lemma follows directly from the construction of the decomposition (8).

4.2.5. *Proof of Theorem* 4.2.3. The proof of uniqueness of P is standard and therefore I omit it. Let us prove existence. Let D, L, n, u be the same objects as in the statement of Lemma 4.2.4. Let $m\geqq 4u+2n+1$ and let $\{r_{a,b}\}$ be the coefficients of the sum (8). Put

$$N(L)=(q-q^{-1})^{-u-n}\sum_{a,b\in\mathbf{Z}} r_{a,b}\,q^a t^b.$$

It follows from Lemma 4.2.4 that $N(L)$ is a Laurent polynomial of the variables q, t which does not depend on the choice of m. Since P_m is an isotopy invariant, $N(L)$ is preserved under the Reidemeister moves. (Note that $m=4u+2n+9$ is suitable both for D and for any diagram obtained from D by a single Reidemeister move). Thus $N(L)$ is an isotopy invariant of L. If follows from (6) that

$$tN(L_+)-t^{-1}N(L_-)=(q-q^{-1})\,N(L_0).$$

If G is the trivial n-component link then

$$N(G)=(t-t^{-1})^n/(q-q^{-1})^n.$$

Hence for any link L the function $N(L)$ is a Laurent polynomial of t and $q-q^{-1}$. Substituting $t=\sqrt{-1}\,x$ and $q-q^{-1}=\sqrt{-1}\,y$ and multiplying the resulting polynomial by $(q-q^{-1})/(t-t^{-1})=y/(x+x^{-1})$ we get $P(L)(x, y)$.

4.2.6. *Remark.* For any link L

$$P_m(L)=(q^m-q^{-m})(q-q^{-1})^{-1}\,P(L)(\sqrt{-1}\,q^m, \sqrt{-1}(q-q^{-1})).$$

4.3. *Series B^1, C^1, D^1 and A^2.* Fix $\nu\in\{1,-1\}$. We shall assume that if m is odd then $\nu=-1$. For $i=1, \ldots, m$ put $i'=m+1-i$ and

$$\bar{i}=\begin{cases} i-\nu/2 & \text{if } \ 1\leqq i<(m+1)/2\\ i & \text{if } \ i=(m+1)/2, \quad m \text{ being odd}\\ i+\nu/2 & \text{if } \ (m+1)/2<i\leqq m\end{cases}$$

$$\varepsilon(i)=\begin{cases} 1 & \text{if } \ 1\leqq i\leqq (m+1)/2\\ -\nu & \text{if } \ (m+1)/2\leqq i\leqq m\end{cases}$$

According to [4] the fundamental representations of simple Lie algebras of series B^1, C^1, D^1, A^2 give rise to the following YB-operator R_ν:

$$\begin{aligned} R_\nu = {} & q\sum_{\substack{i\\ i\neq i'}} E_{i,i}\otimes E_{i,i}+\sum_{\substack{i\\ i=i'}} E_{i,i}\otimes E_{i,i}+\sum_{\substack{i,j\\ i\neq j,j'}} E_{i,j}\otimes E_{j,i}\\ & +q^{-1}\sum_{\substack{i\\ i\neq i'}} E_{i,i'}\otimes E_{i',i}+(q-q^{-1})\sum_{i<j} E_{i,i}\otimes E_{j,j}\\ & +(q^{-1}-q)\sum_{i<j}\varepsilon(i)\,\varepsilon(j)\,q^{\bar{i}-\bar{j}}\,E_{i,j'}\otimes E_{i',j}. \end{aligned}$$

Here for Lie algebras B_n^1, C_n^1, D_n^1, A_n^2 the pair (m, ν) is respectively $(2n+1, -1)$, $(2n, 1)$, $(2n, -1)$, $(n+1, -1)$. It is understood that in the case of odd m the ring $K=\mathbf{Z}[q, q^{-1}]$ is extended to $\mathbf{Z}[q^{1/2}, q^{-1/2}]$.

A direct, purely computational verification of the equality (1) for R_ν seems to be extremely difficult. However, (1) can be verified for R_ν using ideas suggested by a study of link diagrams.

From [4] one can extract a formula for R_v^{-1}:

$$R_v^{-1} = q^{-1} \sum_{\substack{i \\ i \neq i'}} E_{i,i} \otimes E_{i,i} + \sum_{\substack{i \\ i = i'}} E_{i,i} \otimes E_{i,i} + \sum_{\substack{i,j \\ i \neq j,j'}} E_{i,j} \otimes E_{j,i}$$
$$+ q \sum_{\substack{i \\ i \neq i'}} E_{i,i'} \otimes E_{i',i} + (q^{-1} - q) \sum_{i>j} E_{i,i} \otimes E_{j,j}$$
$$+ (q - q^{-1}) \sum_{i>j} \varepsilon(i)\, \varepsilon(j)\, q^{\bar{i} - \bar{j}} E_{i,j'} \otimes E_{i',j}.$$

4.3.1. *Remarks.* (i) The case of even m is somewhat easier since $i \neq i'$ for all i in this case.

(ii) The formula for $\bar{i}$ given in [4] contains erroneous signs $\pm$, the correct signs used above were pointed out to me by N. Reshetikhin.

4.3.2. **Theorem.** *Put* $\mu_i = q^{2\bar{i} - m - 1}$ *for* $i = 1, 2, \ldots, m$. *Put* $\alpha = q^{m+v}$ *and* $\beta = 1$. *Then* $S_v = (R_v, \mu = \mathrm{diag}(\mu_1, \ldots, \mu_m), \alpha, \beta)$ *is a* EYB-*operator.*

Proof. The matrices of R_v and R_v^{-1} have the following property: If $(R_v)_{i,j}^{k,l} \neq 0$ or $(R_v^{-1})_{i,j}^{k,l} \neq 0$ then either the non-ordered pairs $\{i,j\}$, $\{k,l\}$ coincide, or $j = i'$ and $l = k'$ (or both). If $j = i'$ and $l = k'$ then

$$\mu_i \mu_j = \mu_i \mu_{i'} = q^{2(\bar{i} + \bar{i}') - 2m - 2} = 1 = \mu_k \mu_l.$$

Thus, S_v satisfies the first condition (i)$'$ of Theorem 2.3.1. Let us check (ii)$'$. Put $a(i,j) = (R_v)_{i,j}^{i,j}$ and $b(i,j) = (R_v^{-1})_{i,j}^{i,j}$ where $i,j = 1, \ldots, m$. We have to prove that for any i

$$\sum_{j=1}^{m} a(i,j)\, q^{2\bar{j} - m - 1} = q^{m+v}, \tag{9}$$

$$\sum_{j=1}^{m} b(i,j)\, q^{2\bar{j} - m - 1} = q^{-(m+v)}. \tag{10}$$

It is easy to check up that for any i, j, k, l

$$(R_v^{-1})_{i,j}^{k,l} = \varphi((R_v)_{i',j'}^{k',l'})$$

where φ is the automorphism of $\mathbf{Z}[q^{1/2}, q^{-1/2}]$ sending $q^{1/2}$ into $q^{-1/2}$. In particular, $b(i,j) = \varphi(a(i',j'))$. Thus

$$\sum_{j=1}^{m} b(i,j)\, q^{2\bar{j} - m - 1} = \sum_{j=1}^{m} \varphi(a(i',j')\, q^{2\bar{j}' - m - 1}) = \varphi\left(\sum_{j=1}^{m} a(i',j)\, q^{2\bar{j} - m - 1}\right).$$

Therefore, (9) implies (10).

Using the equalities

$$\varepsilon(i)\, \varepsilon(i') = -v \quad \text{and} \quad \bar{i} + \bar{i}' = m + 1$$

it is easy to compute $a(i,j)$:

$$a(i,j)=\begin{cases} 0 & \text{if} \quad i>j \\ q & \text{if} \quad i=j\neq i' \\ 1 & \text{if} \quad i=j=i' \\ q-q^{-1} & \text{if} \quad i<j\neq i' \\ (q-q^{-1})(1+v\,q^{2\bar{i}-m-1}) & \text{if} \quad i<j=i' \end{cases}$$

To verify (9) we shall consider 3 cases: $i<i'$, $i=i'$, $i>i'$. If $i>i'$, then $i>(m+1)/2$ and

$$\sum_{j=1}^{m} a(i,j)\,q^{2\bar{j}-m-1}=q^{2i-m+v}+(q-q^{-1})\sum_{j=i+1}^{m} q^{2j-m-1+v}=q^{m+v}.$$

If $i=i'$, then $i=(m+1)/2$, m is odd, $v=-1$ and the computation is similar. Let $i<i'$. Then $i<(m+1)/2$ and

$$\sum_{j=1}^{m} a(i,j)\,q^{2\bar{j}-m-1}=q^{2\bar{i}-m}+(q-q^{-1})\sum_{j=i+1}^{m} q^{2\bar{j}-m-1}+v(q-q^{-1}). \tag{11}$$

The sequence

$$q^{2\bar{j}-m-1}, \quad j=i+1, i+2, \ldots, m$$

is the geometric progression

$$q^{2\bar{i}+1-m}, \quad q^{2\bar{i}+3-m}, \ldots, q^{m+v-1}$$

with one superfluous member $q^0=1$ in case $v=-1$ or one omitted member $q^0=1$ in case $v=1$. This excess or omission is exactly compensated by $v(q-q^{-1})$. Therefore the right-hand side of (11) equals

$$q^{2\bar{i}-m}+(q-q^{-1})[q^{2\bar{i}+1-m}+q^{2\bar{i}+3-m}+\ldots+q^{m+v-1}]=q^{m+v}.$$

4.3.3. For an oriented link L the invariant $T_S(L)$ with $S=S_v$ will be denoted by $Q_{m,v}(L)$. It follows from (4) that

$$Q_{m,v}(\mathrm{O})=-v+(q^{m+v}-q^{-m-v})/(q-q^{-1}).$$

Apriori, if m is odd then $Q_{m,v}(L)\in\mathbf{Z}[q^{1/2}, q^{-1/2}]$. The next theorem shows among other things that actually $Q_{m,v}(L)\in\mathbf{Z}[q,q^{-1}]$. In the statement of this theorem $\sqrt{-v}=1$, if $v=-1$, and $\sqrt{-v}$ is the complex unit $\sqrt{-1}$ if $v=1$.

4.3.4. **Theorem.** *Let $v\in\{1,-1\}$. For any diagram D of an oriented link L the polynomial*

$$\tilde{Q}_{m,v}(D)=(\sqrt{-v}\,q^{m+v})^{w(D)}\,Q_{m,v}(L)$$

does not depend on the choice of orientation of L. (Here $w(D)$ is the writhe of D – see Sect. 5.1). If D_+, D_-, D_0 and D_∞ are link diagrams coinciding outside some disk and looking as in Fig. 4 inside this disk then

$$\tilde{Q}_{m,v}(D_+)+v\tilde{Q}_{m,v}(D_-)=\sqrt{-v}(q-q^{-1})[\tilde{Q}_{m,v}(D_0)+v\tilde{Q}_{m,v}(D_\infty)]. \tag{12}$$

This theorem is proved in § 6 using the results of § 5.

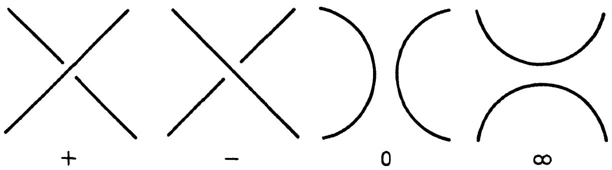

Fig. 4

4.3.5. **Corollary.** *Let* $v\in\{1,-1\}$. *There exists a unique mapping* Q_v *of the set of isotopy classes of oriented links into the ring* $\mathbf{Z}[x,x^{-1},y,y^{-1}]$ *such that* $Q_v(\mathrm{O})=1$ *and*

1) *for any diagram D of an oriented link L the polynomial* $\tilde{Q}_v(D)=x^{w(D)}\,Q_v(L)$ *does not depend on the choice of orientation of L*;

2) *if* D_+, D_-, D_0, D_∞ *are link diagrams as in the statement of Theorem* 4.3.4 *then*

$$\tilde{Q}_v(D_+)+v\,\tilde{Q}_v(D_-)=y[\tilde{Q}_v(D_0)+v\tilde{Q}_v(D_\infty)].$$

This Corollary is deduced from Theorem 4.3.4 exactly in the same fashion as Theorem 4.2.3 was deduced from Theorem 4.2.1. (In particular, Lemma 4.2.4 remains true if one replaces in its statement P_m by $Q_{m,v}$ and all (other) entries of m by $m+v$.)

The polynomial Q_1 was introduced by Kauffman [7]. (It is denoted by F in [7]). It was pointed out to me by W.B.R. Lickorish that the link invariants $Q_-=Q_{-1}$ and $Q_+=Q_1$ are essentially equivalent: If L is an n-component link then

$$Q_-(L)(x,y)=(-1)^{n-1}\,Q_+(L)(\sqrt{-1}\,x,\,-\sqrt{-1}\,y).$$

It is clear that

$$Q_{m,v}(L)=\left[-v+\frac{q^{m+v}-q^{-m-v}}{q-q^{-1}}\right]Q_v(L)(\sqrt{-v}\,q^{m+v},\sqrt{-v}(q-q^{-1})). \tag{13}$$

4.3.6. *Remarks.* (i) Using Q_v or $\tilde{Q}_v$ one can easily define an isotopy invariant of non-oriented links. Namely, if D is a diagram of an oriented link L and if u is the sum of the crossing signs (± 1) over all self-crossing of components of L then the polynomial $x^{-u}\,\tilde{Q}_v(D)$ does not depend on the choice of orientation of L. This polynomial is easily seen to be preserved under Reidemeister moves. Hence, it is an isotopy invariant.

(ii) It is easy to deduce from the properties of Q_v stated in Corollary 4.3.5 that Q_v is annihilated by the polynomial

$$(x^2t-1)(xt^2-yt+vx^{-1})\in(\mathbf{Z}[x,x^{-1},y,y^{-1}])[t].$$

This fact is equivalent to the identity

$$(R_v + v q^{-m-v} I)(R_v + q^{-1} I)(R_v - q I) = 0$$

where $I = \mathrm{Id}_{V \otimes V}$. It is curious to note that the image of $(R_v + q^{-1} I)(R_v - q I)$ is the one-dimensional subspace of $V \otimes V$ generated by $\rho = \sum_{i=1}^{m} \varepsilon(i) q^{\bar{i}} v_i \otimes v_{i'}$. A direct calculation shows that $R_v(\rho) = -v q^{-m-v} \rho$.

(iii) According to [3, 7] the Jones polynomial V_L can be computed from both $P(L)$ and $Q_+(L)$. This computation shows that up to a standard multiple and reparametrization V is the same link invariant as P_2 and $Q_{2,+}$. If one introduced $Q_{m,-}$ with $m<0$ by the formula (13) then one would similarly have that V is equivalent to $Q_{-2,-}$. The polynomial $Q_{2,-}$ stays somewhat aside: It can be completely computed from the linking coefficients of the components of the link. In particular, if L is a knot then $Q_{2,-}(L) = 2$. This follows from the equality $\tilde{Q}_{2,-}(D) = \sum_{\omega \in \Omega} q^{w(\omega)}$ where D is an arbitrary link diagram, Ω is the set of orientations of D, $w(\omega)$ is the writhe of the oriented link diagram (D, ω).

§ 5. State models for the invariants of links

L. Kauffman constructed for the Jones polynomial of links a "state model" of striking beauty and simplicity (see [7]). The construction is based on ideas which came from the study of the Potts model in statistical mechanics (see [1]). Jones [6] constructed a state model for the polynomials $P(L)(\sqrt{-1}\, q^{-m}, \sqrt{-1}(q - q^{-1}))$, $m = 2, 3, \ldots$ For $m = 2$ the Jones model is related to the Kauffman model via "arrow coverings" (see [1, 6]). In this section under certain conditions on the EYB-operator S I construct a state model for T_S generalizing the Jones construction.

Fix a EYB-operator $S = (R, \mu, \alpha, \beta)$. We shall assume that the K-module V is provided with a basis so that μ is diagonal regarding this basis, $\mu = \mathrm{diag}(\mu_1, \ldots, \mu_m)$, $m = \mathrm{rk}_K V$.

5.1. *States of diagrams.* Let D be a diagram of an oriented link L. The diagram D determines a planar graph Γ_D which is obtained from D by identifying each overcrossing point with the corresponding undercrossing point. (Γ_D is the projection of L in R^2; see Fig. 5). The orientation of L induces orientation of all edges of Γ_D so that Γ_D is an oriented graph. By vertices and edges of D I will mean respectively vertices and (oriented) edges of Γ_D. The sets of vertices and edges of D will be denoted respectively by $\mathrm{Vert}\, D$ and $\mathrm{Edg}\, D$. The writhe $w(D)$ of D is defined to be $w_+(D) - w_-(D)$ where $w_+(D)$ and $w_-(D)$ are respectively the numbers of positive and negative vertices of D (see Fig. 6).

A *state* of D is an arbitrary mapping f: $\mathrm{Edg}\, D \to \{1, 2, \ldots, m\}$. The set of all states of D is denoted by $\mathrm{St}(D)$. With each state f of D and each vertex u of D we associate an element $\pi_u(f)$ of K as follows. If a, b, c, d are edges of D incident to u as in Fig. 7 then

$$\pi_u(f) = \begin{cases} R^{f(c), f(d)}_{f(a), f(b)} & \text{if } u \text{ is positive} \\ (R^{-1})^{f(c), f(d)}_{f(a), f(b)} & \text{if } u \text{ is negative} \end{cases}$$

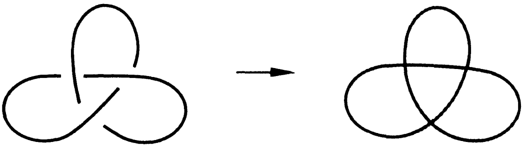

Fig. 5

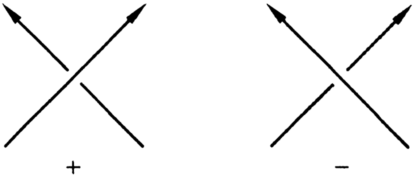

Fig. 6

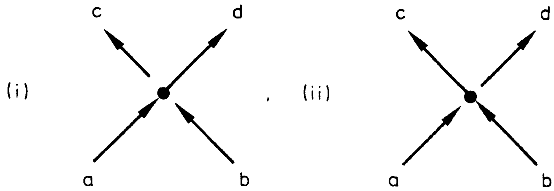

Fig. 7

(Here the matrix elements of R, R^{-1} are taken with respect to the basis in $V^{\otimes 2}$ constructed as usual from the fixed basis in V.)

Put

$$\Pi(f) = \prod_{u \in \operatorname{Vert} D} \pi_u(f) \in K. \tag{14}$$

5.2. *State models for special diagrams.* Let D be the diagram of a link L obtained by closing a diagram of a certain n-string braid ξ. Let $a_1, \ldots, a_n$ be the (oriented) edges of D which tie the top and bottom ends of ξ (see Fig. 1). For a state f of D put

$$\int_D f = \prod_{i=1}^{n} \mu_{f(a_i)} \in K.$$

5.2.1. **Theorem.**

$$T_S(L) = T_S(\xi) = a^{-w(D)} \beta^{-n} \sum_{f \in \operatorname{St}(D)} \Pi(f) \int_D f. \tag{15}$$

Proof. Let $v_1, \ldots, v_m$ be the fixed basis of V. Consider the matrix of $b_R(\xi) \circ \mu^{\otimes n}$: $V^{\otimes n} \to V^{\otimes n}$ with respect to the basis $\{v_{i_1} \otimes \ldots \otimes v_{i_n} | 1 \leqq i_1, \ldots, i_n \leqq m\}$. It is easy to see that the diagonal element of this matrix corresponding to the basis vector $v_{i_1} \otimes \ldots \otimes v_{i_n}$ is equal to

$$\sum_{\substack{f\in \mathrm{St}(D)\\ f(a_1)=i_1,\,\ldots,\,f(a_n)=i_n}} \Pi(f)\,\mu_{i_1}\mu_{i_2}\cdots\mu_{i_n}.$$

This implies (15).

5.3. *Further definitions.* In order to generalize (15) to the case of an arbitrary oriented diagram D we need to generalize $\int_D f$ and the number of strings n. The generalization of n is the number rot $D\in\mathbf{Z}$ defined as $(2\pi)^{-1}\psi$ where ψ is the total rotation angle of the tangent vector of D. (The direction of the clockwise rotation is taken to be positive.) Alternatively, one can define rot D using the "Gauss mapping" associated with D. Namely, let us flatten Γ_D in a small neighbourhood U of its vertices so that two branches incident to any vertex were tangent to each other in this vertex (see Fig. 8). Denote by $\Gamma^o=\Gamma_D^o$ the oriented graph in $\mathbf{R}^2$ obtained by this flattening. Let $\Delta\colon \Gamma^o\to S^1$ be the Gauss mapping which associates with a point $x\in\Gamma^o$ the unit positive tangent vector of Γ^o in x. Then $\operatorname{rot} D=\deg\Delta$. (The unit circle S^1 is provided with the clockwise orientation.) Note that there is a natural homeomorphism $\Gamma_D\to\Gamma^o$ which is identity on $\Gamma_D\setminus U$. In what follows we shall identify the sets of vertices and (oriented) edges of Γ^o respectively with Vert$D$ and Edg$D$ via this homeomorphism.

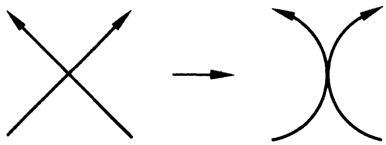

Fig. 8

To define $\int_D f$ I will assume that our EYB-operator $S=(R,\mu,\alpha,\beta)$ satisfies the following two conditions:

(5.3.1) $\mu_1,\ldots,\mu_m\in K^*$;

(5.3.2). If $R_{i,j}^{k,l}\neq 0$ or $(R^{-1})_{i,j}^{k,l}\neq 0$ then $\mu_i\mu_j=\mu_k\mu_l$.

These conditions are not too restrictive. In particular, if the ring K has no zero divisors then 5.3.2 holds for an arbitrary EYB-operator (see § 2).

The integral $\int_D f$ will be defined for the so-called *contributing states* of D. A state f of D is called contributing if $\pi_u(f)\neq 0$ for all $u\in\operatorname{Vert}D$. The set of contributing states of D is denoted by $\mathrm{CSt}(D)$.

Let $f\in\mathrm{CSt}(D)$. If $u\in\operatorname{Vert}D$ and if a, b, c, d are edges incident to u as in Fig. 7 then $\pi_u(f)\neq 0$ and in view of (5.3.2) $\mu_{f(a)}\mu_{f(b)}=\mu_{f(c)}\mu_{f(d)}$. Therefore the formal sum $\sum_{a\in\operatorname{Edg}D}\mu_{f(a)}\,a$ is a one-dimensional cycle in $\Gamma_D\approx\Gamma_D^o$ with coefficients in the multiplicative abelian group K^*. Let $[f]$ be the class of this cycle in $H_1(\Gamma^o;K^*)=H_1(\Gamma^o;\mathbf{Z})\otimes_{\mathbf{Z}}K^*$. Then $\Delta_*([f])\in H_1(S^1;K^*)$. The chosen orientation of S^1 determines an isomorphism $\Psi\colon H_1(S^1;K^*)\to K^*$. Put $\int_D f$

$=\Psi(\Delta_*([f]))$. It is clear that the integral $\int_D f$ is preserved by ambient isotopies of D in $\mathbf{R}^2$.

5.3.3. *Remarks.* 1. It is convenient to calculate $\int_D f$ as follows. Let $p \in S^1$ be a generic value of the mapping $\Delta\colon \Gamma^o \to S^1$. Then $\Delta^{-1}(p)$ is a finite set of non-vertex points of Γ^o in which the tangent vector is parallel (and equally directed) to the unit vector p. For $x \in \Delta^{-1}(p)$ denote by $a(x)$ the (oriented) edge of Γ^o containing x. Put $\varepsilon(x)=1$ if when one goes along $a(x)$ through x the tangent vector rotates in the clockwise direction and put $\varepsilon(x)=-1$ in the opposite case. Then

$$\int_D f = \prod_{x \in \Delta^{-1}(p)} \mu_{f(a(x))}^{\varepsilon(x)}. \tag{16}$$

2. If D is the closure of a braid diagram then the definitions of $\int_D f$ given in Sections 5.2, 5.3 are equivalent. The easiest way to see this is to apply the preceding remark to the vector p directed downwards in the plane of Fig. 1.

5.4. **Theorem.** *Let* $S=(R, \mu=\operatorname{diag}(\mu_1, \dots, \mu_m), \alpha, \beta)$ *be a* EYB-*operator which satisfies Conditions* 5.3.1, 5.3.2 *and the following condition:*

5.4.1. *For any* $i, j, k, l \in \{1, 2, \dots, m\}$

$$\sum_{1 \leqq x, y \leqq m} (R^{-1})_{y,i}^{x,j} R_{x,l}^{y,k} \mu_j \mu_y^{-1} = \delta_l^j \delta_k^i. \tag{17}$$

Then for any diagram D of an oriented link L

$$T_S(L) = \alpha^{-w(D)} \beta^{-\operatorname{rot} D} \sum_{f \in \operatorname{CSt}(D)} \Pi(f) \int_D f. \tag{18}$$

Condition 5.4.1 looks rather unpleasant. However, it is necessary for the right-hand side of (18) to be preserved by the Reidemeister move $\Omega 2b$ (see Fig. 9). Anyway, it is easy to check up that the operator S constructed in Theorem

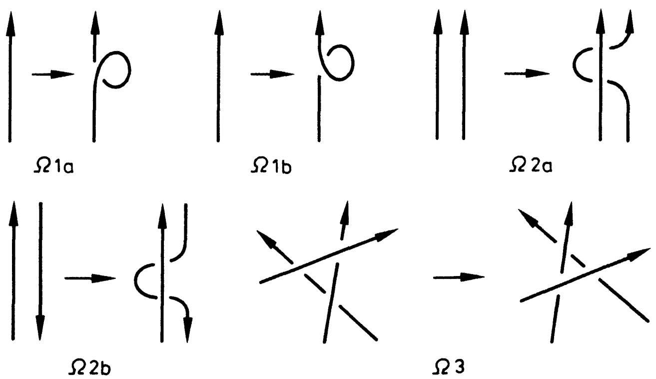

Fig. 9

4.2.1 satisfies 5.4.1. Both operators S_+ and S_- constructed in Theorem 4.3.2 also satisfy 5.4.1; this is verified in § 6. Thus, Theorem 5.4 gives state models for link invariants P_m, $Q_{m,+}$, $Q_{m,-}$ with $m=2, 3, \ldots$

Proof of Theorem 5.4. If D is the closure of a braid diagram then (18) follows straightforwardly from (15). Therefore, it suffices to check up the invariance of the right-hand side of (18) under the Reidemeister moves. It suffices to consider the moves $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, $\Omega 2b$, $\Omega 3$ pictured in Fig. 9. Other Reidemeister moves (obtained from these by a change of orientations of branches) may be presented as compositions of the listed moves and their inverses (see, for example, Fig. 10).

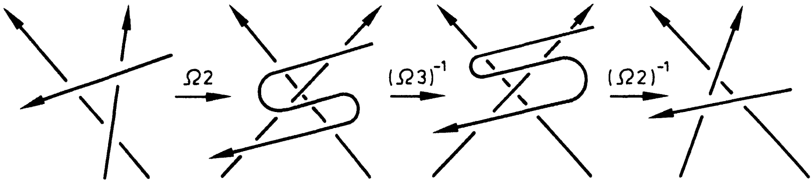

Fig. 10

Let E be a link diagram obtained from D by an application of $\Omega 1a$. Clearly, $w(E)=w(D)+1$ and $\operatorname{rot} E=\operatorname{rot} D+1$. Denote by u the additional vertex of E created by the move. Denote by a, b, c the edges of E incident to u (see Fig. 11).

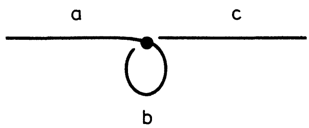

Fig. 11

For a contributing state f of E denote by $M(f)$ the set of contributing states of E $\operatorname{Edg} E \to \{1, 2, \ldots, m\}$ which are equal to f on the set $\operatorname{Edg} E \setminus \{b\}$. If $g \in M(f)$ then

$$\int_E g = \mu_{g(b)} \mu_{f(b)}^{-1} \int_E f.$$

Hence,

$$\sum_{g \in M(f)} \pi_u(g) \int_E g = \mu_{f(b)}^{-1} \int_E f \sum_{j=1}^{m} R_{f(a),j}^{f(c),j} \mu_j. \tag{19}$$

If $f(a) \neq f(c)$ then according to Theorem 2.3.1 the sum in the right-hand side of (19) is equal to zero so that $\sum_{g \in M(f)} \pi_u(g) \int_E g = 0$. If $f(a)=f(c)$ then f evidently gives rise to a state $h_f \in \operatorname{CSt}(D)$ so that $\int_D h_f = \mu_{f(b)}^{-1} \int_E f$. In view of (19) and Theorem 2.3.1,

$$\sum_{g \in M(f)} \Pi(g) \int_E g = \prod_{v \in \operatorname{Vert} D} \pi_v(h_f) \cdot \alpha\beta \int_D h_f = \alpha\beta \Pi(h_f) \int_D h_f.$$

This implies that

$$\alpha^{-w(E)}\beta^{-\mathrm{rot}\,E}\sum_{g\in \mathrm{CSt}(E)}\Pi(g)\int_E g=\alpha^{-w(D)}\beta^{-\mathrm{rot}\,D}\sum_{h\in \mathrm{CSt}(D)}\Pi(h)\int_D h.$$

The move $\Omega 1b$ is considered similarly.

The moves $\Omega 2$ and $\Omega 3$ do not change writhe and rot of link diagrams. Therefore it is sufficient to verify that these moves do not change the sum $\sum_{f\in \mathrm{CSt}(D)}\Pi(f)\int_D f$.

Let E be a link diagram obtained from D by an application of $\Omega 2a$. Denote the additional vertices and (oriented) edges of E respectively by u, v and a, b, c, d, r, s (see Fig. 12). For a contributing state f of E denote by $M(f)$ the set of contributing states of E coinciding with f on $\mathrm{Edg}\,E\setminus\{r,s\}$. If $g\in M(f)$ then $\int_E g=\int_E f$; this easily follows from Remark 5.3.3.1 applied to a vector directed oppositely to the tangent vectors of a, b, c, d. It is clear that

$$\sum_{g\in M(f)}\pi_u(g)\,\pi_v(g)=\sum_{1\leq x,y\leq m}R^{x,y}_{f(a),f(b)}(R^{-1})^{f(c),f(d)}_{x,y}$$

$$=\begin{cases}0, & \text{if } f(a)\neq f(c) \text{ or } f(b)\neq f(d)\\ 1, & \text{if } f(a)=f(c) \text{ and } f(b)=f(d)\end{cases}.$$

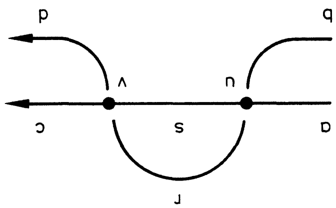

Fig. 12

If $f(a)=f(c)$ and $f(b)=f(d)$ then f gives rise to a state $h_f\in \mathrm{CSt}(D)$ so that $\int_D h_f=\int_E f$. Thus,

$$\sum_{g\in \mathrm{CSt}(E)}\Pi(g)\int_E g=\sum_{h\in \mathrm{CSt}(D)}\Pi(h)\int_D h.$$

The moves $\Omega 2b$ and $\Omega 3$ are considered along the same lines using (instead of the identity $RR^{-1}=1$) respectively formulas (17) and (1).

§ 6. Proof of Theorem 4.3.4

6.1. *Preliminaries.* Let $S=(R\colon V^{\otimes 2}\to V^{\otimes 2},\ \mu\colon V\to V,\ \alpha,\beta)$ be a EYB-operator. Assume that V has a preferred basis and that μ is the diagonal homomorphism diag$(\lambda_1^2, \lambda_2^2, \ldots, \lambda_m^2)$ where $\lambda_1, \ldots, \lambda_m \in K^*$. Assume also that the following holds true:

(6.1.1). $\lambda_{i'} = \lambda_i^{-1}$ for any $i = 1, \ldots, m$ (where $i' = m+1-i$);

(6.1.2). For any $i, j, k, l \in \{1, 2, \ldots, m\}$

$$R_{i,j}^{k,l} = R_{l',k'}^{j',i'}. \tag{20}$$

(6.1.3). If $R_{i,j}^{k,l} \neq 0$, then $\lambda_i \lambda_j = \lambda_k \lambda_l$.

In this setting we shall modify definitions of § 5 to make them applicable to non-oriented link diagrams.

Let E be a non-oriented link diagram. The graph $\Gamma_E \subset \mathbf{R}^2$ and the set Vert$E$ are defined as in Sect. 5.1. By an oriented edge of E we shall mean a pair (an edge of E, an orientation of this edge). The set of oriented edges of E is denoted by Edg E. For an edge $a \in$ Edg$E$ we denote by a' the same edge with the opposite orientation. A state of E (with respect to S) is a mapping f: Edg $E \to \{1, 2, \ldots, m\}$ such that $f(a') = (f(a))'$ for all $a \in$ Edg E. Denote the set of states of E by St(E).

With each state f of E and each vertex u of E we associate $\pi_u(f) \in K$: If a, b, c, d are oriented edges of E incident to u as in Fig. 7 (i) then $\pi_u(f) = R_{f(a), f(b)}^{f(c), f(d)}$. Correctness of this definition follows from (20). For a state f of E define $\Pi(f)$ by the formula (14). A state f of E is called contributing if $\pi_u(f) \neq 0$ for all $u \in$ Vert E. The set of contributing states of E is denoted by CSt(E).

Let us flatten Γ_E in a small neighbourhood of Vert$E$ in the way depicted in Fig. 13. It is important to note that this flattening depends not only on the graph Γ_E but on the diagram E itself. Note also that this flattening differs from the one used in § 5.

Denote the planar graph obtained by this flattening by $\Gamma^\wedge = \Gamma_E^\wedge$. As in Sect. 5.3 the set of oriented edges of $\Gamma^\wedge$ is identified with Edg$E$. Denote by Δ the Gauss mapping $\Gamma^\wedge \to RP^1$ which associates with a point $x \in \Gamma^\wedge$ the line in $\mathbf{R}^2$ tangent to $\Gamma^\wedge$ in x.

Let f be a contributing state of E. Provide every edge of $\Gamma^\wedge$ with an orientation and consider the sum $\Sigma = \Sigma \lambda_{f(a)}\, a$ where a runs through this family of oriented edges. Conditions 6.1.1 and 6.1.3 imply that Σ is a cycle in $\Gamma^\wedge$ with coefficients in K^* and that its class in $H_1(\Gamma^\wedge; K^*)$ does not depend on the choice of orientation in the edges of E. Denote by $\int_E f$ the image of this class under the homomorphism $\Delta_*: H_1(\Gamma^\wedge; K^*) \to H_1(RP^1; K^*) = K^*$. Here the isomorphism $H_1(RP^1; K^*) \to K^*$ is induced by the orientation of RP^1 corresponding to the clockwise rotation of a line in $\mathbf{R}^2$.

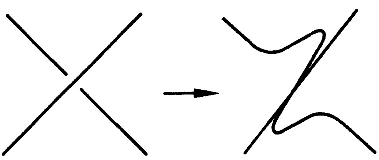

Fig. 13

The next formula is quite analogous to (16): If $p \in RP^1$ is a generic value of $\Delta: \Gamma^\wedge \to RP^1$ then

$$\int_E f = \prod_{x \in \Delta^{-1}(p)} \lambda_{f(a(x))}^{\varepsilon(x,a(x))}. \tag{21}$$

Here for a non-vertex point x of $\Gamma^\wedge$ the symbol $a(x)$ denotes an arbitrarily oriented edge of $\Gamma^\wedge$ containing x. Note that $\lambda_{f(a(x))}^{\varepsilon(x,a(x))}$ does not depend on the choice of orientation in this edge, since $\lambda_{f(a')}^{\varepsilon(x,a')} = \lambda_{f(a)'}^{-\varepsilon(x,a)} = [\lambda_{f(a)}^{-1}]^{-\varepsilon(x,a)} = \lambda_{f(a)}^{\varepsilon(x,a)}$ where $a = a(x)$.

6.2. **Lemma.** *Let* $S = (R, \mu = \operatorname{diag}(\lambda_1^2, \ldots, \lambda_m^2), \alpha, \beta)$ *be a* EYB-*operator with* $\lambda_1, \ldots, \lambda_m \in K^*$. *Suppose that Conditions* 6.1.1, 6.1.2 *and* 6.1.3 *hold true and that for all* $i, j, k, l \in \{1, 2, \ldots, m\}$

$$(R^{-1})_{i,j}^{k,l} = \lambda_i \lambda_l^{-1} R_{k',i}^{l,j'}. \tag{22}$$

If D is a diagram of an oriented link L and if E is the underlying non-oriented diagram then

$$\alpha^{w(D)} \beta^{\operatorname{rot} D} T_S(L) = \sum_{f \in \operatorname{CSt}(E)} \Pi(f) \int_E f. \tag{23}$$

Proof. Put $\mu_i = \lambda_i^2$ for $i = 1, \ldots, m$. It is easy to deduce from 6.1.1, 6.1.3 and the formula (22) that S satisfies Conditions 5.3.1 and 5.3.2. In view of (22) for any $i, j, k, l \in \{1, 2, \ldots, m\}$ $(R^{-1})_{y,i}^{x,j} = \lambda_y \lambda_j^{-1} R_{x',y}^{j,i'}$ and $R_{x,l}^{y,k} = \lambda_y \lambda_l^{-1} (R^{-1})_{l,k'}^{x',y}$. Therefore the equality (17) is satisfied:

$$\sum_{1 \leq x, y \leq m} (R^{-1})_{y,i}^{x,j} R_{x,l}^{y,k} \mu_j \mu_y^{-1}$$
$$= \sum_{1 \leq x, y \leq m} R_{x',y}^{j,i'} (R^{-1})_{l,k'}^{x',y} \mu_j \lambda_j^{-1} \lambda_l^{-1} = \delta_l^j \delta_{k'}^{i'} \lambda_j \lambda_l^{-1} = \delta_l^j \delta_k^i.$$

Now Theorem 5.4 implies that

$$\alpha^{w(D)} \beta^{\operatorname{rot}(D)} T_S(L) = \sum_{f \in \operatorname{CSt}(D)} \Pi(f) \int_D f. \tag{24}$$

We shall prove that the right-hand sides of formulas (23) and (24) are equal. Rewrite first (20, 22) as follows:

$$R_{i,j}^{k,l} = \lambda_k \lambda_j^{-1} (R^{-1})_{j,l'}^{i',k} = \lambda_{j'} \lambda_{k'}^{-1} (R^{-1})_{k',i}^{l,j'} = R_{l',k'}^{j',i'}. \tag{25}$$

Here the first term is equal to the second and fourth ones because of (20, 22); the third and fourth terms are equal because of (22).

Since all edges of D are oriented we have an inclusion $\operatorname{Edg} D \hookrightarrow \operatorname{Edg} E$. If $g \in \operatorname{St}(E)$ then the restriction of g to $\operatorname{Edg} D$ is a state of D denoted by g^*. The mapping $g \mapsto g^*: \operatorname{St}(E) \to \operatorname{St}(D)$ is bijective.

In a neighbourhood of a vertex $u \in \operatorname{Vert} E$ the diagram D has one of 4 possible orientations, given in Fig. 14. Let a, b, c, d be edges of E incident to u and oriented as in Fig. 14 (i). Then $\pi_u(g) = R_{g(a),g(b)}^{g(c),g(d)}$ and $\pi_u(g^*)$ is equal to one of the following 4 matrix elements:

$$R_{g(a),g(b)}^{g(c),g(d)}; \quad (R^{-1})_{g(b),g(d')}^{g(a'),g(c)}; \quad (R^{-1})_{g(c'),g(a)}^{g(d),g(b')}; \quad R_{g(d'),g(c')}^{g(b'),g(a')}.$$

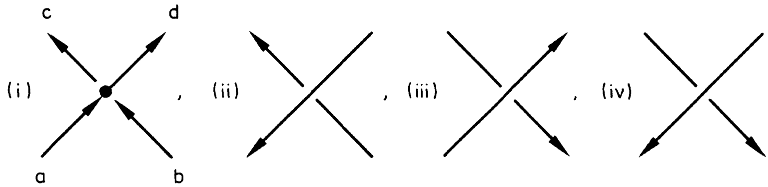

Fig. 14

It follows from (25) and the identity $g(e')=g(e)'$ for $e\in \mathrm{Edg}\, E$ that these 4 elements of K are non-zero iff at least one of them is non-zero. Thus, $\pi_u(g)\neq 0$ iff $\pi_u(g^*)\neq 0$. Therefore, the mapping $g\mapsto g^*: \mathrm{St}(E)\to \mathrm{St}(D)$ transforms the set $\mathrm{CSt}(E)$ bijectively onto $\mathrm{CSt}(D)$. To complete the proof of the lemma we shall show that for any $g\in \mathrm{CSt}(E)$

$$\Pi(g)\int_E g=\Pi(g^*)\int_D g^*. \tag{26}$$

Let z_1, z_2 be the coordinates in $\mathbf{R}^2$. Applying if necessary an ambient isotopy we may assume that in a small disc neighbourhood W_u of any vertex u of D the diagram D looks as in Fig. 14, i.e. the overcrossing branch is a segment of line parallel to the line $z_1=z_2$, and the undercrossing branch is a segment parallel to the line $z_1=-z_2$. We shall also assume that D lies in a generic position with respect to the projection $(z_1, z_2)\mapsto z_2: \mathbf{R}^2\to \mathbf{R}$. Let X be the (finite) set of points of D in which the tangent to D line is parallel to the horizontal line $z_2=0$. Note that $X\subset D\setminus \cup W_u$. Let us flatten Γ_D as in Sect. 5.3. Let Γ^o be the graph in $\mathbf{R}^2$ obtained by this flattening (see Fig. 8). In each disc W_u may lie points of Γ^o in which the tangent line of Γ^o is parallel to the line $z_2=0$. Denote this set of points by Y_u and put $Y=\bigcup_u Y_u$. Deforming, if necessary, Γ^o assume that $u\notin Y_u$ for all vertices u. Let $a(x)$ and $\varepsilon(x)$ be the same objects as in Sect. 5.3.3. To prove (26) we shall need the following formula: If $f\in \mathrm{CSt}(D)$ then

$$\int_D f=\prod_{x\in X\cup Y} \lambda_{f(a(x))}^{\varepsilon(x)} \tag{27}$$

(compare with (16) and (21)). In the proof of (27) (and only here) I will use the additive notation for the group operation (multiplication) in K^*. In particular, $\mu_i=2\lambda_i$ for all i. As in Sect. 6.1, the formal sum $\Sigma \lambda_{f(e)} e$, over $e\in \mathrm{Edg}\, D$ is a 1-cycle in Γ^o with coefficients in K^*. Denote the class of this cycle in $H_1(\Gamma^o; K^*)$ by σ. Clearly, $2\sigma=[f]$ where $[f]$ is the homological class of the sum $\sum_e \mu_{f(e)} e$. Let r: $S^1\to S^1$ be the two-sheeted covering. It is evident that r acts in $H_1(S^1; K^*)$ as multiplication by 2. If $\Delta: \Gamma^o\to S^1$ is the Gauss mapping (see Sect. 5.3) then

$$\int_D f=\Delta_*([f])=2\Delta_*(\sigma)=p_*(\Delta_*(\sigma))=(p\circ\Delta)_*(\sigma).$$

The equality $\int_D f=(p\circ\Delta)_*(\sigma)$ implies (27).

Flatten Γ_E in $\bigcup_u W_u$ according to the instructions of Sect. 6.1. In each disc W_u lie exactly 4 points of the flattened graph in which its tangent line is parallel to the line $z_2=0$, see Fig. 13. It is evident that the product of the expressions $\lambda_{g(a(x))}^{\varepsilon(x,a(x))}$ corresponding to these 4 points is equal to 1. Thus (21) implies that for $g\in \mathrm{CSt}(E)$

$$\int_E g = \prod_{x\in X} \lambda_{g(a(x))}^{\varepsilon(x,a(x))}.$$

A comparison of this formula with (27), where $f=g^*$, shows that to prove (26) we have to prove the local equality

$$\pi_u(g)=\pi_u(g^*) \prod_{x\in Y_u} \lambda_{g(a(x))}^{\varepsilon(x,a(x))} \tag{28}$$

for every $u\in \mathrm{Vert}\, E$.

Let a, b, c, d be the edges of E incident to u and oriented as in Fig. 14 (i). Put $i=g(a)$, $j=g(b)$, $k=g(c)$ and $l=g(d)$. Then $\pi_u(g)=R_{i,j}^{k,l}$. In accordance with 4 possible orientations of D in W_u (see Fig. 14) the right-hand side of (28) is easily computed to be one of 4 terms of (25). Therefore, (25) implies (28).

6.3. *Proof of Theorem* 4.3.4. Put $\lambda_i=q^{\bar{i}-(m+1)/2}$ for $i=1, \dots, m$. It is easy to verify that the EYB-operator (R_v,μ,α,β) constructed in Sect. 4.3 and the sequence $\lambda_1, \dots, \lambda_m$ satisfy Conditions 6.1.1, 6.1.2 and 6.1.3. Instead of (22) we have

$$(R_v^{-1})_{i,j}^{k,l}=\varepsilon(i)\,\varepsilon(l)\,\lambda_i\,\lambda_l^{-1}(R_v)_{k',i}^{l,j'} \tag{29}$$

for any $i, j, k, l\in\{1, 2, \dots, m\}$. I will first consider the case $v=-1$. In this case $\varepsilon(i)\equiv 1$ so that (29) coincides with (22) which enables us to apply Lemma 6.2.

Denote the YB-operator R_{-1} by R and the EYB-operator $(R_{-1}, \mu, \alpha, \beta)$ by S. If D is a diagram of an oriented link L and E is the underlying non-oriented diagram then Lemma 6.2 and equality $\beta=1$ imply

$$\tilde{Q}_{m,-1}(D)=\alpha^{w(D)}\, T_S(L)= \sum_{g\in \mathrm{CSt}(E)} \Pi(g) \int_E g. \tag{30}$$

This directly implies the first statement of the theorem.

Let us prove (12) confining ourselves for simplicity to the case of even m (so that $i'\neq i$ for all i). Denote the sum which stands in the right-hand side of (30) by $Q(E)$. We have to prove that for any non-oriented link diagrams E_+, E_-, E_0, E_∞ coinciding outside some disc and looking as in Fig. 4 inside this disc

$$Q(E_+)-Q(E_-)=(q-q^{-1})(Q(E_0)-Q(E_\infty)).$$

Schematically:

$$Q(\times)-Q(\times=(q-q^{-1})(Q()()-Q(\asymp)).$$

Denote by u the vertex of E_+, E_- pictured in Fig. 4. Let a, b, c, d be edges of E_+, incident to u and oriented as in Fig. 14 (i). Let $g\in \mathrm{CSt}(E_+)$. Put $i=g(a)$, $j=g(b)$, $k=g(c)$ and $l=g(d)$. Since $R_{i,j}^{k,l}=\pi_u(g)\neq 0$ there are 6 mutually exclusive cases: (I) $k=l=j=i$ and then $\pi_u(g)=q$; (II) $k=j$, $l=i\neq j,j'$ and $\pi_u(g)=1$; (III)

$k=l'=j=i'$ and $\pi_u(g)=q^{-1}$; (IV) $k=i<l=j\neq i'$ and $\pi_u(g)=q-q^{-1}$; (V) $k=i<l=j=i'$ and $\pi_u(g)=(q-q^{-1})(1-\lambda_i^2)$; (VI) $i=j'<l=k'$, $i\neq k$ and $\pi_u(g)=(q^{-1}-q)\lambda_i\lambda_l^{-1}$. The set $\mathrm{CSt}(E_+)$ splits in the disjoint union of 6 subsets, say, $A_1, \ldots, A_6$ singled out respectively by these possibilites for $\pi_u(g)$. This splitting induces a splitting of $Q(E_+)$ in a sum of 6 summands. Schematically:

$$Q(E_+)=Q\begin{pmatrix} i & & i \\ & \times & \\ i & & i \end{pmatrix}+\underset{j\neq i,\, i'}{Q\begin{pmatrix} j & & i \\ & \times & \\ i & & j \end{pmatrix}}+Q\begin{pmatrix} i' & & i \\ & \times & \\ i & & i' \end{pmatrix}$$
$$+\underset{i<j\neq i'}{Q\begin{pmatrix} i & & j \\ & \times & \\ i & & j \end{pmatrix}}+\underset{i<i'}{Q\begin{pmatrix} i & & i' \\ & \times & \\ i & & i' \end{pmatrix}}+\underset{k\neq i<k'}{Q\begin{pmatrix} k & & k' \\ & \times & \\ i & & i' \end{pmatrix}}$$

(It is understood that we sum up over all permitted i, j, k.) Each state $g\in A_1$ determines in the evident fashion a (contributing) state g^* of E_0. This gives all contributing states of E_0 whose values on two pictured in Fig. 4 and oriented upwards edges of E_0 are equal. Clearly

$$\int_{E_0} g^*=\int_{E} g, \qquad \Pi(g)=\pi_u(g)\,\Pi(g^*)=q\,\Pi(g^*).$$

Thus,

$$Q\begin{pmatrix} i & & i \\ & \times & \\ i & & i \end{pmatrix}=q\,Q(i\upharpoonleft\!\upharpoonright i).$$

Analogously, using (21),

$$Q\begin{pmatrix} i' & & i \\ & \times & \\ i & & i' \end{pmatrix}=q^{-1}\,Q\begin{pmatrix} i \\ \rightleftarrows \\ i \end{pmatrix};$$

$$\underset{i<j\neq i'}{Q\begin{pmatrix} i & & i' \\ & \times & \\ i & & i' \end{pmatrix}}=(q-q^{-1})\underset{i<j\neq i'}{Q(i\upharpoonleft\!\upharpoonright j)};$$

$$\underset{i<i'}{Q\begin{pmatrix} i & & i' \\ & \times & \\ i & & i' \end{pmatrix}}=(q-q^{-1})\left[\underset{i<i'}{Q(i\upharpoonleft\!\upharpoonright i')}-\underset{i<i'}{Q\begin{pmatrix} i' \\ \rightleftarrows \\ i \end{pmatrix}}\right];$$

$$\underset{k\neq i<k'}{Q\begin{pmatrix} k & & k' \\ & \times & \\ i & & i' \end{pmatrix}}=(q^{-1}-q)\underset{i<j\neq i'}{Q\begin{pmatrix} j \\ \rightleftarrows \\ i \end{pmatrix}}$$

Here each expression $Q(\ldots)$ denotes the sum of products $\Pi(g)\int g$ over those contributing stages g of the corresponding non-oriented (!) link diagram, whose values on the pictured oriented edges satisfy the pointed out equalities and inequalities.

Summing it up, we obtain

$$Q(\times)=qQ(i\,\curvearrowleft\curvearrowright\,i)+\underset{j\neq i,i'}{Q\begin{pmatrix} j & & i\\ & \times & \\ i & & j\end{pmatrix}}+q^{-1}Q\begin{pmatrix} i\\ \asymp \\ i\end{pmatrix}$$

$$+(q-q^{-1})\underset{i<j}{Q(i\,\curvearrowleft\curvearrowright\,j)}+(q^{-1}-q)\underset{i<j}{Q\begin{pmatrix} j\\ \asymp \\ i\end{pmatrix}}. \tag{31}$$

An analogous formula holds for $Q(\times)$. Namely, rotate E_- around u to the angle 90°, apply (31) and then rotate all the diagrams involved in the right-hand side of (31) to $-90°$. After some evident renotation we get

$$Q(\times)=qQ\begin{pmatrix} i\\ \asymp \\ i\end{pmatrix}+\underset{j\neq i,i'}{Q\begin{pmatrix} j & & i\\ & \times & \\ i & & j\end{pmatrix}}+q^{-1}Q(i\,\curvearrowleft\curvearrowright\,i)$$

$$+(q-q^{-1})\underset{j<i}{Q\begin{pmatrix} j\\ \asymp \\ i\end{pmatrix}}+(q^{-1}-q)\underset{i>j}{Q(i\,\curvearrowleft\curvearrowright\,j)}. \tag{32}$$

Here I used the obvious equalities

$$\underset{i<j}{Q(i\,\curvearrowright\curvearrowleft\,j)}=\underset{i<j}{Q(i'\,\curvearrowleft\curvearrowright\,j')}=\underset{i>j}{Q(i\,\curvearrowleft\curvearrowright\,j)}.$$

It is easy to understand that

$$\underset{j\neq i,i'}{Q\begin{pmatrix} j & & i\\ & \times & \\ i & & j\end{pmatrix}}=\underset{j\neq i,i'}{Q\begin{pmatrix} j & & i\\ & \times & \\ i & & j\end{pmatrix}}.$$

(Here it is important that for $g\in A_2$, $\pi_u(g)=1$). Therefore, subtracting (32) from (31) we get

$$Q(\times)-Q(\times)=(q-q^{-1})\Bigg[Q(i\,\curvearrowleft\curvearrowright\,i)+\underset{i<j}{Q(i\,\curvearrowleft\curvearrowright\,j)}+\underset{i>j}{Q(i\,\curvearrowleft\curvearrowright\,j)}$$

$$-Q\begin{pmatrix} i\\ \asymp \\ i\end{pmatrix}-\underset{i<j}{Q\begin{pmatrix} j\\ \asymp \\ i\end{pmatrix}}-\underset{i>j}{Q\begin{pmatrix} j\\ \asymp \\ i\end{pmatrix}}\Bigg]$$

$$=(q-q^{-1})[Q(\;)(\;)-Q(\asymp)].$$

Consider now the case $\nu=1$. We shall slightly modify R_1 to satisfy (22). Put $R=R_1$ and put

$$\tilde{R}=q\sum_i E_{i,i}\otimes E_{i,i}+\sum_{\substack{i,j\\ i\neq j,j'}} \varepsilon(i)\,\varepsilon(j)\,E_{i,j}\otimes E_{j,i}$$

$$-q^{-1}\sum_i E_{i,i'}\otimes E_{i',i}+(q-q^{-1})\sum_{i<j}[E_{i,i}\otimes E_{j,j}+q^{\bar{i}-\bar{j}}\,E_{i,j'}\otimes E_{i',j}].$$

(Here $i, j=1, 2, \ldots, m$). The matrices of R and $\tilde{R}$ are related by the formula

$$\tilde{R}^{k,l}_{i,j}=\varepsilon(i)\,\varepsilon(k)\,R^{k,l}_{i,j}.$$

This implies that $\tilde{R}$ is invertible and

$$(\tilde{R}^{-1})^{k,l}_{i,j}=\varepsilon(i)\,\varepsilon(k)(R^{-1})^{k,l}_{i,j}.$$

If $R^{k,l}_{i,j}\neq 0$ then either the ordered pairs $\varepsilon(k)$, $\varepsilon(l)$ and $\varepsilon(i)$, $\varepsilon(j)$ are equal or $\varepsilon(i)=-\varepsilon(j)$ and $(\varepsilon(k), \varepsilon(l))=(\varepsilon(j), \varepsilon(i))$. In the first case $\tilde{R}^{k,l}_{i,j}=R^{k,l}_{i,j}$, in the second case $\tilde{R}^{k,l}_{i,j}=-R^{k,l}_{i,j}$. An analogous statement is valid for R^{-1}. Therefore, if z: $V^{\otimes n}\to V^{\otimes n}$ is a composition of several homomorphisms $R_i(n)$, $R_i(n)^{-1}$ and if $\tilde{z}$: $V^{\otimes n}\to V^{\otimes n}$ is the corresponding composition of $\tilde{R}_i(n)$, $\tilde{R}_i(n)^{-1}$, then the matrix elements of $z, \tilde{z}$ are related as follows. Let $I=(i_1, \ldots, i_n)$ and $J=(j_1, \ldots, j_n)$ be two sequences of elements of the set $\{1, 2, \ldots, m\}$. If $z^J_I=0$, then $\tilde{z}^J_I=0$. If $z^J_I\neq 0$ then the sequences $(\varepsilon(i_1), \ldots, \varepsilon(i_n))$ and $(\varepsilon(j_1), \ldots, \varepsilon(j_n))$ may be obtained from each other by several, say, $p(I, J)$ transposition $(1, -1)\leftrightarrow(-1, 1)$. Then

$$\tilde{z}^J_I=(-1)^{P(I,J)}\,p^J_I.$$

This easily implies that $\tilde{R}$ is a YB-operator and that the same μ, α, β as in Theorem 4.3.2 (the case $\nu=1$) enhance $\tilde{R}$ to a EYB-operator. The identity $p(I, I)=0$ implies that (R, μ, α, β) and $(\tilde{R}, \mu, \alpha, \beta)$ give rise to the same invariant of braids and links. The formula (29) gives the equalities

$$(\tilde{R}^{-1})^{k,l}_{i,j}=\varepsilon(k)\,\varepsilon(k')\,\lambda_i\,\lambda_l^{-1}\,\tilde{R}^{l,j'}_{k',i}=-\lambda_i\,\lambda_l^{-1}\,\tilde{R}^{l,j'}_{k',i}.$$

We see, finally, that the EYB-operator $(\sqrt{-1}\,\tilde{R}, \mu, \sqrt{-1}\,\alpha, \beta)$ and the sequence $\lambda_1, \ldots, \lambda_m$ satisfy Conditions 6.1.1, 6.1.2, 6.1.3 and the equality (22). This EYB-operator gives rise to the same link invariant as (R, μ, α, β), namely, to $Q_{m,1}$. Now the same argument as in the case $\nu=-1$ can be applied to this EYB-operator which proves the theorem for $\nu=1$.

6.4. *Remark.* In Sect. 6.1 and 6.2 one could use instead of the involution $i\mapsto i'=m+1-i$ an arbitrary involution of the set $\{1, 2, \ldots, m\}$.

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