$*$ (A, μ, ι) is an algebra if

$\mu : A \otimes A \to A$, $\iota : k \to A$ are linear maps s.t.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes A \arrow[r, "\iota\otimes id"] \arrow[dr, "\simeq"'] & A\otimes A \arrow[d, "\mu"] & A\otimes k \arrow[l, "id\otimes\iota"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}$$

$*$ (C, Δ, ε) is a coalgebra if

$\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ are linear maps s.t.

$$\begin{tikzcd}
C\otimes C\otimes C & C\otimes C \arrow[l, "\Delta\otimes id"'] \\
C\otimes C \arrow[u, "id\otimes\Delta"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes C \arrow[dr, "\simeq"'] & C\otimes C \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & C\otimes k \arrow[dl, "\simeq"] \\
& C \arrow[u, "\Delta"] &
\end{tikzcd}$$

Let $\Delta(x) = \sum x_i \otimes y_i$. Then coassociativity says

$$\sum \Delta(x_i) \otimes y_i = \sum x_i \otimes \Delta(y_i)$$

$\rightsquigarrow$ we have general coassociativity.

And $\sum \varepsilon(x_i)\, y_i = \sum x_i\, \varepsilon(y_i) = x$ by counit property.

$*$ For a bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$ $\left(\Delta, \varepsilon : \text{alg hom} \overset{\text{or}}{\Longleftrightarrow} \mu, \iota : \text{coalg hom}\right)$

An antipode $S : H \to H$ is a linear map such that

$$\sum_i x_i\, S(y_i) = \varepsilon(x)\, 1 = \sum_i S(x_i)\, y_i \quad \text{if } \Delta(x) = \sum x_i \otimes y_i .$$

$\boxed{\text{e.g}}$ $\Delta^{(5)}(x) = \sum x_i \otimes y_i \otimes z_i \otimes u_i \otimes w_i \xleftarrow[\text{write}]{\text{we}} \sum_{(x)_5} x' \otimes x'' \otimes x''' \otimes x'''' \otimes x'''''$.

what is $\sum x_i \otimes y_i S(z_i) \otimes u_i \otimes w_i$?

$= (id \otimes m\circ(id\otimes S) \otimes id \otimes id)(id \otimes \Delta \otimes id \otimes id)\Delta^{(4)}(x) = (id \otimes \underbrace{\varepsilon(x)\, 1} \otimes id \otimes id)\Delta^{(4)}(x) = \cancel{id\otimes 1\otimes}\ \sum u_i \otimes 1 \otimes v_i \otimes w_i$

if $\Delta^{(3)}(x) = \sum u_i \otimes v_i \otimes w_i$. $= \sum_{(x)_3} x' \otimes 1 \otimes x'' \otimes x'''$

* Review

$$U_q = U_q(\mathfrak{sl}_2) = \left\langle E, K^{\pm}, F \;;\; \begin{array}{l} KE = qEK, \quad KK^{-1} = K^{-1}K = 1, \quad KF = q^{-1}FK \\ EF - FE = (K - K^{-1})/(q^{1/2} - q^{-1/2}) \end{array} \right\rangle$$

We regard $K = q^{H/2}$.

$$[m] = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} = q^{-(m-1)/2} \frac{q^m - 1}{q - 1} = (m)_q \, q^{(1-m)/2}$$

U_q has a Hopf algebra structure via

$$\Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}, \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

$$\varepsilon(K^{\pm}) = 1, \quad \varepsilon(E) = 0, \quad \varepsilon(F) = 0$$

$$S(K^{\pm}) = K^{\mp}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF$$

* $f : U \longrightarrow V$ linear map

$\Rightarrow f^* : V^* \longrightarrow U^*$ the transpose

$$\alpha \longmapsto \alpha \circ f = f^*(\alpha)$$

i.e. $\langle u, f^*\alpha \rangle := \langle fu, \alpha \rangle$

* $\bar{\lambda} : V^* \otimes V^* \longrightarrow (V \otimes V)^*$

$$\alpha \otimes \beta \longmapsto \bar{\lambda}(\alpha \otimes \beta) : V \otimes V \longrightarrow \mathbb{C}$$

$$(u \otimes v) \longmapsto \alpha(u)\beta(v)$$

i.e. We regard $\alpha \otimes \beta \in V^* \otimes V^*$ as $\langle u \otimes v, \alpha \otimes \beta \rangle := \langle u, \alpha \rangle \langle v, \beta \rangle$.

Prop $\bar{\lambda}$ is an isomorphism if $\dim V < \infty$.

* Sweedler's notation

If $\Delta(x) = \sum_i x_i \otimes x^i$, we just write $\Delta(x) = \sum_{(x)} x' \otimes x''$.

$\boxed{\text{Def}}$ A pair (X, A) of bialgebras is called 'matched' if there exist linear maps

$$\alpha : A \otimes X \longrightarrow X \qquad \text{and} \qquad \beta : A \otimes X \longrightarrow A$$
$$a \otimes x \longmapsto a \cdot x \qquad\qquad a \otimes x \longmapsto a^x$$

such that

① X is a (left) A-module-coalgebra via α, that is,

$\begin{cases} \alpha \text{ gives a left } A\text{-module structure on the v.s. } X. \ (A\text{: algebra}) \\ \alpha \text{ is a coalg morphism.} \end{cases}$

② A is a (right) A-module-coalgebra via β.

③
$$a \cdot (xy) = \sum_{(a),(x)} (a' \cdot x')(a''^{x''} \cdot y)$$

$$a \cdot 1 = \varepsilon(a)\, 1$$

$$(ab)^x = \sum_{(b),(x)} a^{b' \cdot x'}\, b''^{x''}$$

$$1^x = \varepsilon(x)\, 1$$

$$\sum_{(a),(x)} a'^{x'} \otimes a'' \cdot x'' = \sum_{(a),(x)} a''^{x''} \otimes a' \cdot x'$$

for all $a, b \in A$ and $x, y \in X$

Rmk ① $A \otimes X$ has a coalgebra structure as follows.

$$\begin{cases} \Delta_{A\otimes X} = (1 \otimes \tau \otimes 1) \circ (\Delta_A \otimes \Delta_X) \\ \varepsilon_{A\otimes X} = \varepsilon_A \otimes \varepsilon_B \end{cases}$$

that is,
$$\begin{cases} \Delta_{A\otimes X}(a \otimes x) = \sum_{(a),(x)} a' \otimes x' \otimes a'' \otimes x'' \\ \varepsilon_{A\otimes X}(a \otimes x) = \varepsilon(a)\,\varepsilon(x) \end{cases}$$

② For two coalgebras P, Q, a linear map $\varphi : P \to Q$ is a coalgebra map, if

$$\Delta_Q \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_P \quad \text{and} \quad \varepsilon_Q \circ \varphi = \varepsilon_P$$

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\Delta_P"'] & Q \arrow[d, "\Delta_Q"] \\ P \otimes P \arrow[r, "\varphi \otimes \varphi"] & Q \otimes Q \end{tikzcd} \qquad \begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon_P"'] & & Q \arrow[dl, "\varepsilon_Q"] \\ & k & \end{tikzcd}$$

So in our case, α satisfies

$$\begin{cases} \Delta_X(a \cdot x) = \sum_{(a),(x)} a' \cdot x' \otimes a'' \cdot x'' \\ \varepsilon_X(a \cdot x) = \varepsilon(a)\,\varepsilon(x) \end{cases} \quad \text{and} \quad (ab) \cdot x = a \cdot (b \cdot x)$$

$a, b \in A$, $x \in X$

And β satisfies

$$\begin{cases} \Delta_A(a^x) = \sum_{(a),(x)} a'^{x'} \otimes a''^{x''} \\ \varepsilon_A(a^x) = \varepsilon(a)\,\varepsilon(x) \end{cases} \quad \text{and} \quad (a^x)^y = a^{xy}$$

$a \in A$, $x, y \in X$

Thm (X, A) : matched pair of bialgebras.

$\Rightarrow$ The vector space $X \otimes A$ has a bialgebra structure as follows:

[multiplication] $\quad (x \otimes a)(y \otimes b) = \sum_{(a),(y)} x(a' \cdot y') \otimes a''^{\,y''} b$

[Unit] $\quad 1 \otimes 1$

[Comultiplication] $\quad \Delta(x \otimes a) = \sum_{(a),(x)} (x' \otimes a') \otimes (x'' \otimes a'')$

[Counit] $\quad \varepsilon(x \otimes a) = \varepsilon(x)\, \varepsilon(a)$

Rmk ① ~~$i_X : X \otimes A \longrightarrow$~~ We write $X \bowtie A$ or $X \underset{(\alpha,\beta)}{\bowtie} A$.

② $i_X : X \longrightarrow X \bowtie A$, $\quad i_A : A \longrightarrow X \bowtie A$

$\quad x \longmapsto x \otimes 1 \qquad\qquad a \longmapsto 1 \otimes a$

are bialgebra morphism.

And we have $\quad x \otimes a = (x \otimes 1)(1 \otimes a) = i_X(x)\; i_A(a)$.

③ If X, A has antipodes, respectively.

$X \bowtie A$ has an antipode given by

$$S(x \otimes a) = \sum_{(a),(x)} S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')}$$

⟨Proof⟩ [Associativity]

$$((x\otimes a)(y\otimes b))(z\otimes c) = \sum_{(a),(y)} (x(a'\cdot y')\otimes a''^{y''}b)(z\otimes c)$$

$$= \sum_{(a)_3,(y)_3,(a''^{y''}b),(z)} x(a'\cdot y')((a''^{y''}b)'\cdot z')\otimes (a''^{y''}b)''^{z''}c$$

$$\downarrow \beta,\Delta \qquad \downarrow \Delta,\beta$$

$$= \sum_{(a)_3,(y)_3,(b),(z)} x(a'\cdot y')((a''^{y''}b')\cdot z')\otimes (a'''^{y'''}b'')^{z''}c$$

$$\downarrow (ab)^x$$

$$= \sum_{(a)_3,(b),(y)_3,(z)} x(a'\cdot y')((a''^{y''}\boxed{b'})\cdot\boxed{z'})\otimes (a'''^{y'''})^{(b'')'\cdot(z'')'}((b'')'')^{(z'')''}c$$

$$= \sum_{(a)_3,(b)_3,(y)_3,(z)_3} x(a'\cdot y')((a''^{y''}b')\cdot z')\otimes (a'''^{y'''})^{b''\cdot z''}b'''^{z'''}c$$

$$\downarrow \beta$$

$$= \sum x(a'\cdot y')((a''^{y''}b')\cdot z')\otimes a'''^{y'''(b''\cdot z'')}b'''^{z'''}c$$

$$(x\otimes a)((y\otimes b)(z\otimes c)) = \sum_{(b),(z)} (x\otimes a)(y(b'\cdot z')\otimes b''^{z''}c)$$

$$= \sum_{(a),(b),(y(b'\cdot z')),(z)} x(a'\cdot(y(b'\cdot z'))')\otimes a''^{(y(b'\cdot z'))''}b''^{z''}c$$

$$= \sum_{(a),(b),(y),(z)} x(a'\cdot(y'(b'\cdot z')'))\otimes a''^{(y''(b'\cdot z')'')}\boxed{b''}^{\boxed{z''}}c$$

$$= \sum_{(a),(b)_3,(y),(z)_3} x(a'\cdot(y'(b'\cdot z')))\otimes a''^{(y''(b''\cdot z''))}b'''^{z'''}c$$

$$\downarrow a\cdot(xy)$$

$$= \sum_{(a),(b)_3,(y),(z)_3} x((a')'\cdot(y')')\cancel{((a')''\cdot(b'\cdot z')'')}(a')''^{(y')''}\cdot(b'\cdot z')\otimes(\boxed{a''}^{\boxed{y''}})^{b''\cdot z''}b'''^{z'''}c$$

$$= \sum_{(a)_3,(b)_3,(y)_3,(z)_3} x((a'\cdot y')(a''^{y''}\cdot(b'\cdot z')))\otimes a'''^{y'''(b''\cdot z'')}b'''^{z'''}c$$

$$\downarrow \alpha$$

$$= \sum_{(a),(b),(y),(z)} x((a'\cdot y')((a''^{y''}b')\cdot z'))\otimes a'''^{y'''(b''\cdot z'')}b'''^{z'''}c$$

* H : finite dim'l Hopf alg w/ invertible antipode

$\rightsquigarrow X = (H^{op})^* = (H^*, \Delta^* \circ \bar{\lambda}^{-1}, \varepsilon^*, \bar{\lambda}^{-1} \circ (\mu^{op})^*, \iota^*, (S^{-1})^*, S^*)$: Hopf alg.

[Multiplication] $X \otimes X = H^* \otimes H^* \xrightarrow{\bar{\lambda}^{-1}} (H \otimes H)^* \xrightarrow{\Delta^*} H^* = X$

$$((\Delta^* \circ \bar{\lambda}^{-1})(f \otimes g))(u) = \bar{\lambda}^{-1}(f \otimes g)(\Delta(u)) = \sum_{(u)} \langle u', f \rangle \langle u'', g \rangle = \langle u, \mu_X (f \otimes g) \rangle$$

[Unit] $k \simeq k^* \xrightarrow{\varepsilon^*} H^* = X$

$c \longmapsto m_c \longrightarrow \varepsilon^*(m_c)$

$\varepsilon^*(m_c)(h) = m_c(\varepsilon(h)) = c\,\varepsilon(h)$ $\quad\therefore\ \varepsilon^*(m_c) = c\,\varepsilon$

$(\varepsilon^*(1) = \varepsilon)$

So we regard ε^* as ε.

[Comultiplication] $X = H^* \xrightarrow{(\mu^{op})^*} (H \otimes H)^* \xrightarrow{\bar{\lambda}^{-1}} H^* \otimes H^* = X \otimes X$

$f \longmapsto \sum_{(f)} f' \otimes f'' =: \Delta_X(f)$

By $\bar{\lambda}$, $\langle u \otimes v, \Delta(f) \rangle \overset{\bar{\lambda}}{=\!=} \sum_{(f)} \langle u, f' \rangle \langle v, f'' \rangle$

$\| \ (\mu^{op})^*$

$\langle vu, f \rangle$

So, $\Delta_X(f) = \sum_{(f)} f' \otimes f'' \iff \sum_{(f)} \langle u, f' \rangle \langle v, f'' \rangle = \langle vu, f \rangle$.

[Counit] $\iota^* : H^* \xrightarrow{\sim} k^* \simeq k$

$f \longmapsto \iota^*(f) \iff \iota^*(f)(1) = f(\iota(1)) = f(1)$.

So $\iota^*(f) = f(1)$

[Antipode] $(S^{-1})^* : H^* \longrightarrow H^*$

$f \longmapsto (S^{-1})^* f$

$\langle u, (S^{-1})^* f \rangle = \langle S^{-1}(u), f \rangle$

* Matching between H and $X = (H^{op})^*$

Define
$$\alpha : H \otimes X \longrightarrow X$$
$$a \otimes f \longmapsto a \cdot f := \sum_{(a)} f(S^{-1}(a'') \_\_ a')$$

$$\beta : H \otimes X \longrightarrow H$$
$$a \otimes f \longmapsto a^f := \sum_{(a)} f(S^{-1}(a''')\, a')\, a''$$

Thm/Def The pair (X, H) is matched by α, β.

We define the quantum double $D(H)$ of H by ~~$H \bowtie X$~~ $X \bowtie H$

<pf> $\langle u, a \cdot f \rangle := \sum_{(a)} \langle S^{-1}(a'') u a', f \rangle$ by definition.

$$\langle u, a \cdot (b \cdot f) \rangle = \sum_{(a)} \langle S^{-1}(a'') u a', b \cdot f \rangle = \sum_{(a),(b)} \langle S^{-1}(b'') S^{-1}(a'') u a' b', f \rangle$$
$$= \sum_{(ab)} \langle S^{-1}((ab)'') u (ab)', f \rangle = \langle u, (ab) \cdot f \rangle .$$

Recall $\Delta(f) = \sum f' \otimes f'' \iff \sum_{(f)} \langle u, f' \rangle \langle v, f'' \rangle = \langle vu, f \rangle$

We will show $\Delta(a \cdot f) = \sum_{(a),(f)} a' \cdot f' \otimes a'' \cdot f''$ to show that α is a coalg map.

Hence we will show

$$\sum_{(a),(f)} \langle u, a' \cdot f' \rangle \langle v, a'' \cdot f'' \rangle = \langle vu, a \cdot f \rangle .$$

$$左 = \sum_{(a),(f)} \langle S^{-1}((a')'') u (a')', f' \rangle \langle S^{-1}((a'')'') v (a'')', f'' \rangle$$

$(\Delta \otimes \Delta) \circ \Delta(a)$

$$= \sum_{(a)_4,(f)} \langle S^{-1}(a'') u a', f' \rangle \langle S^{-1}(a'''') v a''', f'' \rangle$$
$$= \sum_{(a)_4} \langle (S^{-1}\boxed{a''''}) v a''')(S^{-1}(a'') u a'), f \rangle$$
$$= \sum_{(a)} \langle S^{-1}(a'') v u a', f \rangle = \langle vu, a \cdot f \rangle, \text{ etc.}$$

Lem The multiplication in $D(H)$ is given by

$$(f\otimes a)(g\otimes b) = \sum_{(a)} fg(S^{-1}(a''')\_\, a')\otimes a''b$$

<pf> $(f\otimes a)(g\otimes b) = \sum_{(a),(g)} f(a'\cdot g')\otimes a''g''b$

$$= \sum_{\substack{(a),(g)\\(a'),(a'')}} f\, g'(S^{-1}((a')'')\_\,(a')'))\otimes g''(S^{-1}((a'')''')\_\,(a'')')(a'')''b$$

$$= \sum_{(a),(g)} f(g'(S^{-1}(a'')\_\, a'))\otimes g''(S^{-1}(a''''')\_\, a''')a''''b$$

$$= \sum_{(a)} f(g(S^{-1}(a''''')\_\, \underset{\sim\sim\sim}{a''' S^{-1}(a'')}\_\, a'))\otimes a''''b$$

$$= \sum_{(a)} \underset{\sim\sim}{\varepsilon(a'')}\, fg(S^{-1}(a'''')\_\, a')\otimes \underset{\sim}{a'''}\, b$$

$$= \sum_{(a)} fg(S^{-1}(a''')\_\, a')\otimes a''b .$$

Rmk We can write the above equation as follows

$$(1\otimes a)(f\otimes 1) = \sum_{(a)} f(S^{-1}(a''')\_\, a')\otimes a''$$

$*$ $\lambda : H\otimes X \xrightarrow{\sim} \mathrm{End}(H)$ if $\dim H<\infty$

$a\otimes f \longmapsto \lambda(a\otimes f) : H \longrightarrow H$

$b \longmapsto f(b)a$

$\rho := \lambda^{-1}(\mathrm{id}_H) \in H\otimes X$. (If H has basis $\{e_i\}$ and $\{e^i\}$ be the dual basis, clearly $\rho = \sum e_i\otimes e^i$.

Define $R = (i_H\otimes i_X)(\rho) = \sum (1\otimes e_i)\otimes(e^i\otimes 1)$

$\boxed{\text{Thm}}$ $(D(H), R)$ is a quasi-triangular Hopf algebra.

<proof> First R has inverse $\bar{R} = \sum (1 \otimes e_i) \otimes ((e^i \circ S) \otimes 1)$

$\because$ $R\bar{R} = \sum_{i,j} (1 \otimes e_i e_j) \otimes (e^i (e^j \circ S) \otimes 1) \in D(H) \otimes D(H) = (X \bowtie H) \otimes (X \bowtie H)$

Evaluate at $a \otimes f \otimes b \otimes g$.

$\Rightarrow \sum_{i,j} \varepsilon(a) \langle e_i e_j, f \rangle \langle e^i (e^j \circ S), b \rangle g(1)$

[Next hour]

Example $H = k[G]$; the group algebra ($|G| < \infty$)

Recall that H has basis $\{g\}_{g\in G}$ and have coalg / Hopf structure

$$\Delta(g) = g\otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1} \qquad (g\in G)$$

Let $X = (H^{op})^*$. And $\{e_g\}_{g\in G}$ be the dual basis defined by

$$e_g(h) = \delta_{gh} \quad \text{for } h\in G.$$

[Multiplication] $\langle u, e_g e_h\rangle \underset{\substack{\uparrow \\ \Delta(u)=u\otimes u}}{=} \langle u, e_g\rangle\langle u, e_h\rangle = \delta_{gu}\delta_{hu} = \delta_{gh}\langle u, e_g\rangle.$

$$\Rightarrow \quad e_g e_h = \delta_{gh} e_g.$$

[Unit] $\underset{\substack{\shortparallel \\ (\varepsilon^*(1))(h)}}{\varepsilon^*(h)} = \varepsilon(h) = 1, \ \forall h\in G \quad \Rightarrow \quad \varepsilon^*(1) = \sum_{g\in G} e_g$ is the unit.

[Comultiplication] Let $\Delta(e_g) = \sum_{u,v} a_{uv}\, e_u\otimes e_v$. Then

$$\underbrace{\sum_{u,v} a_{uv}\langle x, e_u\rangle\langle y, e_v\rangle}_{=\,a_{xy}} = \langle yx, e_g\rangle = \delta_{g,yx}$$

$$\therefore \quad \Delta(e_g) = \sum_{vu=1} e_u\otimes e_v$$

[Counit] $\iota^*(e_g) = e_g(1) = \delta_{g1} = \varepsilon_X(e_g)$

[Antipode] $\langle h, (S^{-1})^* e_g\rangle = \langle S^{-1}(h), e_g\rangle = \langle h^{-1}, e_g\rangle = \delta_{g,h^{-1}} = \delta_{g^{-1},h}$

$$S_X(e_g) = (S^{-1})^*(e_g) = e_{g^{-1}}.$$

* The quantum double of $H = k[G]$.

$\Rightarrow D(H) = X \bowtie H$ has basis $\{e_g \otimes h\}_{g,h\in G}$.

Let's calculate the multiplication.

$$(e_g \otimes u)(e_h \otimes v) = \sum_{(u)} e_g\, e_h(S^{-1}(u''')\_\, u') \otimes u'' v$$

$$= e_g\, e_h(u^{-1}\_\, u) \otimes uv$$

$$= e_g\, e_{uhu^{-1}} \otimes uv = \delta_{g,\,uhu^{-1}}\, e_g \otimes uv$$

$$\Delta(e_g \otimes u) = \sum_{(e_g),(u)} (e_g' \otimes u') \otimes (e_g'' \otimes u'')$$

$$= \sum_{qp=g} (e_p \otimes u) \otimes (e_q \otimes u)$$

$$S(e_g \otimes u) = \sum_{(e_g),(u)} S(u'')\cdot S(e_g'') \otimes S(u')^{S(e_g')}$$

$$= \sum_{qp=g} u^{-1}\cdot e_{q^{-1}} \otimes (u^{-1})^{e_{p^{-1}}}$$

Recall $\begin{cases} g\cdot e_h = \sum_{(g)} e_h(S^{-1}(g'')\_\, g') = e_h(g^{-1}\_\, g) = e_{ghg^{-1}} \\ g^{e_h} = \sum_{(g)} e_h(S^{-1}(g''')g')\, g'' = e_h(g^{-1}g)\, g = \delta_{h1}\, g \end{cases}$

Hence $S(e_g \otimes u) = \sum_{qp=g} e_{u^{-1}q^{-1}u} \otimes \delta_{p1}\, u^{-1} = e_{u^{-1}g^{-1}u} \otimes u^{-1}$.

<u>Rmk</u> $(1 \otimes u)(e_h \otimes 1) = \sum_{g\in G} (e_g \otimes u)(e_h \otimes 1) = e_{uhu^{-1}} \otimes u$

$\Delta(1 \otimes u) = \sum_g \sum_{qp=g} (e_p \otimes u) \otimes (e_q \otimes u) = \sum_{p,q} (e_p \otimes u) \otimes (e_q \otimes u) = (1 \otimes u) \otimes (1 \otimes u)$.

* The universal R-matrix on $D(k[G])$.

We know $R = \sum_{g\in G} (1\otimes g)\otimes(e_g\otimes 1)$. $e_g\circ S : h \mapsto e_g(h^{-1}) = e_{g^{-1}}(h)$

① Put $\bar{R} = \sum_{g\in G}(1\otimes g)\otimes(e_g\circ S\otimes 1) = \sum_{g\in G}(1\otimes g)\otimes(e_{g^{-1}}\otimes 1)$

$$\Rightarrow R\bar{R} = \sum_{g,h\in G}(1\otimes gh)\otimes(e_g e_{h^{-1}}\otimes 1) = \sum_{g\in G}(1\otimes gg^{-1})\otimes(e_g\otimes 1)$$

$$= \sum_{g\in G}(1\otimes 1)\otimes(e_g\otimes 1) = \mathbb{1}\otimes\mathbb{1} \in D(H)\otimes D(H) \quad \left(\because \sum_{g\in G} e_g = \mathbb{1}_X\right)$$

Similarly $\bar{R}R = 1$.

③ $$\Delta^{op}(e_g\otimes h)\, R = \left(\sum_{qp=g}(e_q\otimes h)\otimes(e_p\otimes h)\right)\left(\sum_{u\in G}(1\otimes u)\otimes(e_u\otimes 1)\right)$$

$$= \sum_{\substack{qp=g\\ u}}(e_q\otimes h)(1\otimes u)\otimes\underbrace{(e_p\otimes h)(e_u\otimes 1)}_{p=huh^{-1}}$$

$$= \sum_{\substack{qhuh^{-1}=g\\ u}}(e_q\otimes hu)\otimes(e_{huh^{-1}}\otimes h) = \sum_{\substack{qv=g\\ v}}(e_q\otimes vh)\otimes(e_v\otimes h)$$

$$= \sum_{v}(e_{gv^{-1}}\otimes vh)\otimes(e_v\otimes h)$$

$$R\Delta(e_g\otimes h) = \left(\sum_{u\in G}(1\otimes u)\otimes(e_u\otimes 1)\right)\left(\sum_{qp=g}(e_p\otimes h)\otimes(e_q\otimes h)\right)$$

$$= \sum_{\substack{u\\ qp=g}}\big((1\otimes u)(e_p\otimes h)\big)\otimes\underbrace{\big((e_u\otimes 1)(e_q\otimes h)\big)}_{q=u}$$

$$= \sum_{u}\big((1\otimes u)(e_{u^{-1}g}\otimes h)\big)\otimes(e_u\otimes h)$$

$$= \sum_{u}(e_{gu^{-1}}\otimes uh)\otimes(e_u\otimes h)$$

② $(\Delta\otimes id)(R) = \sum_g \Delta(1\otimes g)\otimes(e_g\otimes 1) = \sum_g (1\otimes g)\otimes(1\otimes g)\otimes(e_g\otimes 1)$

$$R_{13}R_{23} = \sum_{g,h}\big((1\otimes g)\otimes(1\otimes 1)\otimes(e_g\otimes 1)\big)\big((1\otimes 1)\otimes(1\otimes h)\otimes(e_h\otimes 1)\big)$$

$$= \sum_{g,h}(1\otimes g)\otimes\cancel{(e_g\otimes h)}(1\otimes h)\otimes(e_g e_h\otimes 1) = \sum_g (1\otimes g)\otimes(1\otimes g)\otimes(e_g\otimes 1).$$

Similarly, $(id\otimes\Delta)(R) = R_{13}R_{12}$

Now $\quad u = \sum_{g\in G} S(e_g \otimes 1)(1\otimes g) = \sum_{g\in G} (e_{g^{-1}}\otimes 1)(1\otimes g) = \sum_{g\in G} e_{g^{-1}}\otimes g$

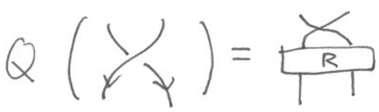
So $\quad S(u) = \sum_{g\in G} S(e_{g^{-1}}\otimes g) = \sum_{g\in G} e_{g^{-1}gg}\otimes g^{-1} = \sum_{g\in G} e_g \otimes g^{-1} = u.$

Hence we may choose v as u. $\quad (\because\ v^2 = u S(u))$

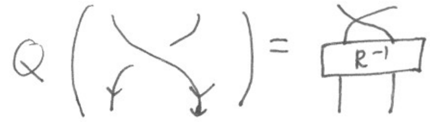
<u>Exercise</u> ① u is central in $D(H)$.

② $S^2(x) = u x u^{-1}$

* Recall the assignment of $Q^{D(H);*}$. Since $uv^{-1}=1$, only nontrivial assignments are



and



Hence Q gives the invariant 'Writh'.

* Application to $\bar{U}_q(sl_2)$

$q^d = 1$ (d : odd integer >1)

$\Rightarrow$ $[n] \neq 0$ for $n = 1, \cdots, d-1$ and $[d] = 0$, and $(d) = 0$.

Define $\bar{U}_q = U_q(sl_2) \Big/ \langle E^d, F^d, K^d - 1 \rangle$

$\rightsquigarrow$ $\bar{U}_q$ has a basis $\{ E^i F^j K^l \}_{0 \le i, j, l \le d-1}$. ($\dim \bar{U}_q \overset{d^3}{=} < \infty$)

$\boxed{\text{Prop}}$ $\pi : U_q \longrightarrow \bar{U}_q$ the canonical projection gives a Hopf structure on $\bar{U}_q$.

<pf> We must show that

$$\begin{cases} \Delta(E)^d = \Delta(F)^d = \Delta(K)^d - 1 = 0 \\ \varepsilon(E)^d = \varepsilon(F)^d = \varepsilon(K)^d - 1 = 0 \\ S(E)^d = S(F)^d = S(K)^d - 1 = 0 \end{cases}$$

(e.g. $(S(E))^d = (-EK^{-1})^d = \cancel{}\, E^d K^{-d} = 0$.)

Only nontrivials are $\Delta(E)^d = \Delta(F)^d$.

Recall $\Delta(E) = \overset{1 \otimes E \;+\; E \otimes K}{\cancel{E \otimes K + 1 \otimes E}}$. Also note $\overset{y}{(E \otimes K)} \overset{x}{(1 \otimes E)} = q \overset{x}{(1 \otimes E)} \overset{y}{(E \otimes K)}$.

$$\Rightarrow \quad \Delta(E)^n = \sum_{r=0}^{n} \binom{n}{r}_q (1 \otimes E)^r (E \otimes K)^{n-r} = \sum_{r=0}^{n} \binom{n}{r}_q E^{n-r} \otimes E^r K^{n-r}$$

Note that $\binom{d}{r}_q = 0$ for $r = 1, \cdots, d-1$ since $\dfrac{(d)!}{(r)!(d-r)!} = 0$

Hence $\Delta(E)^d = E^d \otimes K^d + E^0 \otimes E^d = 0$.

$\boxed{\text{Def}}$ $B_q := \bar{U}_q^{\ge 0} = \langle E^m K^n ;\ 0 \le m, n \le d-1 \rangle$.

<u>Rmk</u> B_q is a Hopf subalgebra of $\bar{U}_q$.

(Closed under, Δ, S, ε)

* We now construct $D(B_q)$.

Let's determine $X = (B_q^{op})^*$. Define linear forms on B_q by

$$\langle \alpha, E^m K^n \rangle = \delta_{m0}\, q^n, \qquad \langle \eta, E^m K^n \rangle = \delta_{m1}$$

Recall the product on X. For $\beta, \gamma \in X$, we have $\beta\gamma \in X$ s.t.

$$\langle \beta\gamma, E^m K^n \rangle = \sum_{(E^m K^n)} \langle \beta, (E^m K^n)'' \rangle \langle \gamma, (E^m K^n)' \rangle .$$

Note that $\Delta(E^m K^n) = \left(\sum_{r=0}^{m} \binom{m}{r}_q E^{m-r} \otimes E^r K^{m-r} \right)(K^n \otimes K^n)$

$$= \sum_{r=0}^{m} \binom{m}{r}_q E^{m-r} K^n \otimes E^r K^{m+n-r} .$$

Hence $\langle \beta\gamma, E^m K^n \rangle = \sum_{r=0}^{m} \binom{m}{r}_q \langle \beta, E^r K^{m+n-r} \rangle \langle \gamma, E^{m-r} K^n \rangle$

In particular, we have

$$\langle \beta\eta, E^m K^n \rangle = \binom{m}{m-1}_q \langle \beta, E^{m-1} K^{m+n-(m-1)} \rangle = (m)_q \langle \beta, E^{m-1} K^{n+1} \rangle \quad \text{(A)}$$

$$\langle \beta\alpha, E^m K^n \rangle = q^n \binom{m}{m}_q \langle \beta, E^m K^n \rangle = q^n \langle \beta, E^m K^n \rangle \quad \text{(B)}$$

By induction, we obtain

$$\langle \eta^i, E^m K^n \rangle = \delta_{m,i}\, (m)_q^! \quad \text{and} \quad \langle \alpha^j, E^m K^n \rangle = q^{jn} \delta_{m,0}$$

Hence $\langle \alpha^j \eta^i, E^m K^n \rangle = \delta_{m,i}\, (m)_q^! \cdot q^{(n+i)j}$

$\boxed{\text{Prop}}$ $\alpha^d = 1, \quad \eta^d = 0, \quad \alpha\eta\alpha^{-1} = q^{+1}\eta .$

<pf> $\langle \alpha\eta, E^m K^n \rangle \overset{\text{(A)}}{=} (m)_q \langle \alpha, E^{m-1} K^{n+1} \rangle = (1)_q\, q^{n+1} \cdot \delta_{m,1}$

$\langle \eta\alpha, E^m K^n \rangle \overset{\text{(B)}}{=} q^n \langle \eta, E^m K^n \rangle = q^n \cdot \delta_{m,1}$

Prop
$$\begin{cases} \Delta(\alpha) = \alpha \otimes \alpha \\ \Delta(\eta) = 1 \otimes \eta + \eta \otimes \alpha \end{cases} \qquad \begin{cases} \varepsilon(\alpha) = 1 \\ \varepsilon(\eta) = 0 \end{cases} \qquad \begin{cases} S(\alpha) = \alpha^{d-1} = \alpha^{-1} \\ S(\eta) = -\eta\alpha^{-1} \end{cases}$$

Moreover, $\{\alpha^j \eta^i\}$ forms a basis of X.

<pf> $\Delta(\eta) = 1\otimes\eta + \eta\otimes\alpha \iff \langle E^i K^j \otimes E^m K^n, \Delta(\eta)\rangle = \langle E^m K^n E^i K^j, \eta\rangle$.

$$右 = q^{ni}\langle E^{m+i}K^{n+j}, \eta\rangle = \delta_{m+i,1}\, q^{ni} = \delta_{m,0}\delta_{i,1} q^n + \delta_{m,1}\delta_{i,0}$$

$$左 = \langle E^i K^j \otimes E^m K^n, 1\otimes\eta + \eta\otimes\alpha\rangle = \delta_{i0}\delta_{m1} + \delta_{i1}\delta_{m0} q^n$$

Similarly, we have $\Delta(\alpha)$.

The formulas on ε, S comes easily from the properties of ε, S w/ Δ.

$$\begin{cases} \varepsilon(\alpha)\alpha = \alpha \\ \varepsilon(1)\eta + \varepsilon(\eta)\alpha = \eta \end{cases} \qquad \begin{cases} S(\alpha)\alpha = \varepsilon(\alpha)1 = \alpha S(\alpha) \\ S(1)\eta + S(\eta)\alpha = \varepsilon(\eta)1 = 1S(\eta) + \eta S(\alpha) \end{cases}$$

Now $\langle E^m K^n, \sum \lambda_{ij}\alpha^j\eta^i\rangle = 0$ $\forall\, m,n$ implies $\lambda_{ij} = 0$, $\forall i,j$.

$\Rightarrow \{\alpha^j\eta^i\}_{0\le i,j\le d-1}$ independent w/ d^2-elements.

$$左 = \sum_{i,j} \delta_{m,i}\,(m)_q!\, q^{(n+i)j}\lambda_{ij} = \sum_j (m)_q!\, q^{(n+m)j}\lambda_{mj} = 0, \quad \forall\, 0\le m,n\le d-1$$

$\boxed{\text{Prop}}$ In $D = D(B_q) = \langle \alpha^j \eta^i E^k K^l \rangle$, $\quad \alpha^j \eta^i E^k K^l$

we have $\quad K\alpha = \alpha K, \quad K\eta = q^{-1}\eta K$

$$E\alpha = q^{-1}\alpha E, \quad E\eta = -q^{-1}(1 - \eta E - \alpha K)$$

<u>Thm</u> $\chi : D(B_q) \longrightarrow \bar{U}$

$$g^j \eta^i E^k K^l \longmapsto \left(\frac{q^{1/2} - q^{-1/2}}{q}\right)^i q^{(i+j)k - \frac{i(i-1)}{2}} F^i E^k K^{i+j+l}$$

is a surjective Hopf alg morphism.